EVEN AND ODD INTEGRAL PARTS OF POWERS OF A REAL NUMBER

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Abstract. We define a subset $Z$ of $(1, +\infty)$ with the property that for each $\alpha \in Z$ there is a nonzero real number $\xi = \xi(\alpha)$ such that the integral parts $[\xi \alpha^n]$ are even for all $n \in \mathbb{N}$. A result of Tijdeman implies that each number greater than or equal to 3 belongs to $Z$. However, Mahler’s question on whether the number $3/2$ belongs to $Z$ or not remains open. We prove that the set $\mathcal{S} := (1, +\infty) \setminus Z$ is nonempty and find explicitly some numbers in $\mathcal{S} \cap (5/4, 3)$ and in $\mathcal{S} \cap (1, 2)$.

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1. Introduction. Suppose that $\alpha > 1$ is a real number. Then either for any real number $\xi \neq 0$ the fractional parts $[\xi \alpha^n]$ are greater than or equal to $1/2$ for infinitely many $n \in \mathbb{N}$ or there exists a real number $\xi = \xi(\alpha) \neq 0$ such that $[\xi \alpha^n] < 1/2$ for every $n \in \mathbb{N}$. The set of $\alpha > 1$ for which the first possibility holds will be denoted by $\mathcal{S}$. Similarly, we will denote by $\mathcal{Z}$ the set of $\alpha > 1$ for which the second possibility holds, so $\mathcal{S} \cap \mathcal{Z} = \emptyset$ and $\mathcal{S} \cup \mathcal{Z} = (1, +\infty)$. Equivalently, $\alpha \in \mathcal{Z}$ if and only if there is a real number $\xi = \xi(\alpha) \neq 0$ such that the integral parts $[\xi \alpha^n]$, $n = 1, 2, \ldots$, are all even.

In 1968, Mahler [16] asked whether the number $3/2$ belongs to $\mathcal{S}$ or to $\mathcal{Z}$ (see also [20]). There are many reasons to believe that $3/2 \in \mathcal{S}$. Although for a ‘random’ pair $\xi, \alpha$ the fractional parts $[\xi \alpha^n]$, $n = 1, 2, \ldots$, are uniformly distributed in $[0, 1)$ (see [13], [21]), the behavior of the sequence $[\xi \alpha^n]$, $n = 1, 2, \ldots$, for almost any ‘specific’ pair $\xi, \alpha$, where $\xi \neq 0$, $\alpha \notin \mathbb{N}$, is not known. See, however, [3], [4], [6]–[12], [22], [23] for some results in this direction. In this note, we shall collect several results about the sets $\mathcal{S}$ and $\mathcal{Z}$.

Trivially, $\{2, 3, 4, \ldots\} \subset \mathcal{Z}$, because, for any integer $g \geq 2$, by taking $\xi = 2$ we see that the numbers $[2g^n] = 2g^n$, $n = 1, 2, \ldots$, are all even. In general, one may expect that ‘large’ $\alpha$ belong to the set $\mathcal{Z}$ (see, for instance, Tijdeman’s result [19] stated in Theorem 1(i) below) whereas ‘small’ $\alpha$ lie in $\mathcal{S}$. In connection with this, one can ask whether there exist $\alpha > \alpha' > 1$ such that $\alpha \in \mathcal{S}$ and $\alpha' \in \mathcal{Z}$. If the answer to this question were negative then the sets $\mathcal{S}$ and $\mathcal{Z}$ would simply be two intervals. Unfortunately, the situation is not that simple, because such $\alpha$ and $\alpha'$ do exist. We will show, for instance, that the set $\mathcal{Z}$ contains a number smaller than 1.26 and that the set $\mathcal{S}$ contains the golden mean $(1 + \sqrt{5})/2 = 1.61803 \ldots$.

In order to state our theorems, we recall first that $\alpha > 1$ is called a Pisot number if it is an algebraic integer whose conjugates over $\mathbb{Q}$ different from $\alpha$ itself lie in the open unit disc. A Pisot number is called a strong Pisot number if it is not a rational integer and if its second largest (in modulus) conjugate is positive (see [2] and [6]). Also,
\( \alpha > 1 \) is called a Salem number if it is an algebraic integer whose conjugates over \( \mathbb{Q} \) different from \( \alpha \) itself lie in the closed unit disc \( |z| \leq 1 \) with at least one conjugate lying on the unit circle \( |z| = 1 \). Finally, let \( P_\alpha(x) \in \mathbb{Z}[x] \) denote the minimal polynomial of an algebraic number \( \alpha \). Note that \( P_\alpha(1) \leq -1 \) for each \( \alpha \) which is a Pisot number or a Salem number.

2. Results.

**Theorem 1.** We have

(i) \( [3, +\infty) \subset \mathbb{Z} \),

(ii) \( 3 - 2/q \in \mathbb{Z} \) for any integer \( q \geq 2 \),

(iii) \( \alpha \in \mathbb{Z} \) for any strong Pisot number \( \alpha \),

(iv) \( \alpha \in \mathbb{Z} \) for any Pisot or Salem number \( \alpha \) whose minimal polynomial satisfies \( P_\alpha(1) \leq -3 \).

By Theorem 1(iv), the thirteenth smallest known Salem number \( \alpha = 1.2527759 \ldots \) whose minimal polynomial is

\[
P_\alpha(x) = x^{18} - x^{12} - x^{11} - x^{10} - x^9 - x^8 - x^7 - x^6 + 1
\]

belongs to the set \( \mathbb{Z} \), because \( P_\alpha(1) = -5 < -3 \). (See Mossinho’s page on Lehmer’s problem http://www.cecm.sfu.ca/~mjm/Lehmer/lists/SalemList.html for a list of small Salem numbers.) Most of the results stated in Theorem 1 have been published earlier or follow easily from [19], [10], [11], [22]. Nevertheless, for the sake of completeness, we will give the proofs of (i)–(iii) below and derive (iv) from [10], [22].

The first example of a number \( \alpha > 1 \) lying in the set \( S \) was given recently by the author in [8]: for any \( d \geq 4 \) one can take \( \alpha > 1 \) that satisfies \( \alpha^d - \alpha - 1 = 0 \). So the set \( S \) is nonempty. Note that if \( \alpha \in S \) then \( \alpha^{1/q} \in S \) for each \( q \in \mathbb{N} \), because the set of fractional parts \( \{ \xi \alpha^n \} \), where \( n \in \mathbb{N} \), is a subset of the set \( \{ \xi \alpha^{m/q} \} \), \( n \in \mathbb{N} \). The next theorem not only contains the example given above but also describes some new numbers in \( S \).

**Theorem 2.** We have

(i) \( 2^{1/q} \in S \) for any integer \( q \geq 2 \),

(ii) \( \alpha \in S \) for any \( \alpha > 1 \) which is a root of an irreducible polynomial \( x^d - x^r - 1 \), where \( 0 < r < d \),

(iii) \( \alpha \in S \) for any \( \alpha > 1 \) which is a root of the polynomial \( x^d - x^m - x^r + 1 \), where \( 0 < r \leq m < d \), but is not a Pisot number.

Note that in case (iii) the polynomial \( x^d - x^m - x^r + 1 \) is reducible. Hence the degree of \( \alpha > 1 \) over \( \mathbb{Q} \) is smaller than \( d \). The requirement that \( \alpha \) is not a Pisot number is necessary. If, for instance, \( m = r = d - 1 \) then

\[
 x^d - 2x^{d-1} + 1 = (x - 1)(x^{d-1} - x^{d-2} - \cdots - x - 1).
\]

The polynomial \( P_\alpha(x) = x^{d-1} - x^{d-2} - \cdots - x - 1 \) is irreducible and defines a Pisot number \( \alpha > 1 \) for each \( d \geq 3 \). Since \( P_\alpha(1) \leq -3 \) for every \( d \geq 5 \). Theorem 1(iv) implies that \( \alpha \in \mathbb{Z} \). However, for \( d = 3 \), \( \alpha = (1 + \sqrt{3})/2 \) belongs to \( S \) by Theorem 2(ii). All irreducible polynomials of the form \( x^d - x^r - 1 \) have been described in [15].

Since \( \sqrt{2} \in S \) and \( \sqrt{m} \in \mathbb{Z} \) for each integer \( m \geq 4 \), it is natural to ask the following:

**Problem 3.** Determine whether \( \sqrt{3} \) belongs to \( S \) or to \( \mathbb{Z} \).
We remark that if $\sqrt{3} \in S$ then writing $\sqrt{3}$ in its base $3$ expansion $\sqrt{3} = 1 + \sum_{j=1}^{\infty} b_j 3^{-j}$, where $b_1, b_2, \ldots \in \{0, 1, 2\}$, and taking $\xi = 1$ we would derive that

$$\{\sqrt{3} \cdot 3^j\} = b_{j+1} 3^{-1} + b_{j+2} 3^{-2} + \cdots \geq 1/2$$

for infinitely many $j \in \mathbb{N}$. Hence $b_m = 2$ for infinitely many $m \in \mathbb{N}$. Such results, however, are completely out of reach. See, for instance, [1] for a recent progress on the distribution of digits in the expansions of algebraic irrational numbers in base $b \geq 2$.

The next problem seems to be quite difficult too.

**Problem 4.** Is it true that if $\alpha \in S$ then for each nonzero real number $\xi$ the sequence $[\xi \alpha^n]$, $n = 1, 2, \ldots$, contains infinitely many even numbers?

By the definition of $S$, the sequence $[\xi \alpha^n]$, $n = 1, 2, \ldots$, contains infinitely many odd numbers. It is easy to see that the answer to Problem 4 is affirmative precisely when there is a nonzero real number $\xi$ such that $(\xi \alpha^n) \geq 1/2$ for each $n \in \mathbb{N}$, where $(\xi \alpha^n) > 1/2$ for infinitely many $n$ and $(\xi \alpha^n) = 1/2$ for infinitely many $n \in \mathbb{N}$. Taking, for instance, $\xi = 1/2$ and $\alpha = \sqrt{3}$ we are back to a similar question about the distribution of digits in base $3$ expansion again. This time, the number in question is $\sqrt{3}/2$.

Note that all numbers of $S$ described in Theorem 2 are algebraic integers and lie in the interval $(1, 2)$. We thus conclude this section with the following problem.

**Problem 5.** Is there an element of $S$ greater than $2$?

### 3. Proof of Theorem 1.

We shall prove $(i)$ and $(ii)$ using the method of nested intervals as in [19]. Suppose first that $\alpha > 3$. We claim that there is a $\xi > 0$ such that $(\xi \alpha^n) \leq \beta := 1/(\alpha - 1)$ for every $n \in \mathbb{N}$. Clearly, $(\xi \alpha^n) \leq \beta$ if and only if there is an integer $k_n$ such that $k_n \alpha^{-n} \leq \xi \leq (k_n + \beta) \alpha^{-n}$. Let $k_1$ be an arbitrary integer greater than $\alpha$. Set $I_1 = [k_1 \alpha^{-1}, (k_1 + \beta) \alpha^{-1}]$. The sequence of intervals $I_j = [k_j \alpha^{-j}, (k_j + \beta) \alpha^{-j}]$, where $k_j \in \mathbb{N}$, $j = 1, 2, \ldots$, $n$ is nested if and only if $(k_j + \beta) \alpha^{-j} \leq (k_{j+1} + \beta) \alpha^{-j-1}$ for each $j \in \mathbb{N}$. This happens precisely when for each $j \in \mathbb{N}$ the interval $[ak_j, \alpha(k_j + \beta) - \beta]$ contains the integer $k_{j+1}$. Since the length of this interval is $\alpha \beta - \beta = 1$, such an integer $k_{j+1}$ exists for every $j \in \mathbb{N}$. Hence, setting $\xi := \cap_{j=1}^{\infty} I_j$, we have that $(\xi \alpha^n) \leq 1/(\alpha - 1) < 1/2$ for every $n \in \mathbb{N}$. Therefore, each $\alpha > 3$ lies in $\mathcal{Z}$. Trivially, $2, 3 \in \mathcal{Z}$. This proves $(i)$ and also $(ii)$ for $\alpha = 2$, where $q = 2$.

Next, we will prove $(ii)$ for $\alpha = (3q - 2)/q$, where $q \geq 3$ is an integer. As above, $(\xi(3 \cdot 2/q^n)) \leq 1/2$ if and only if there is an integer $k_n$ such that $k_n(3 - 2/q) - n \leq \xi \leq (k_n + 1/2)(3 - 2/q) - n$. Fix an integer $k_1$ greater than $3$. Set $I_1 = [k_1(3 - 2/q)^{-1}, (k_1 + 1/2)(3 - 2/q)^{-1}]$. The sequence of intervals $I_j = [k_j(3 - 2/q)^{-j}, (k_j + 1/2)(3 - 2/q)^{-j}]$, where $k_j \in \mathbb{N}$, $j = 1, 2, \ldots$, $n$ is nested if and only if

$$(3 - 2/q) k_j \leq k_{j+1} \leq (3 - 2/q)(k_j + 1/2) - 1/2 = (3 - 2/q) k_j + 1 - 1/q.$$  

It is easy to see that the interval $[(3q - 2)k_j, (3q - 2)k_j + q - 1]$ contains an integer divisible by $q$, say $qu$. So we can take $k_{j+1} := u$. Hence, setting $\xi := \cap_{j=1}^{\infty} I_j$, we derive that $(\xi(3 \cdot 2/q^n)) \leq 1/2$ for each $n \in \mathbb{N}$. However, since $3 - 2/q, q \geq 3$, is not an integer, there are only finitely many $n \in \mathbb{N}$ (or no such $n$ at all) for which $(\xi(3 \cdot 2/q^n)) = 1/2$. (See, for instance, Lemma 4 in [11].) If $n_0$ is the largest among those $n$ we can replace...
\( \xi \) by \( (3 - 2/q)^{\alpha_0} \). With this new \( \xi \), the inequality \( \{\xi(3 - 2/q)^{\alpha_0}\} < 1/2 \) holds for every \( n \in \mathbb{N} \). This completes the proof of (ii).

For a strong Pisot number \( \alpha \), we have \( \{\alpha^n\} \to 1 \) as \( n \to \infty \) (see [6]). Indeed, since \( S_n := \alpha^n + \alpha_2^n + \cdots + \alpha_d^n \in \mathbb{Z} \), where \( \alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d \) are the conjugates of \( \alpha \) labelled so that \( \alpha > 1 > \alpha_2 > |\alpha_j| \) for \( j > 2 \), we deduce that \( S_n - \alpha^n \) is positive for each \( n \) sufficiently large. Clearly, \( S_n - \alpha^n \to 0 \) as \( n \to \infty \). It follows that \( \{\alpha^n\} = 1 - \alpha_2^n - \cdots - \alpha_d^n \) for each sufficiently large integer \( n \). Hence \( \{\alpha^n\} \to 1 \) as \( n \to \infty \). In particular, by taking \( \xi = -\alpha^n \) with \( n_0 \) sufficiently large, we obtain that \( \{-\alpha^n\} < 1/2 \) for each \( n \in \mathbb{N} \). This proves (iii).

The proof of (iv) for Pisot and Salem numbers follows [10] and [22], respectively. To be precise, it was shown in [10] that if the minimal polynomial of a Pisot number \( \alpha \) satisfies \( P_\alpha(1) \leq -2 \) then, setting \( \xi = 1/(P_\alpha(\alpha)(\alpha - 1)) \), we have \( \lim_{n \to \infty} \{\xi \alpha^n\} = 1/|P_\alpha(1)| \). Similarly, Zaimi [22] showed that if the minimal polynomial of a Salem number \( \alpha \) satisfies \( P_\alpha(1) \leq -2 \) then, for any \( \varepsilon > 0 \), there is a nonzero \( \xi = \xi(\alpha, \varepsilon) \in \mathbb{Q}(\alpha) \) such that \( 1/|P_\alpha(1)| - \varepsilon < \{\xi \alpha^n\} < 1/|P_\alpha(1)| + \varepsilon \) for each \( n \in \mathbb{N} \) large enough. So in both (Pisot and Salem) cases one can find a positive integer \( m_0 \) such that, by taking \( \xi \alpha^{m_0} \in \mathbb{Q}(\alpha) \) instead of \( \xi \), we obtain that \( 1/|P_\alpha(1)| - \varepsilon < \{\xi \alpha^n\} < 1/|P_\alpha(1)| + \varepsilon \) for each \( n \in \mathbb{N} \). Clearly, this implies the inequality \( \{\xi \alpha^n\} < 1/2 \) for each \( n \in \mathbb{N} \) under the stronger condition \( P_\alpha(1) \leq -3 \) if \( \varepsilon < 1/6 \). This proves (iv). The proof of Theorem 1 is completed.

Since in [10] and in [22] the statements concerning the fractional parts \( \{\xi \alpha^n\} \) mentioned in the proof of (iv) are not given explicitly, let us summarize them here as follows.

**Theorem 6.** Suppose that \( \alpha \) is a Pisot number or a Salem number with minimal polynomial \( P_\alpha(x) \in \mathbb{Z}[x] \). If \( P_\alpha(1) \leq -2 \) then for any \( \varepsilon > 0 \) there is a real number \( \xi \in \mathbb{Q}(\alpha) \) (which depends on \( \varepsilon \) in the case \( \alpha \) is a Salem number) such that

\[
1/|P_\alpha(1)| - \varepsilon < \{\xi \alpha^n\} < 1/|P_\alpha(1)| + \varepsilon
\]

for any \( n \in \mathbb{N} \).

We remark that the fractional parts \( \{\xi \alpha^n\}, n \in \mathbb{N}, \) can be quite small for some Salem numbers that are not too large. Take, for instance, the Salem number \( \alpha = 1.6733248 \ldots \) given in [14] whose minimal polynomial is

\[
P_\alpha(x) = x^{14} - x^{12} - x^{11} - x^{10} - x^9 - 2x^8 - 3x^7 - 2x^6 - x^5 - x^4 - x^3 - x^2 + 1,
\]

so that \( P_\alpha(1) = -13 \). Then, by Theorem 6, for any \( \varepsilon > 0 \), there exists a real number \( \xi = \xi(\alpha, \varepsilon) \in \mathbb{Q}(\alpha) \) such that \( 1/13 - \varepsilon < \{\xi \alpha^n\} < 1/13 + \varepsilon \) for each \( n \in \mathbb{N} \). This not only implies that \( \alpha \in \mathbb{Z} \) but also that every integral part \( \{\xi \alpha^n\} \), where \( n \in \mathbb{N} \) and \( \xi = 12\xi \), is divisible by 12.

4. **Proof of Theorem 2.** In all three cases it suffices to show that, for any \( \xi \neq 0 \), the integral parts \( x_n := \lfloor \xi \alpha^n \rfloor, n = 1, 2, \ldots \), cannot all be even. Suppose they are, i.e. \( \alpha \in \mathbb{Z} \). Setting \( y_n := \lfloor \xi \alpha^n \rfloor \) for \( n \in \mathbb{N} \), we have \( x_{n+q} - 2x_n = 2y_n - y_{n+q} \) (case (i)) or \( x_{n+d} - x_{n+r} - x_n = y_n + y_{n+r} - y_{n+d} \) (case (ii)) or \( x_{n+d} - x_{n+m} - x_{n+r} + x_n = -y_n + y_{n+r} + y_{n+m} - y_{n+d} \) (case (iii)). A fractional part is a non-negative number smaller than 1. So the right-hand sides of all three equalities belong to the interval \((-2, 2)\). But all left-hand sides are even integers. Hence, for every \( n \in \mathbb{N} \), we have \( x_{n+q} - 2x_n = 2y_n - y_{n+q} = 0 \).
Similarly, from the linear recurrence $\alpha n + \alpha n + r - \alpha n + d = 0$ (case (ii)), $x_{n+d} - x_{n+m} - x_{n+r} + x_n = -y_n + y_{n+r} + y_{n+m} - y_{n+d}$ (case (iii)).

In case (i) we deduce that $y_{n+qm} = 2^m y_n$ for any $m \in \mathbb{N}$. Taking $m$ arbitrarily large we obtain that $y_n = \{\xi 2^{m/q}\} = 0$ for every $n \in \mathbb{N}$. Next, by considering, firstly, the subsequence $n = qk$, $k = 1, 2, \ldots$, and, secondly, the subsequence $n = qk + 1$, $k = 1, 2, \ldots$, we derive that $\xi 2^{m/q}$ is an integer for every $n \in \mathbb{N}$ if and only if $\xi = 0$, a contradiction. Hence $2^{m/q} \notin \mathcal{S}$ for each integer $q \geq 2$. This proves (i).

In case (iii) the sequence $s_n := -y_n + y_{n+r} + y_{n+m} - y_{n+d} = 0$ is periodic. So, by Lemma 3 of [8], $\alpha > 1$ must be a Pisot number or a Salem number. It cannot be a Pisot number by the condition of (iii). Hence $\alpha$ is a Salem number. But from $\alpha^d - \alpha^m - \alpha^r + 1 = 0$ on replacing $\alpha \rightarrow \alpha^{-1}$ (Salem numbers are reciprocal) we obtain that $\alpha^d - \alpha^{d-r} - \alpha^{d-m} + 1 = 0$. Observe that if $m + r = d$ then $\alpha^d - \alpha^m - \alpha^r + 1 = (\alpha^m - 1)(\alpha^r - 1) = 0$, a contradiction with $\alpha > 1$. If $m + r \neq d$ then

$$\alpha^d - \alpha^m - \alpha^r + 1 - (\alpha^d - \alpha^{d-r} - \alpha^{d-m} + 1) = \alpha^{d-m} - \alpha^r - \alpha^{d-r} - \alpha^m = (\alpha^r + \alpha^m)(\alpha^{d-r-m} - 1) = 0,$$

a contradiction again. This proves (iii). (Note that we proved the following statement: each irreducible reciprocal factor of $x^d - x^m - x^r + 1$ is cyclotomic. See [15] for more about irreducible factors of such quadrinomials.)

In case (ii) the sequence $s_n := -y_n + y_{n+r} + y_{n+d} = 0$ is periodic. As above, Lemma 3 of [8] implies that $\alpha$ is a Pisot number or a Salem number and $\xi \in \mathbb{Q}(\alpha)$. Since $\alpha > 1$ is a root of an irreducible nonreciprocal polynomial $x^d - x^r - 1$, it can only be a Pisot number. (Indeed it can: for instance, if $r = 1$ and $d = 2$ or $d = 3$.) Note that it is not a strong Pisot number, because the polynomial $x^d - x^r - 1$ has no roots in the interval $[0, 1]$. Suppose that the conjugates of $\alpha = \alpha_1 > 1$ over $\mathbb{Q}$ are $\alpha_2, \ldots, \alpha_d$, where $|\alpha_1| > |\alpha_2| \geq |\alpha_3| \geq \cdots \geq |\alpha_d|$. Since $x_{n+d} - x_{n+r} - x_n = 0$ for every $n \in \mathbb{N}$, we have that $x_n = \xi_1 \alpha_1^n + \cdots + \xi_d \alpha_d^n$. Moreover (see [5] or the proof of Theorem 3 in [8]), $\xi_j \in \mathbb{Q}(\alpha_j)$, $j = 1, \ldots, d$, and the numbers $\xi_1, \ldots, \xi_d$ are conjugate over $\mathbb{Q}$. Similarly, from the linear recurrence $x_{n+d} - x_{n+r} - x_n = 0$, $n = 1, 2, \ldots$, we obtain that there exist certain complex numbers $\eta_1, \ldots, \eta_d$ such that $y_n = \eta_1 \alpha_1^n + \cdots + \eta_d \alpha_d^n$ for each $n \in \mathbb{N}$. But $x_n + y_n = \xi \alpha_1^n$, so that $\eta_1 = \xi - \xi_1, \eta_2 = -\xi_2, \ldots, \eta_d = \xi - \xi_d$. If $\eta_1 \neq 0$ then $|y_n| \rightarrow \infty$ as $n \rightarrow \infty$, a contradiction. It follows that $\eta_1$ must be equal to zero, so $\xi_1 = \xi$. Summarizing, we have that $y_n = -\xi_2 \alpha_2^n - \xi_3 \alpha_3^n - \cdots - \xi_d \alpha_d^n$, where $\xi_2 \in \mathbb{Q}(\alpha_2), \ldots, \xi_d \in \mathbb{Q}(\alpha_d)$ are conjugate over $\mathbb{Q}$ and $\xi_2 \neq 0$.

In order to get a contradiction it suffices to show that the sums $y_n = -\xi_2 \alpha_2^n - \xi_3 \alpha_3^n - \cdots - \xi_d \alpha_d^n$ are negative for infinitely many $n \in \mathbb{N}$. Indeed, since every Pisot number $\alpha = \alpha_1$ has at most two conjugates of largest modulus in the unit disc (see [18]) which is $|\alpha_2|$, but $\alpha$ is not a strong Pisot number, i.e. $\alpha_2 \notin (0, 1)$ there are only two possibilities. Either $\alpha_2$ is a real negative number in $(-1, 0)$ and $|\alpha_2| > |\alpha_1|$ for $j > 2$ or $\alpha_2$ and $\alpha_3$ are complex conjugate numbers, i.e. $\alpha_3 = \overline{\alpha_2}$ and $|\alpha_3| > |\alpha_1|$ for $j > 3$. In both cases, since $-\xi_2 \alpha_2^n - \xi_3 \alpha_3^n = -2\Re(\xi_2 \alpha_2^n)$, the sign of $y_n$ is the same as that of $-\Re(\xi_2 \alpha_2^n)$ for each $n$ sufficiently large. Of course, if $\alpha_2 \in (-1, 0)$ then $-\xi_2 \alpha_2^n$ is positive for infinitely many $n \in \mathbb{N}$ and negative for infinitely many $n \in \mathbb{N}$. Assume that $\alpha_2$ is complex. Let us write $\alpha_2 = \rho e^{i\theta}$ and $\xi_2 = \rho' e^{i\theta}$. Then $\Re(\xi_2 \alpha_2^n) = \rho \rho' \cos(n\theta + \vartheta)$. Since $\theta / \pi$ is irrational (see [17] or derive a contradiction from $\alpha_3^n = \alpha_1^n$, where $m \in \mathbb{N}$, by mapping $\alpha_2$ to $\alpha$), Kronecker’s theorem [5] yields that the fractional parts $\{n\theta / \pi + \vartheta/n\}$, $n = 1, 2, \ldots$, are dense in $[0, 1)$. It follows that $\cos(n\theta + \vartheta)$ is positive for infinitely many $n \in \mathbb{N}$ and negative for infinitely many $n \in \mathbb{N}$. This completes the proof of Theorem 2.
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