The pair \((S, M)\) is a matroid if \(S\) is a finite set and \(M\) a collection of subsets of \(S\) such that (1) every subset of a set of \(M\) is in \(M\), and (2) all maximal sets in \(M\) have a common cardinality. The span of a set \(A \subseteq S\) is \(\Gamma(A)\) where \(y \in \Gamma(A)\) if and only if \(y \in A\) or there is \(A' \subseteq A\), \(A' \in M\) and \(\{y\} \cup A' \notin M\). A maximal set in \(M\) is called a base. For each base \(B\) in \(M\) define \(B:S \rightarrow M\) by: \(B(x)\) is \(\{x\}\) if \(x \in B\); otherwise \(B(x)\) is the unique minimal subset of \(B\) such that \(B(x) \cup \{x\} \notin M\).

Remark. Using the natural correspondence between bases and circuits the set \(B(x) \cup \{x\}\) is just the atom \(J(D; x)\) of Tutte [1] where \(D\) is the dendroid \(E - B\) when \(B\) is a base and \(x \in S - B\).

For \(A \subseteq S\) and \(x \in S\), \(x\) is \(B\)-orthogonal to \(A\), written \(x \in O_B(A)\), if \(B(x) \cap B(a) = \emptyset\) for all \(a \in A\).

A problem posed by Welsh [2] is the following: If \(A \in M\), \(B\) is a base and \(x \in O_B(A)\) is \(x \in O_B(\Gamma(A))\)? The answer is yes and is in fact true for any \(A \subseteq S\). To prove this two lemmas are required.

LEMA 1. Let \((S, M)\) be a matroid and \(b \in B\), \(B\) a base in \(M\). Then \(B' = (B - \{b\}) \cup \{a\}\) is a base if and only if \(b \in B(a)\).

Proof. Necessity. If \(a \in B\) then \(B' = B\) and \(\{b\} = B(a)\). If \(a \notin B\) then \(B'(b) \subseteq B'\) and \(B'(b) \cup \{b\} \notin M\). Thus \(a \in B'(b)\) and \(B(a) = (B'(b) \cup \{b\}) - \{a\}\) by uniqueness.

Sufficiency. If \(a \in B\) then \(b = a\) and so \(B' = B\). If \(a \notin B\) and \(B'\) is dependent then there is \(A \subseteq B', A\) minimal dependent. Since \(B\) is independent, \(A \subseteq (B - \{b\}) \cup \{a\}\) so that \(A - \{a\} = B(a)\) and \(b \notin B(a)\), a contradiction. \(B'\) is independent and \(|B| = |B'|\). Hence \(B'\) is a base. (\(|X|\) denotes the cardinality of \(X\).)

LEMA 2. Let \((S, M)\) be a matroid, \(B\) a base, \(a \in S - B\), \(b \in B(a)\), and \(B' = (B - \{b\}) \cup \{a\}\). Then for \(p \in S\),

(i) \(B'(p) = B(p)\) if \(b \notin B(p)\),

(ii) \(B'(b) = (B(a) \cup \{a\}) - \{b\}\).

(iii) \( B'(a) = \{ a \} \), and

(iv) \( B(a) + B(p) \subseteq B'(p) \subseteq B(a) \cup B(p) \cup \{ a \} \) if \( b \in B(p), b \neq p \).

\( X + Y \) denotes the symmetric difference of \( X \) and \( Y \).

**Proof.** \( B' \) is a base by Lemma 1.

(i) If \( b \notin B(p) \) then \( B(p) \subseteq B' \). Hence \( B'(p) = B(p) \).

(ii) The proof of Lemma 1 shows that \( B'(b) = (B(a) \cup \{ a \}) - \{ b \} \).

(iii) By definition.

(iv) Right hand side. The minimal dependent sets form a "circuit" matroid (Edmonds [3, Prop. 2]).

Since \( b \in B(p) \cup \{ p \} \) and \( b \in B(a) \cup \{ a \} \), there is a minimal dependent set \( C \) with \( p \in C \subseteq B(a) \cup B(p) \cup \{ a, p \} - \{ b \} \subseteq B' \cup \{ p \} \).

Hence \( B'(p) = C - \{ p \} \subseteq B(a) \cup B(p) \cup \{ a \} \). Also \( a \in B'(p) \) by Lemma 1 since \( B = (B - \{ a \}) \cup \{ b \} \).

Left hand side. (a) For \( z \in B(p) - B(a) \), \( B'' = (B - \{ z \}) \cup \{ p \} \) is a base by Lemma 1 and since \( z \notin B(a) \), \( B'' = B'(a) \) by Lemma 2, (i).

\( a \in B'(p) \cup \{ p \} \) is a circuit, and if \( z \notin B'(p) \) then \( B'(p) \cup \{ p \} \subseteq B'' \cup \{ a \} \), giving \( b \in B(a) = B''(a) = B'(p) \cup \{ p \} - \{ a \} \), a contradiction. (b) For \( z \in B(a) - B(p) \), \( B'' = (B - \{ z \}) \cup \{ a \} \) is a base by Lemma 1 and since \( z \notin B(p) \), \( B''(p) = B(p) \) by Lemma 2, (i). If \( z \notin B'(p) \) then \( B'(p) \subseteq (B(p) \cup B(a) \cup \{ a \}) - \{ z \} \subseteq B''(p) \). Since \( B'(p) \cup \{ p \} \) is a circuit in \( B'' \cup \{ p \} \) we have \( b \in B'(p) = B''(p) = B'(p) \), a contradiction. Thus \( B(p) + B(a) \subseteq B'(p) \).

**THEOREM.** Let \( (S, M) \) be a matroid, \( B \) a base, \( A \subseteq S \) and \( x \in C_B(A) \). Then \( x \in C_{B'}(\Gamma(A)) \).

**Proof.** If \( B(x) = \phi \) there is nothing to prove. If \( y \in \Gamma(A) \) with \( B(x) \cap B(y) \neq \phi \) then \( y \notin A \) and there is a circuit \( C \), \( y \in C = A' \cup \{ y \} \) with \( A' \subseteq A, A' \in M \). Then \( x \in C \subseteq C_{B'}(A') \), \( x \notin C_{B'}(\Gamma(A')) \) so that it is sufficient to prove the theorem for sets \( A \in M \).

We now have: if the theorem is not true there is a triple \( (x, A, B) \) where \( x \in S, B \) is a base, \( A \in M, 0 \leq |A - B|, x \notin C_{B}(A) \) and \( x \notin C_{B}(\Gamma(A)) \). Suppose \( (x, A, B) \) such a triple. If \( A \subseteq B \) then \( \bigcup_{a \in A} B(a) = A \) and for \( y \in \Gamma(A) - A \) there is a circuit \( C \) with \( y \in C = A' \cup \{ y \} \), \( A' \subseteq A \). Hence \( B(y) = A' \subseteq A \) and \( B(x) \cap B(y) = \phi, \) a contradiction. Thus for such triple \( 0 < |A - B| \) and we may choose one with \( |A - B| \) least.

Take \( a_0 \in A - B \). Since \( a_0 \in M \) there is \( b \in B(a_0) \) and by hypothesis
b \notin B(x). By Lemma 1, \( B' = (B - \{ b \}) \cup \{ a_0 \} \) is a base and by Lemma 2 (i) \( B(x) = B'(x) \). Now \( a_0 \notin B'(x) \), for otherwise by Lemma 2 (iv) (with \( B' \) and \( B \) interchanged) \( B'(a_0) + B'(x) = B(x) = B'(x) \) giving \( B'(a_0) = \{ a_0 \} = \emptyset \), a contradiction. For the remaining \( a \in A \) again using Lemma 2 (iv), \( B'(a) \subseteq B(a) \cup B(a_0) \cup \{ a \} \) so that \( B'(x) \cap B'(a) = \emptyset \) for all \( a \in A \). Thus \( x \in O_{B'}(A) \) and by the minimality of \( |A - B| \), \( x \in O_{B'}(\Gamma(A)) \).

Now take \( y \in \Gamma(A) \). If \( b \notin B(y) \) then by Lemma 2 (i), \( \emptyset = B'(x) \cap B'(y) = B(x) \cap B(y) \). If \( b \in B(y) \), \( B'(y) \supseteq B(a_0) \cup B(y) \) by Lemma 2 (iv) and \( \emptyset = B'(x) \cap B'(y) = B(x) \cap (B(a_0) + B(y)) = B(x) \cap B(y) \). In all cases, \( B(x) \cap B(y) = \emptyset \) and so \( x \in O_B(\Gamma(A)) \), a contradiction to the existence of a triple \( (x, A, B) \). Thus the theorem is proved.

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