ARTINIAN BAND SUMS OF RINGS

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Abstract

Band sums of associative rings were introduced by Weissglass in 1973. The main theorem claims that the support of every Artinian band sum of rings is finite. This result is analogous to the well-known theorem on Artinian semigroup rings.


Let B be a band, that is, a semigroup consisting of idempotents. An associative ring R is called a band sum of its subrings $R_b$, $b \in B$, if $R = \bigoplus_{b \in B} R_b$, the sum is direct for additive groups, and $R_a R_b \subseteq R_{ab}$ for every $a, b \in B$. Band sums of rings were introduced in [11]. A discussion of relations between band sums and other ring and semigroup constructions is contained in [5], where several examples of effective applications of band sums are presented. Many articles have been devoted to the investigations of various properties of band sums. In particular, a wide range of properties were considered in [1]. For a fairly complete list of relevant references the reader may turn to [5]. Here we mention only the more recent articles [6, 8].

The aim of this paper is to investigate Artinian band sums of rings. Looking for a way of research it seems natural to consider the well-known results on Artinian semigroup rings as a guide. It was proved in [12] that if a semigroup ring $RS$ is Artinian then $R$ is Artinian and $S$ is finite. (The proof was simplified in [7, 9].) If $R$ and $S$ are commutative, a complete description of Artinian $RS$ was obtained in [4]. We shall deal with an Artinian band sum $R = \bigoplus_{b \in B} R_b$ and deduce analogous results. Our main theorem asserts that the set $\{b \mid R_b \neq 0\}$ is finite. Clearly, $B$ can be infinite since there may be infinitely many zero rings among the $R_b$.

Two examples will be given to show in general that each $R_b$ being Artinian is neither a necessary nor sufficient condition for $R$ to be Artinian. In the important case
where $B$ is commutative a corollary to the main theorem will show that $R$ is Artinian if all $R_b$ are Artinian and the set $\{b \mid R_b \neq 0\}$ is finite. The converse implication is not valid. Now let us state the main result.

**THEOREM.** If $B$ is a band and $R = \bigoplus_{b \in B} R_b$ is an Artinian band sum of rings, then all but a finite number of the $R_b$ are equal to zero.

By Artinian we mean right Artinian. Some definitions are needed for the proof. A band is called a semilattice (right zero band; left zero band; rectangular band) if it satisfies the identity $xy = yx$ ($xy = y$; $xy = x$; $xyx = x$). For each band $B$ there exist a semilattice $S$ and rectangular bands $H_s, s \in S$, such that $B = \bigoplus_{s \in S} H_s$, $H_s H_t \subseteq H_{st}$ for every $s, t \in S$, and $H_s \cup H_t = \phi$ whenever $s \neq t$ (cf. [2]). In this case $B$ is said to be a semilattice $S$ of rectangular bands $H_s$. It is easily seen that each rectangular band $H$ is isomorphic to a direct product of a left zero band and a right zero band. Note that every semilattice $S$ is a partially ordered set under the natural order $\leq$ defined by the rule $s \leq t$ if and only if $st = s$ (cf. [2, §1.8]).

Let $R = \bigoplus_{b \in B} R_b$ be a band sum of rings. Each element $r$ in $R$ has a unique representation of the form $r = \sum_{b \in B} r_b$, where $r_b \in R_b$. The set

$$\supp(r) = \{b \in B \mid r_b \neq 0\}$$

is called the support of $r$. For any $I \subseteq R$ the union $\bigcup_{i \in I} \supp(r)$ will be denoted by $\supp(I)$ and called the support of $I$. An ideal $I$ of $R$ is said to be homogeneous if $I = \bigoplus_{b \in B} I \cap R_b$. Putting $I_b = I \cap R_b$ we see that the homogeneous ideal $I$ is a band sum of the $I_b$. In proving the theorem, we shall several times use the easy fact that the quotient ring $R/I$ is a band sum of $R_b/I_b : R/I = \bigoplus_{b \in B} R_b/I_b$ provided that $I$ is homogeneous.

**LEMMA 1.** Let $R = \bigoplus_{b \in B} R_b$ be a band sum, $E$ an infinite set of ideals of $R$, $|\supp(\sum_{i \in E} I)| = \infty$, and $k$ a natural number such that $|\supp(I)| \leq k$ for every $I \in E$. Then $R$ is not Artinian.

**PROOF.** We will construct an infinite sequence of ideals $I_1, I_2, \ldots$ in $E$ such that $\supp(I_i)$ is not included in $\bigcup_{j \geq i} \supp(I_j)$ for any $i \geq 1$.

Take a finite subset $E_1$ of $E$ such that $|\bigcup_{i \in E_1} \supp(I)| > k$. For any $J \in E_1$ let $G_J$ denote the set of all $I \in E$ such that $\supp(I) \not\supseteq \supp(J)$. For each $I \in E$, $\supp(I)$ does not contain at least one of the $\supp(J), J \in E_1$. Therefore $E = \bigcup_{J \in E_1} G_J$. Since $\supp(\sum_{i \in E_1} I) = \bigcup_{J \in E_1} \supp(\sum_{i \in G_J} I)$, there exists $I_1$ in $E_1$ such that the set $G_1 = \{I \in E \mid \supp(I) \not\supseteq \supp(I_1)\}$ is infinite and, moreover, $|\supp(\sum_{i \in G_1} I)| = \infty$.

In $G_1$ we choose a finite subset $E_2$ such that $|\bigcup_{i \in E_2} \supp(I)| > k$. As above, there exists $I_2$ in $E_2$ such that the set $G_2 = \{I \in G_1 \mid \supp(I) \not\supseteq \supp(I_2)\}$ is infinite. Again we can pick $I_3$ such that $I_3 \in G_2$ and $G_3 = \{I \in G_2 \mid \supp(I) \not\supseteq \supp(I_3)\}$ is
infinite, and so on. For each \( i > 1 \) the ideals \( I_{i+1}, I_{i+2}, \ldots \) are taken from \( G_i \), whence \( \text{supp}(I_i) \) is not contained in \( \text{supp}(I_j) \) for every \( j > i \). Setting \( P_i = \sum_{j \geq i} I_j \) for \( i \geq 1 \), we get \( \text{supp}(P_{i+1}) = \bigcup_{j > i} \text{supp}(I_j) \not\supseteq \text{supp}(I_i) \) and so \( P_i \neq P_{i+1} \). Therefore \( P_1 \supset P_2 \supset P_3 \supset \ldots \), implying that \( R \) is not Artinian.

**Lemma 2.** Let \( R = \bigoplus_{b \in B} R_b \) be a band sum, \( I \) a nilpotent homogeneous ideal of \( R \) such that \( \text{supp}(I) \) is infinite and only a finite number of the \( R_b \) are not contained in \( I \). Then \( R \) is not Artinian.

**Proof.** Since \( I \) is nilpotent and \( \text{supp}(I) \) is infinite, there exists a natural number \( n > 1 \) such that the number of the \( R_b \) which do not lie in \( I^n \) is infinite. Assume that \( n \) is the least natural number of this sort. It is enough to show that \( R/I^n \) is not Artinian.

Evidently, \( I^n \) is homogeneous, and therefore we may pass to the quotient ring \( R/I^n \) and assume that \( I^n = 0 \).

By the choice of \( n \) only a finite number of the \( R_b \) are not contained in \( I^{n-1} \). Hence \( \text{supp}(I^{n-1}) \) is infinite. Thus \( I^{n-1} \) satisfies all the conditions of Lemma 2. To simplify the notation we assume that from the very beginning \( I = I^{n-1} \), that is, \( n = 2 \).

Let \( C \) denote the subsemigroup generated in \( B \) by all \( b \) such that \( R_b \) is not contained in \( I \). It is well-known that every band is locally finite. Therefore \( C \) is finite. Set \( m = |C|, \quad D = \{ b \in B \mid R_b \neq 0, \ b \not\in C \} \). Clearly \( D \) is infinite and \( R_d \subseteq I \) for every \( d \in D \).

Take any \( d \) in \( D \) and consider the ideal \( I_d \) generated in \( R \) by \( R_d \). We get

\[
I_d = R_d + RR_d + R_dR + RR_dR = R_d + \bigoplus_{b \in B} R_bR_d + \bigoplus_{c \in B} R_dR_c + \bigoplus_{b,c \in B} R_bR_dR_c.
\]

If \( b \) and \( c \) are not in \( C \), then \( R_b \subseteq I \) and \( R_c \subseteq I \), whence \( R_bR_d + R_dR_c \subseteq I^2 = 0 \), because \( R_d \subseteq I \). Therefore

\[
I_d = R_d + \bigoplus_{b \in C} R_bR_d + \bigoplus_{c \in C} R_dR_c + \bigoplus_{b,c \in C} R_bR_dR_c.
\]

Hence \( \text{supp}(I_d) \) cannot contain more than \( k = 1 + m + m + m^2 \) elements, where \( m = |C| \). So the set \( E = \{ I_d \mid d \in D \} \) satisfies the conditions of Lemma 1. Thus \( R \) is not Artinian.

**Proof of the Theorem.** Let \( B \) be a semilattice \( S \) of rectangular bands \( H_s : B = \bigcup_{s \in S} H_s \). Then \( R = \bigoplus_{s \in S} Q_s \), where \( Q_s = \bigoplus_{b \in H_s} R_b \). Set \( T = \{ t \in S \mid Q_t \neq 0 \} \). First we will prove that only a finite number of the \( Q_s \) are not equal to zero, that is \( T \) is finite. (Note that in the special case where every \( Q_s \) is not radical, the claim follows from [10, Proposition 1.3].)
Suppose the contrary: let $T$ be infinite. If $T$ contains a descending chain of elements $s_1 > s_2 > \ldots$, then we consider ideals $D_k$ generated in $R$ by $\sum_{i \geq k} S_i$, $k = 1, 2, \ldots$. Clearly $D_1 \supset D_2 \supset \ldots$, which gives a contradiction. Thus $T$ satisfies the descending chain condition.

By transfinite induction we shall construct an infinite ascending chain of ideals $I_1 \subset I_2 \subset \ldots$ of $S$ such that $T$ is contained in $\bigcup_{k=1}^{\infty} I_k$. Since $T$ satisfies the descending chain condition, there exists a minimal element $t_1$ in $T$. Denote by $I_1$ the ideal generated in $S$ by $t_1$. Suppose that for some ordinal number $v$ all the ideals $I_u$, $u < v$ are defined. Set $U_v = \bigcup_{u < v} I_u$. If $v$ is a limit ordinal, then put $I_v = U_v$. If $v$ is not limit, that is, if there exists $v - 1$, then choose any minimal element $t_v$ in $T \setminus I_{v-1}$ and denote by $I_v$ the ideal generated in $S$ by $I_{v-1} \cup \{t_v\}$. Having defined $I_v$, we check whether $T$ is contained in $\bigcup_k I_k$, and if not, then we continue the process. Clearly, the chain of ideals terminates, and so for a certain ordinal number $w$ we have $T \subseteq I_w$. Since $T$ is infinite, $w$ is infinite too. Easy induction shows that $I_v \cap T = \{t_u \mid u < v, u \text{ a non-limit ordinal}\}$ for each $v$.

Consider the chain of ideals $A_1 \subset A_2 \subset \ldots \subset A_w = R$, where $A_v = \bigoplus_{u \in I_u} Q_u$. It is well-known that every semisimple Artinian ring is Noetherian. Therefore in $R/J(R)$ the ascending chain of ideals

$$A_1 + J(R) \subseteq A_2 + J(R) \subseteq \ldots \subseteq A_w + J(R)$$

breaks. Hence there exists a positive integer $n$ such that

$$A_n + J(R) = A_{n+1} + J(R) = \ldots = A_w + J(R) = R + J(R).$$

This implies that the quotient ring $F = R/A_n$ is radical. Since $A_n$ is homogeneous, $F$ is a semilattice sum of its subrings $F_s = Q_s/(A_n \cap Q_s)$, $s \in S$. Evidently, $A_v = \bigcup_{u \leq v} A_u$ for a limit $v \leq w$. If $v$ is not limit, then $I_v \cap T = \{t_v\} \cup \{I_{v-1} \cap T\}$, and so $A_v = Q_{t_v} + A_{v-1}$, $A_v \neq A_{v-1}$. Therefore supp $(F)$ is infinite. Obviously, $F$ is a homogeneous ideal in $F$ containing all $F_s$. Since $F$ is radical, it is nilpotent. By Lemma 2 $F$ is not Artinian, whence neither is $R$. This contradiction shows that $T$ is finite.

Now suppose that there are infinitely many non-zero rings among the $R_b$, $b \in B$. We will show that this leads to a contradiction.

Clearly $J_B(R) = \bigoplus_{b \in B} J(R)R_b$ is a homogeneous ideal of $R$. Moreover, it is nilpotent because $J_B(R) \subseteq J(R)$. If only a finite number of the $R_b$ are not included in $J_B(R)$, then supp $(J_B(R))$ is infinite, and Lemma 2 yields that $R$ is not Artinian. Hence an infinite number of the components $R_b$ do not lie in $J_B(R)$. Therefore $R/J_B(R)$ is a band sum with an infinite number of non-zero components $R_b/(R_b \cap J_B(R))$. Thus we may assume that from the very beginning $J_B(R) = 0$.

For $s \in S$ set $T_s = \{b \in H_s \mid R_b \neq 0\}$. Since we have proved that $T$ is finite, only a finite number of the $T_s$ are not empty. So we can choose a maximal element $t$ in $S$.
such that $T_t$ is infinite. It is routine to verify that the set $V = \bigcup \{H_s \mid s \in S, st \neq t \}$ is an ideal of $S$, $W = \bigoplus_{b \in V} R_b$ is a homogeneous ideal in $R$, and the quotient ring $F = R/W$ satisfies exactly the same conditions as $R$. Therefore we can replace $R$ by $F$. In order to preserve simple notation we just assume that $W = 0$ instead of substituting $F$ for $R$. Then $t$ becomes a minimal element such that $T_t$ is not empty. (Indeed, $H_s \subseteq V$ for every $s < t$. Since $W = 0$, then $Q_s = 0$ for $s < t$, and so $T_s \neq \emptyset$.) Moreover, we may delete all $s$ such that $st \neq t$ from $S$ and assume that $t$ is the least element of $S$.

The rectangular band $H_t$ is isomorphic to a direct product of a left zero band $C$ and a right zero band $D$. Identifying $H_t$ with this product, we have $H_t = C \times D$. Moreover we may assume that $C, D \subseteq H_t$. For $d \in D$ set $C_d = \bigoplus_{c \in C} R_{(c,d)}$. Take any $b \in B$. There exists $s \in S$ such that $b(e) \in H_s$. By the assumption above $st = t$. Therefore $b(c,d) \in H_t$ for any $c \in C, d \in D$. Let $b(c,d) = (c_1, d_1) \in H_t$. Consequently, $b(c,d) = b(c_1, d_1)(c, d) = (c_1, d_1)(c, d) = (c_1, d)$ implying $R_{b(c,d)} \subseteq C_d$. Thus $C_d$ is a left ideal in $R$. Similarly, $D_c = \bigoplus_{d \in D} R_{(c,d)}$ is a right ideal of $R$ for every $c \in C$.

Given that $R$ is right Artinian, only a finite number of the right ideals $D_c$ are non-zero. Since $D_c \cap C_d = R_{(c,d)}$, it follows that each $C_d$ is a sum of a finite number of non-zero $R_{(c,d)}$'s. As $T_t$ is infinite, there exist an infinite number of $C_d \neq 0, d \in D$.

Define on the union $A = D \cup (\bigcup_{s \neq t} H_s)$ a multiplication $*$ by setting

$$a \ast b = \begin{cases} ab & \text{if } ab \neq H_t \\ d & \text{if } ab = (c, d) \in H_t. \end{cases}$$

It is easy to check that $*$ is associative. To illustrate this let us take any $x, y, z \in A$ and consider the case when $xyz \in H_t, x, yz \notin H_t$. Let $xyz = (c, d) \in H_t$. Then $x \ast (y \ast z) = x \ast (yz) = d$. On the other hand, if $x \not\in H_t$, we get $(x \ast y) \ast z = (xy) \ast z = d$. If $x = (c_1, d_1) \in H_t$, then $(x \ast y) \ast z = d_1 \ast z = d$, because $d_1 = d_1(c_1, d_1)$, and $d_1z = d_1xyz = d_1(c, d) = (c_2, d)$ for some $c_2 \in C$. Thus $x \ast (y \ast z) = (x \ast y) \ast z$ in the case we are considering. Other cases are similar.

For $a \in A$ set $P_a = R_a$, when $a \in B$, and set $P_a = C_a$ when $a \in D$. It is routine to verify that $A$ is a band under $*$, and $R = \bigoplus_{a \in A} P_a$ is a band sum. It has been proved that an infinite number of $P_a, a \in A$, are not zero. Therefore we may consider $A$ and $P_a$ instead of $B$ and $R_b$. To make further notation simpler assume that $A = B$, and so $H_t = D, C_d = R_d$.

Denote by $J_t$ the set $\bigoplus_{d \in D} J(R) \cap R_d$. Because $t$ is minimal with $Q_t \neq 0, Q_t$ is an ideal in $R$. For any $b \in B, d \in D$ it is easily seen that $R_b(J(R) \cap R_d) \subseteq J(R) \cap R_{bd}$ and $(J(R) \cap R_d)R_b \subseteq R_{db}$. Therefore $J_t$ is an ideal of $R$. Since $J(R)$ is nilpotent, it follows that $J_t$ is a homogeneous nilpotent ideal of $R$. By Lemma 2 an infinite number of the $R_b, b \in B$, do not lie in $J_t$. Since $t$ is maximal with infinite $T_t$, it follows that the number of $R_b, b \in H_t$, which are not contained in $J_t$, is infinite. Therefore we may pass to the quotient ring $R/J_t$ and assume that $J_t = 0$. 


The semisimple Artinian ring $R/J(R)$ is also left Noetherian. Hence its left ideal $(Q_t + J(R))/J(R)$ is finitely generated. Since every $R_d, d \in D$, is a left ideal, there exists a finite subset $F$ of $D$ such that

$$Q_t + J(R) = J(R) + \bigoplus_{f \in F} R_f.$$ 

For each $d \in T_i \setminus F$ choose any $x$ in $R_d$ and fix elements $r(d) \in \bigoplus_{f \in F} R_f, y(d) \in J(R)$ such that $x = r(d) + y(d)$. Since $y(d) = x - r(d)$, then $	ext{supp}(y(d)) \subseteq F \cup \{d\}$, and so $|\text{supp}(y(d))| \leq m$, where $m = |F| + 1$.

Let $I(d)$ denote the ideal generated in $R$ by $y(d)$. Put $E = \{I(d) \mid d \in T_i \setminus F\}$. We claim that $E$ satisfies all the conditions on the set of ideals in Lemma 1. Indeed, the union $\bigcup_{t \in E} \text{supp}(I)$ contains $T_i \setminus F$, and so it is infinite. It remains to evaluate $|\text{supp}(I(d))|$. To this end note that

$$I(d) = Z y(d) + R y(d) + y(d)U + y(d)Q_t + R y(d)U + R y(d)Q_t,$$

where $Z$ is the set of integers, and $U = \bigoplus_{b \in B \setminus D} R_b$. Evidently, $\text{supp}(Z y(d) + R y(d)) \subseteq \text{supp}(y(d))$ implying $|\text{supp}(Z y(d) + R y(d))| \leq m$. Since only a finite number of $T_s, s \in S$, are not empty and $T_i$ is the only infinite set among them, we obtain that $\bigcup_{s \neq i} T_s$ is finite. Therefore $\text{supp}(U)$ contains a finite number of elements, say $n$. Then $|\text{supp}(y(d)U)| \leq mn$. Similarly, $|\text{supp}(R y(d)U)| \leq |\text{supp}(y(d))\text{supp}(U)| \leq mn$. Further, for $b \in D$ we have $y(d)R_b \subseteq J(R) \cap R_b \subseteq J_i = 0$. This implies $R y(d)Q_t = 0$, too. Thus $|\text{supp}(I(d))| \leq m + 2mn$ for every $d$. Lemma 1 completes the proof.

**Example.** Let $Q$ be the field of rational numbers, $D$ a right zero band, $|D| > 1$. Then the semigroup ring $QD = \bigoplus_{b \in D} Qd$ is a band sum, where all the $Qd$ are Artinian, since they are isomorphic to $Q$. However, $QD$ is not Artinian. It has an infinite descending chain of right ideals $I_2 \supseteq I_3 \supseteq \ldots$, where $I_k = \{m(c - d)/n \mid m, n$ are integers: $2, \ldots, k$ do not divide $n\}$ for fixed $c, d \in D, c \neq d$.

**Example.** Let $R = Q_n$ be the ring of $n \times n$ matrices over $Q$. Denote by $e_{ij}$, where $1 \leq i, j \leq n$, the standard matrix units. Put $R_{(i,j)} = Qe_{ij}$. On the set $B = \{(i, j) \mid 1 \leq i, j \leq n\}$ define a multiplication by $(i_1, j_1)(i_2, j_2) = (i_1, j_2)$. This makes $B$ a rectangular band. It is easy to see that $R = \bigoplus_{b \in B} R_b$ is an Artinian band sum. However, $R_{(i,j)}$ is a ring with zero multiplication for $i \neq j$, and so it is not Artinian.

**Proposition.** Let $B$ be a semilattice, $R = \bigoplus_{b \in B} R_b$ a semilattice sum such that all $R_b$ are Artinian and the set $\{b \in B \mid R_b \neq 0\}$ is finite. Then $R$ is Artinian.
PROOF. If $B$ has only two elements, say $a < b$, then $R$ is an extension of $R_a$ by $R_b$, and so $R$ is Artinian. The general case follows by induction on $|B|$ as in the proof of [3, Theorem 1].

The following example appeared in [3, proof of Theorem 5]:

EXAMPLE. Let $Q^0$ be the ring with zero multiplication and the same additive group $Q^+$ as $Q$. The ring $R$ defined by

$$R^+ = Q^+ \bigoplus Q^+$$

$$(p, q)(r, s) = (ps + qr, qs)$$

is commutative and is a semilattice sum of $Q^0$ and $Q$, with $Q^0$ an ideal of $R$. It is easy to check that $R$ has only three ideals: 0, $Q^0$ and $R$. Therefore $R$ is Artinian, although $Q^0$ is not.

References