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PERCOLATION OF WORDS ON $\mathbb{Z}^d$ WITH LONG-RANGE CONNECTIONS

B. N. B. DE LIMA * AND
R. SANCHIS,* ** *Universidade Federal de Minas Gerais*
R. W. C. SILVA,*** *Universidade Federal de Minas Gerais and Universidade Federal de Ouro Preto*

Abstract

Consider an independent site percolation model on $\mathbb{Z}^d$, with parameter $p \in (0, 1)$, where all long-range connections in the axis directions are allowed. In this work we show that, given any parameter p, there exists an integer $K(p)$ such that all binary sequences (words) $\xi \in \{0, 1\}^{\mathbb{N}}$ can be seen simultaneously, almost surely, even if all connections with length larger than $K(p)$ are suppressed. We also show some results concerning how $K(p)$ should scale with p as p goes to 0. Related results are also obtained for the question of whether or not almost all words are seen.

Keywords: Percolation of words; truncation question

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1. Introduction and notation

The problem of percolation of words was introduced in [2] and is formulated as follows. Let $G = (\mathbb{V}, \mathbb{E})$ be a graph with a countably infinite vertex set $\mathbb{V}$. Consider site percolation on G; to each site $v \in \mathbb{V}$ we associate a Bernoulli random variable $X(v)$, which takes the values 1 and 0 with probability p and $1 - p$, respectively. We work with the probability space $(\Omega, \mathcal{F}, \mathrm{P}_p)$, where $\Omega = \{0, 1\}^{\mathbb{V}}$, $\mathcal{F}$ is the σ-algebra generated by the cylinder sets in Ω, and $\mathrm{P}_p = \prod_{v\in\mathbb{V}} \mu(v)$ is the product of Bernoulli measures with parameter p, in which the configurations $\{X(v),\ v \in \mathbb{V}\}$ take place. We denote a typical element of Ω by ω, and sometimes we write $X(\omega, v)$ instead of $X(v)$ to indicate that $X(v)$ depends on the configuration. When $X(v) = 1$ or $X(v) = 0$, we say that v is 'occupied' or 'vacant', respectively. A path γ on G is a sequence $v_1, v_2, \dots$ of vertices in $\mathbb{V}$ such that $v_i \neq v_j$ for all $i \neq j$ and v_{i+1} is a nearest neighbor of v_i for all i; that is, the edge $\langle v_i, v_{i+1}\rangle$ belongs to $\mathbb{E}$.

Let $\Xi = \{0, 1\}^{\mathbb{N}}$. A semi-infinite binary sequence $\xi = (\xi_1, \xi_2, \dots) \in \Xi$ will be called a *word*. Given a word $\xi \in \{0, 1\}^{\mathbb{N}}$, a vertex $v \in \mathbb{V}$, and a configuration $\omega \in \Omega$, we say that the word ξ *is seen in the configuration* ω *from the vertex* v if there is a self-avoiding path $\langle v = v_0, v_1, v_2, \dots\rangle$ such that $X(v_i) = \xi_i$ for all $i = 1, 2, \dots$. Note that the state of v is irrelevant. For fixed $\omega \in \Omega$ and $v \in \mathbb{V}$, we will consider the random sets $S_v(\omega) = \{\xi \in \Xi;\ \xi$ is seen in ω from $v\}$ and $S_\infty(\omega) = \bigcup_{v\in\mathbb{V}} S_v(\omega)$. An interesting problem is to describe the

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* Postal address: Departamento de Matemática, Universidade Federal de Minas Gerais, Av. Antônio Carlos 6627, CP 702, CEP 30123-970 Belo Horizonte, MG, Brazil.
** Email address: rsanchis@mat.ufmg.br
*** Postal address: Departamento de Estatística, UFMG, Av. Antônio Carlos 6627, CP 702, CEP 30123-970 Belo Horizonte, MG, Brazil.

circumstances in which the events $\{\omega \in \Omega;\ S_\infty(\omega) = \Xi\}$ and $\{\omega \in \Omega;$ there exists $v \in \mathbb{V}$ with $S_v(\omega) = \Xi\}$ occur almost surely. Whenever either one of these events occurs, we say that *all words are seen*.

From a different perspective, if we suppose that the sequence of digits in the word ξ is a sequence of independent Bernoulli random variables with parameter α, i.e. each word ξ take its values in the probability space $(\Xi, \mathcal{A}, \mu_\alpha)$, where $\mathcal{A}$ is the σ-algebra generated by the cylinder sets in Ξ, and $\mu_\alpha = \prod_{n\in\mathbb{N}} \mu(n)$ is the product of Bernoulli measures with parameter α, another question arises, namely whether the event

$$\{\omega \in \Omega;\ \mu_\alpha(S_\infty(\omega)) = 1\}$$

occurs almost surely. Whenever this occurs, we say that *almost all words are seen* or that the *random word percolates*.

In general, the problem of seeing all words is significantly harder than the problem of seeing almost all words. For instance, it is known that, for $d \geq 3$ and $p = \frac{1}{2}$, almost all words are seen on $\mathbb{Z}^d$ with nearest neighbors, whereas, in [2], it was shown that it is possible to see all words on $\mathbb{Z}^d$, $\mathrm{P}_{1/2}$-almost surely ($\mathrm{P}_{1/2}$-a.s.) for $d \geq 10$, but, for $d < 10$, the problem of seeing all words remains open (see Theorem 1 and Open Problem 2 of [2]). We should remark that, in general, seeing almost all words does not imply that all words are seen. For instance, Theorem 5 of [2] gives an example of a tree where we can see $\mu_{1/2}$-almost all words but not all words are seen $\mathrm{P}_{1/2}$-a.s.

In [9] it was shown that μ_α-almost all words are seen (with $\alpha \in (0, 1)$) on the triangular lattice $\mathrm{P}_{1/2}$-a.s. (remember that in the triangular lattice $p_\mathrm{c} = \frac{1}{2}$, so it is not possible to see all words). In [10] it was proved that on the closed packed graph of $\mathbb{Z}^2$ (the ordinary square lattice decorated with both diagonal bonds in each square), for $p \in (1 - p_\mathrm{c}(\mathbb{Z}^2), p_\mathrm{c}(\mathbb{Z}^2))$, all words are seen P_p-a.s.

On the one hand, percolation of words is a natural generalization of the usual percolation and, as such, is a source of interesting mathematical questions. On the other hand, research in theoretical computer science has raised several deep and interesting problems, such as the compatibility of binary sequences and the clairvoyant demon problem (see [3] and [12]), which are conceptually closely related to percolation of words. At the same time control over the behavior of truncated long-range sequences may shed some light on the issue of quasi-isometries between random spatial objects (as discussed in [11]). See [5] for a more accurate description of the relations between percolation of words, quasi-isometries, and compatibility of binary sequences.

In the present paper we are concerned with the graph $G_K = (\mathbb{V}, \mathbb{E}_K)$, in which $\mathbb{V} = \mathbb{Z}^d$, $d \geq 2$, and where all long-range edges parallel to the coordinate axes are allowed, that is,

$$\mathbb{E}_K = \{\langle(x, y)\rangle \subset \mathbb{Z}^d \times \mathbb{Z}^d : \text{there exists } i \in \{1, \dots, d\} \text{ such that } 0 < |x_i - y_i| \leq K \\ \text{and } x_j = y_j \text{ for all } j \neq i\}.$$

The graph G_K can be seen as a truncation of the (nonlocally finite) graph $G = (\mathbb{V}, \mathbb{E})$, where

$$\mathbb{E} = \{\langle(x_1, \dots, x_d)(y_1, \dots, y_d)\rangle \subset \mathbb{Z}^d \times \mathbb{Z}^d : \text{there exists } i \in \{1, \dots, d\} \text{ such that } x_i \neq y_i \\ \text{and } x_j = y_j \text{ for all } j \neq i\},$$

that is, G_K can be obtained from G by erasing all bonds with length larger than K.

In an earlier paper, de Lima [4] showed that, for all $p \in (0, 1)$, there exists a positive integer $K = K(p)$ such that, on the graph G_K,

$$\mathrm{P}_p\Bigg(\bigcup_{v\in\mathbb{V}}\{\omega \in \Omega;\, \xi \text{ is seen in } \omega \text{ from } v \text{ on } G_K\}\Bigg) = 1 \quad \text{for all } \xi \in \Xi. \tag{1}$$

Moreover, (1) implies that, for the same $K(p)$, it is possible to see μ_α-almost all words (with $\alpha \in (0, 1)$) on G_K, but (1) does not imply that it is possible to see all words.

We would like to point out that Grimmett *et al.* [7] treated similar questions to those considered in this paper, but on $\mathbb{Z}$ (in the oriented case) instead of $\mathbb{Z}^d$, $d \geq 2$. One of the results proved therein was that, when $K = 2$, not all words are seen from the origin $\mathrm{P}_{1/2}$-a.s.

In Section 2 we prove that there is a constant $K(p)$ such that, with positive probability, all words are seen on G_K from a given vertex and state some results on the scaling of the constant $K(p)$ as $p \searrow 0$. In Section 3 we state a result on the scaling of the constant $K(p)$ for which μ_α-almost all words are seen on G_K. In Section 4 we make some final remarks concerning the scaling behavior for ordinary percolation, and state some conjectures and open questions.

2. All words can be seen

Our first result generalizes that of [4], showing that *all* words are seen on G_K for sufficiently large K.

Theorem 1. *For all* $p \in (0, 1)$*, there exists a positive integer* $K = K(p)$ *such that*

$$\mathrm{P}_p\{\omega \in \Omega;\, S_0(\omega) = \Xi \text{ on } G_K\} > 0.$$

Equivalently,

$$\mathrm{P}_p\Bigg(\bigcup_{v\in\mathbb{V}}\{\omega \in \Omega;\, S_v(\omega) = \Xi \text{ on } G_K\}\Bigg) = 1.$$

Proof. For any given $n \in \mathbb{N}$ and $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$, let

$$\Lambda_x(n) = \{y = (y_1, \ldots, y_d) \in \mathbb{Z}^d;\, 0 \leq y_i - nx_i \leq n - 1 \text{ for all } i = 1, \ldots, d\} \tag{2}$$

be a hypercubic box of side n. We observe that, for any $n \in \mathbb{N}$, the set of boxes $\{\Lambda_x(n);\, x \in \mathbb{Z}^d\}$ forms a partition of $\mathbb{Z}^d$.

Consider a renormalized lattice, isomorphic to $\mathbb{Z}^d$, whose sites are the boxes $\{\Lambda_x(n);\, x \in \mathbb{Z}^d\}$. Given a configuration $\omega \in \Omega$, we declare each box as 'good', in the configuration ω, if all lines have at least one occupied site and one vacant site. To be precise, the box $\Lambda_x(n)$ will be 'good' if, for all $i \in \{1, \ldots, d\}$ and all finite sequences $(l_j)_j$ with $l_j \in \{0, \ldots, n-1\}$ and $j \in \{1, \ldots, d\} - \{i\}$, there exists $z, w \in L(i, (l_j)_j)$ such that $X(\omega, z) = 1$ and $X(\omega, w) = 0$, where

$$L(i, (l_j)_j) = \{y \in \Lambda_x(n);\, y_j = l_j + x_j \text{ for all } j \in \{1, \ldots, d\} - \{i\}\}$$

are the lines of $\Lambda_x(n)$.

Consider the events

$$A_x(n) = \{\omega \in \Omega;\ \text{the box } \Lambda_x(n) \text{ is good in } \omega\}.$$

It is clear that all events of the collection $\{A_x(n);\, x \in \mathbb{Z}^d\}$ are independent and have the same probability. A rough estimate for a lower bound of this probability gives

$$\mathrm{P}_p(A(n)) = 1 - \mathrm{P}_p(A(n)^{\mathrm{c}}) \geq 1 - dn^{d-1}(p^n + (1-p)^n).$$

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Then,

$$\lim_{n\to\infty} \mathrm{P}_p(A(n)) = 1 \quad \text{for all } p \in (0,1).$$

Now, for fixed $p \in (0,1)$, let $N = N(p) = \min\{n \in \mathbb{N}; \mathrm{P}_p(A_n) > p_\mathrm{c}(\mathbb{Z}^d)\}$, where $p_\mathrm{c}(\mathbb{Z}^d)$ is the ordinary independent nearest-neighbor site percolation threshold for $\mathbb{Z}^d$. Then the origin of the renormalized lattice will percolate with strictly positive probability, that is, there is an infinite path $(\Lambda_{x_0}(N), \Lambda_{x_1}(N), \Lambda_{x_2}(N), \dots)$ of renormalized 'good' sites, with $x_k = (x_{k,1}, \dots, x_{k,d}) \in \mathbb{Z}^d$, $\|x_{k+1} - x_k\|_1 = 1$ for all $k \in \mathbb{N}$, and $x_0 = (0, \dots, 0)$. From now on, we fix some configuration $\omega \in \Omega$ for which this infinite path $(\Lambda_{x_0}(N), \Lambda_{x_1}(N), \Lambda_{x_2}(N), \dots)$ of renormalized 'good' sites occurs.

Given any word $\xi = (\xi_1, \xi_2, \dots) \in \Xi$, we can see its digits along some path $\gamma = \langle v_0 = 0, v_1, v_2, \dots\rangle$ starting from the origin of the original lattice in the following way. Define v_0 as being the origin; we will define the other vertices inductively. Given the vertex $v_{k-1} \in \Lambda_{x_{k-1}}(N)$, let $i_k \in \{1, \dots, d\}$ be the unique integer such that $|x_{k-1,i_k} - x_{k,i_k}| = 1$. Since the box $\Lambda_{x_k}(N)$ is good, there exists at least one vertex $v \in \Lambda_{x_k}(N)$ along the line $L(i_k, (l_j)_j)$ with $l_j = v_{k-1,j}$ for all $j \neq i_k$ such that $X(v) = \xi_k$. Choose one of these vertices and call it v_k. Observe that v_{k-1} and v_k belong to the same line and that $\|v_{k-1} - v_k\|_1 \le 2N - 1$ for all $k \in \mathbb{N}$.

Then, by construction, on this fixed configuration ω, we have $\xi_k = X(\omega, v_k)$ for all $k \in \mathbb{N}$. So, taking $K(p) = 2N(p) - 1$ we have

$$\mathrm{P}_p\{\omega \in \Omega; S_0(\omega) = \Xi \text{ on } G_K\} > 0.$$

The last statement of the theorem follows by observing that the event $\bigcup_{v\in\mathbb{V}}\{\omega \in \Omega; S_v(\omega) = \Xi \text{ on } G_K\}$ is translation invariant, so its probability must be 0 or 1.

From now on, we can suppose that $K(p)$ is the minimal truncation constant such that Theorem 1 holds. A natural question to ask is how the magnitude of $K(p)$ varies as a function of p. We can easily see that $K(p)$ is symmetric around $p = \frac{1}{2}$ and a coupling argument can be used to show that $K(p)$ is nonincreasing on the interval $(0, \frac{1}{2}]$, achieving its minimum at $p = \frac{1}{2}$ (when $d = 2$, the constant $K(\frac{1}{2})$ could be taken as 11). It is natural to expect that $K(p)$ increases to ∞ as p approaches 0, as we will show in Theorem 2 below, and a problem of relevance is to determine how $K(p)$ scales as p goes to 0. Without loss of generality (by symmetry), we consider only the situation where $p \in (0, \frac{1}{2}]$.

Related problems on other models have been extensively studied, for example, in [1], the authors determined the right finite-size scaling as p goes to 0 for the critical threshold in two-dimensional bootstrap percolation. This is the setup of the next theorem and lemmas.

Lemma 1. *If $K = K(p) = 2\lfloor\lambda/p\rfloor$ then, for $\lambda > -3\ln(1 - p_\mathrm{c}(\mathbb{Z}^d))$, it holds that*

$$\lim_{p\to 0} \mathrm{P}_p\bigg(\bigcup_{v\in\mathbb{V}}\{\omega \in \Omega; S_v(\omega) = \Xi \text{ on } G_K\}\bigg) = 1.$$

Proof. By translation invariance, it is enough to prove that there exists some $p^* > 0$ such that, for large λ and all $p \in (0, p^*)$,

$$\mathrm{P}_p\{\omega \in \Omega; S_0(\omega) = \Xi \text{ on } G_K\} > 0. \tag{3}$$

We say that there is a *seed* at vertex $v \in \mathbb{Z}^d$ if $X(v) = 1$ and $X(u) = 0$ for all u with $\|v - u\|_1 = 1$. We call the vertex v the *center* of the seed. Observe that

$$\mathrm{P}_p\{\text{there is a seed located at } v\} = (1-p)^{2d} p,$$

and that the events {there is a seed located at v_1} and {there is a seed located at v_2} are independent if $\|v_1 - v_2\|_1 \geq 3$.

As $\lim_{p\to 0}[1 - [1 - p(1-p)^{2d}]^{\lfloor 1/3(\lfloor\lambda/p\rfloor - 2)\rfloor}] = 1 - \exp(-\lambda/3)$, we can choose some large $\lambda > -3\ln(1 - p_c(\mathbb{Z}^d))$ and some small p^* such that

$$1 - [1 - p(1-p)^{2d}]^{\lfloor 1/3(\lfloor\lambda/p\rfloor - 2)\rfloor} > p_c(\mathbb{Z}^d) \quad \text{for all } p \in (0, p^*).$$

For this large λ and $p \in (0, p^*)$, define $n = \lfloor\lambda/p\rfloor$ and consider the partition of $\mathbb{Z}^d$, $\{\Lambda_x(n);\ x \in \mathbb{Z}^d\}$, as defined in (2). We will use the letters x and y to denote vertices of the renormalized lattice. The idea is to construct, dynamically, a sequence $(R_x,\ x \in U \subset \mathbb{Z}^d)$ of $\{0, 1\}$-valued random variables and a sequence (D_i, E_i), $i = 0, 1, \dots$, of ordered pairs of subsets of $\mathbb{Z}^d$, defined as follows.

First, let $f\colon \mathbb{N} \to \mathbb{Z}^d$ be a fixed ordering of the vertices of $\mathbb{Z}^d$ and define $(D_0, E_0) = (\varnothing, \varnothing)$. Let $x_0 = 0$ be the origin of $\mathbb{Z}^d$. We say that $R_{x_0} = 1$ if at least one of the $d(n-1)$ vertices in the set

$$T_{x_0} = \{v = (v_1, \dots, v_d) \in \mathbb{Z}^d;\ \text{there exists } i \in \{1, \dots, d\} \text{ with } v_i \in \{1, \dots, n-1\} \\ \text{and } v_j = 0 \text{ for all } j \neq i\}$$

is the center of some seed, that is, if there exists $v \in T_{x_0}$ with $X(v) = 1$ and $X(u) = 0$ for all u with $\|v - u\|_1 = 1$. Otherwise, we say that $R_{x_0} = 0$. Observe that

$$\mathrm{P}_p(R_{x_0} = 1) \geq 1 - [1 - p(1-p)^{2d}]^{\lfloor d(n-1)/3\rfloor} \geq 1 - [1 - p(1-p)^{2d}]^{\lfloor n/3\rfloor}.$$

Now, define

$$(D_1, E_1) = \begin{cases} (D_0 \cup \{x_0\}, E_0) & \text{if } R_{x_0} = 1, \\ (D_0, E_0 \cup \{x_0\}) & \text{if } R_{x_0} = 0, \end{cases}$$

and if $R_{x_0} = 1$, define $z(x_0)$ as the center of some seed belonging to T_{x_0}.

Let (D_i, E_i) be given. If $\partial_e(D_i) \cap E_i^{\mathrm{c}} = \varnothing$, define $(D_j, E_j) = (D_i, E_i)$ for all $j > i$, where

$$\partial_e(A) = \{v \in \mathbb{Z}^d;\ v \in A^{\mathrm{c}} \text{ and there exists } u \in A \text{ with } \|v - u\|_1 = 1\}.$$

Otherwise, let x_i be the first vertex in the fixed order belonging to $\partial_e(D_i) \cap E_i^{\mathrm{c}}$ and define y_i as any vertex belonging to D_i such that $\|x_i - y_i\|_1 = 1$ (observe that $y_1 = x_0$).

We say that $R_{x_i} = 1$ if at least one of the $n - 2$ vertices of the set

$$T_{x_i} = \widetilde{\Lambda}_{x_i} \cap \{z(y_i) + j\bar{e}_{x_i - y_i};\ j \in \mathbb{Z}\}$$

is the center of some seed, that is, if there exists $v \in T_{x_i}$ with $X(v) = 1$ and $X(u) = 0$ for all u with $\|v - u\|_1 = 1$. Here, $\bar{e}_l$ denotes the unit vector of $\mathbb{Z}^d$ in the lth direction and $\widetilde{\Lambda}_x = \{y = (y_1, \dots, y_d) \in \mathbb{Z}^d; 0 \leq y_i - nx_i \leq n - 3 \text{ for all } i = 1, \dots, d\}$ (this ensures the independence of the sequence $(R_{x_i},\ i = 0, 1, \dots))$. Otherwise, we say that $R_{x_i} = 0$. Observe that $\mathrm{P}_p(R_{x_i} = 1 \mid R_{x_j} \text{ for all } j < i) \geq 1 - [1 - p(1-p)^{2d}]^{\lfloor (n-2)/3\rfloor}$. Define

$$(D_{i+1}, E_{i+1}) = \begin{cases} (D_i \cup \{x_i\}, E_i) & \text{if } R_{x_i} = 1, \\ (D_i, E_i \cup \{x_i\}) & \text{if } R_{x_i} = 0, \end{cases}$$

and if $R_{x_i} = 1$, define $z(x_i)$ as the center of some seed belonging to T_{x_i}. Owing to our choices of λ, p, and n, the process $(R_{x_i},\ i = 0, 1, \dots)$ dominates an independent and identically

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distributed (i.i.d.) $\{0, 1\}$-valued process with parameter larger than $p_c(\mathbb{Z}^d)$. Comparison with ordinary site percolation shows that (see Lemma 1 of [6]) $\mathrm{P}_p(\#(\bigcup_{i\in\mathbb{N}} D_i) = \infty) > 0$ and, by construction, on the event $(\#(\bigcup_{i\in\mathbb{N}} D_i) = \infty)$, all words $\xi \in \Xi$ can be seen along some self-avoiding path $\langle 0, v_1, v_2, \dots\rangle$ with v_i belonging to some seed for all i, as we will now show. Then, (3) is proved with $K(p) = 2\lfloor\lambda/p\rfloor$.

When the event $\{\#(\bigcup_{i\in\mathbb{N}} D_i) = \infty\}$ occurs, it is possible to take a sequence of adjacent boxes $\Lambda_{x_{i_0}}, \Lambda_{x_{i_1}}, \Lambda_{x_{i_2}}, \dots$, with $x_{i_0} = x_0 = 0$, such that $R(x_{i_j}) = 1$ for all j and $z(x_{i_j}) - z(x_{i_{j-1}}) = m\bar{e}_l$ for some $m \in \mathbb{Z}^*$ and $l \in \{1, \dots, d\}$. That is, seeds in adjacent boxes have their centers belonging to the same line. To simplify the notation, let us denote x_{i_j} by w_j.

Given any word $\xi \in \Xi$, define $l_1 = \min\{i; \xi_i = 1\}$ and $l_j = \min\{i > l_{j-1}; \xi_i = 1\}$ for $j \ge 2$. If $l_1 = 1$, define $v_1 = z(w_0)$; if $l_1 > 1$, define $v_i = z(w_{i-1}) - \bar{e}_b$ for all $i < l_1$ and $v_{l_1} = z(w_{l_1-2})$, where b is the unique direction such that the inner product $\langle \bar{e}_b \cdot z(w_0)\rangle$ is not 0. Then, by construction, the finite word $(\xi_1, \dots, \xi_{l_1})$ is seen along the path $\langle 0, v_1, \dots, v_{l_1}\rangle$. Define $I(1)$ as the index such that $v_{l_1} = z(w_{I(1)})$ (observe that $I(1) = 0$ if $l_1 = 1$ and $I(1) = l_1 - 2$ if $l_1 \ge 2$).

Now, we describe the induction step. Suppose that the finite word $(\xi_1, \dots, \xi_{l_k})$ is seen along the path $\langle 0, v_1, \dots, v_{l_k}\rangle$ for all $k \ge 1$. If $l_{k+1} = l_k + 1$, define $v_{l_{k+1}} = z(w_{I_k+1})$; if $l_{k+1} > l_k + 1$, define $v_i = z(w_{I(k)+i-l_k}) - \bar{e}_{w_{I(k)} - w_{I(k)+1}}$ for all $l_k < i < l_{k+1}$ and $v_{l_{k+1}} = z(w_{I(k)+l_{k+1}-1-l_k})$. Then, by construction, the finite word $(\xi_1, \dots, \xi_{l_{k+1}})$ is seen along the path $\langle 0, v_1, \dots, v_{l_{k+1}}\rangle$. Define $I(k+1)$ as the index such that $v_{l_{k+1}} = z(w_{I(k+1)})$ (observe that $I(k+1) = I(k) + 1$ if $l_{k+1} = l_k + 1$ and $I(k+1) = I(k) + l_{k+1} - l_k - 1$ if $l_{k+1} > l_k + 1$). Thus, we define the path $\langle 0, v_1, v_2, \dots\rangle$ in such way that $X(v_i) = \xi_i$ for all i. This completes the proof.

Lemma 2. *If $K = K(p) = \lfloor\lambda/p\rfloor$ with $\lambda < 1/2d$, it holds that*

$$\lim_{p\to 0} \mathrm{P}_p\Bigg(\bigcup_{v\in\mathbb{V}} \{\omega \in \Omega; S_v(\omega) = \Xi \text{ on } G_K\}\Bigg) = 0.$$

Proof. For the subcritical behavior, with a standard argument we show that, for $\lambda < (2d)^{-1}$, the word $\bar{1} = (1, 1, \dots)$ does not percolate. Let σ_m^K be the number of self-avoiding paths of length m starting from the origin on the graph G_K, and let M_m^K be the number of such paths which are occupied. It is clear that if we see the word $\bar{1}$ from the origin then there are occupied paths of all lengths starting from the origin. This implies that, for all $m \in \mathbb{N}$,

$$\begin{aligned}
\mathrm{P}_p\{\omega \in \Omega; \bar{1} \in S_0(\omega) \text{ on } G_K\} &\le \mathrm{P}_p\{\omega \in \Omega; M_m^K(\omega) \ge 1 \text{ on } G_K\} \\
&\le \mathrm{E}(M_m^K) \\
&= p^m \sigma_m^K \\
&\le (p2dK)^m.
\end{aligned}$$

This last inequality follows from the fact that, in order to have a self-avoiding path, each new step has at most $2dK$ choices. Therefore,

$$\mathrm{P}_p\{\omega \in \Omega; \bar{1} \in S_0(\omega) \text{ on } G_K\} \le \lim_{m\to\infty} (p2dK)^m.$$

Thus, if $K < 1/p2d$, it holds that

$$\mathrm{P}_p\{\omega \in \Omega; \bar{1} \in S_0(\omega) \text{ on } G_K\} = 0,$$

that is,

$$\mathrm{P}_p\Bigg(\bigcup_{v\in\mathbb{V}}\{\omega\in\Omega;\ S_v(\omega)=\Xi \text{ on } G_K\}\Bigg)=0.$$

Theorem 2. *There exists a constant* $\lambda_0 \in (1/2d, -6\ln(1-p_{\mathrm{c}}(\mathbb{Z}^d)))$ *such that, if* $K(p)=\lfloor\lambda/p\rfloor$, *it holds that*

$$\lim_{p\to 0}\mathrm{P}_p\Bigg(\bigcup_{v\in\mathbb{V}}\{\omega\in\Omega;\ S_v(\omega)=\Xi \textit{ on } G_K\}\Bigg)=\begin{cases}0 & \textit{if } \lambda<\lambda_0,\\ 1 & \textit{if } \lambda>\lambda_0.\end{cases}$$

Proof. Observe that $\mathrm{P}_p(\bigcup_{v\in\mathbb{V}}\{\omega\in\Omega;\ S_v(\omega)=\Xi \text{ on } G_K\})$ is increasing in λ and must be 0 or 1 by translation invariance. Therefore, this theorem follows by Lemmas 1 and 2.

Observe that in Lemma 1 we made a more involved construction than in Theorem 1. The reason is that the right scale for $K(p)$ is different if we consider the event percolation of good boxes, as shown in the next theorem.

Theorem 3. *Let* $A_0(n)=\{\omega\in\Omega;$ *the box* $\Lambda_0(n)$ *is good in* $\omega\}$. *Then, for* $n=n(p)=\lfloor-\beta\ln p/p\rfloor$, *we have*

$$\lim_{p\to 0}\mathrm{P}_p(A_0(n))=\begin{cases}1 & \textit{if } \beta>d-1,\\ 0 & \textit{if } \beta\le d-1.\end{cases} \tag{4}$$

Proof. For $i\in\{1,\dots,d\}$, define the events

$$\begin{aligned}C_0^i(n)=\{\omega\in\Omega\colon &\text{ for all } (l_j)_j \text{ with } l_j\in\{0,1,\dots,n-1\} \text{ and } j\in\{1,\dots,d\}-\{i\},\\ &\text{there exists } z\in L(i,(l_j)_j) \text{ such that } X(\omega,z)=1\}\end{aligned}$$

and

$$B_0(n)=\bigcap_{i=1}^{d}C_0^i(n),$$

where

$$L(i,(l_j)_j)=\{y\in\Lambda_0(n);\ y_j=l_j \text{ for all } j\in\{1,\dots,d\}-\{i\}\}$$

are the lines of $\Lambda_0(n)$.

By the definitions of $A_0(n)$ and $B_0(n)$, observe that

$$\begin{aligned}\mathrm{P}_p(B_0(n)\setminus A_0(n)) &\le \mathrm{P}_p\{\text{there exists in the box } \Lambda_0(n) \text{ at least one line without } 0\}\\ &\le dn^{d-1}p^n.\end{aligned}$$

Then we have

$$\lim_{p\to 0}\mathrm{P}_p(B_0(n)\setminus A_0(n))=0. \tag{5}$$

For all $i\in\{1,\dots,d\}$, the $C_0^i(n)$ are increasing events, so, by the Fortuin–Kasteleyn–Ginibre inequality and rotational invariance, we have

$$[\mathrm{P}_p(C_0^1(n))]^d\le \mathrm{P}_p(B_0(n))\le \mathrm{P}_p(C_0^1(n)). \tag{6}$$

Thus, using (5) and (6), it is enough to prove (4) by replacing the event $A_0(n)$ by $C_0^1(n)$.

Observe that $\mathrm{P}_p(C_0^1(n)) = [1-(1-p)^n]^{n^{d-1}}$. Then, when $n = n(p) = \lfloor -\beta \ln p/p \rfloor$, we have

$$\begin{aligned}\lim_{p\to 0} \mathrm{P}_p(C_0^1(n)) &= \lim_{p\to 0} [1-(1-p)^{-\beta \ln p/p}]^{(-\beta \ln p/p)^{d-1}} \\ &= \lim_{p\to 0} \exp[-(-\beta \ln p)^{d-1} p^{\beta-(d-1)}] \\ &= \begin{cases} 1 & \text{if } \beta > d-1, \\ 0 & \text{if } \beta \le d-1. \end{cases}\end{aligned}$$

Remarks. (i) The result of Theorem 1 can be generalized if we consider any finite alphabet instead of the binary alphabet, that is, $\Xi = \{0, 1, \dots, n-1\}^{\mathbb{N}}$, in the following sense. Suppose that we have i.i.d. random variables indexed by the sites $v \in \mathbb{Z}^d$ such that $\mathrm{P}(X(v) = i) = p_i$ for $i \in \{0, 1, \dots, n-1\}$. Then the result of Theorem 1 should hold taking $p = \min\{p_0, \dots, p_{m-1}\}$ with minor modifications to the proof.

(ii) The statement of Theorem 2 remains the same replacing the event $\bigcup_{v\in\mathbb{V}}\{\omega \in \Omega;\ S_v(\omega) = \Xi \text{ on } G_K\}$ by $\{\omega \in \Omega;\ S_\infty(\omega) = \Xi \text{ on } G_K\}$. Nevertheless, the constant λ_0 should be different.

3. Percolation of random words

Now we consider the same kind of scaling question, but concerning the probability

$$\mathrm{P}_p\biggl(\bigcup_{v\in\mathbb{V}}\{\omega \in \Omega;\ \mu_\alpha(S_v(\omega)) = 1 \text{ on } G_K\}\biggr),$$

i.e. the probability that almost all words are seen on G_K from one vertex.

We aim to prove an analogue of Theorem 2. Observe that, when $\alpha = 0$, we have ordinary percolation of 0s, and so the constant K can be taken to be equal to 1. When $\alpha = 1$, the right scale of $K(p)$ is the same as in Theorem 2 (see Corollary 1 below). We are not yet able to determine the right scale; actually, we do not even know if the scale itself changes (as the next theorem might suggest) or if only the constant λ_0 would change, but we can give a lower bound. We should observe that, although de Lima [4] guaranteed the existence of a truncation that allows almost all words to be seen, the proof does not give an explicit upper bound.

Theorem 4. *Given $0 < \alpha < 1$, it holds that, for all $\varepsilon > 0$ and $K(p) < (4dp^{\alpha-\varepsilon})^{-1}$,*

$$\lim_{p\to 0} \mathrm{P}_p\biggl(\bigcup_{v\in\mathbb{V}}\{\omega \in \Omega;\ \mu_\alpha(S_v(\omega)) = 1 \textit{ on } G_K\}\biggr) = 0.$$

Proof. Given $\varepsilon > 0$ and $N_0 \in \mathbb{N}$, consider the following subset of words:

$$A_{N_0}^\varepsilon = \biggl\{\xi \in \Xi;\ \biggl|\frac{\sum_{i=1}^n \xi_i}{n} - \alpha\biggr| < \varepsilon \text{ for all } n \ge N_0\biggr\}.$$

We claim that $\mu_\alpha(A_{N_0}^\varepsilon) \to 1$ as $N_0 \to \infty$. To see this, note that, for all N_0, $A_{N_0}^\varepsilon \subset A_{N_0+1}^\varepsilon$. This implies that $A_{N_0}^\varepsilon \uparrow A_\infty^\varepsilon = (\bigcup_{N_0=1}^\infty A_{N_0}^\varepsilon)$ and $\mu_\alpha(A_{N_0}^\varepsilon) \to \mu_\alpha(A_\infty^\varepsilon)$ as $N_0 \to \infty$. By the strong law of large numbers, for all ξ, there exists, μ_α-a.s., some $n_0(\xi) \in \mathbb{N}$ such that

$$\biggl|\frac{\sum_{i=1}^n \xi_i}{n} - \alpha\biggr| < \varepsilon \quad \text{for all } n \ge n_0.$$

This implies that $\mu_\alpha(A_\infty^\varepsilon) = 1$.

On the set $A^{\varepsilon}_{N_0}$, we have

$$(\alpha-\varepsilon)n \leq \sum_{i=1}^{n} \xi_i \leq (\alpha+\varepsilon)n \tag{7}$$

for all $n \geq N_0$.

Given any $\xi \in \Xi$, we will define $\xi^{(n)} = (\xi_1, \dots, \xi_n)$. Then, for any $n \geq N_0$, we have

$$\begin{aligned}&\{\omega \in \Omega;\, S_0(\omega) \cap A^{\varepsilon}_{N_0} \neq \varnothing \text{ on } G_K\}\\ &\quad \subset \bigcup_{\substack{\gamma;|\gamma|=n\\ \xi^{(n)};\xi\in A^{\varepsilon}_{N_0}}} \{\omega \in \Omega;\, \xi^{(n)} \text{ is seen in } \omega \text{ along the path } \gamma \text{ on } G_K\},\end{aligned}$$

where the union is over all self-avoiding paths on G_K of size n, having the origin as its starting point. Hence, for all $n \geq N_0$,

$$\mathrm{P}_p\{\omega \in \Omega;\, S_0(\omega) \cap A^{\varepsilon}_{N_0} \neq \varnothing \text{ on } G_K\} \leq \sum_{\substack{\gamma;|\gamma|=n\\ \xi^{(n)};\xi\in A^{\varepsilon}_{N_0}}} p^{\sum_{i=1}^{n}\xi_i}(1-p)^{n-\sum_{i=1}^{n}\xi_i}.$$

Using (7), we have, for all $n \geq N_0$,

$$\mathrm{P}_p\{\omega \in \Omega;\, S_0(\omega) \cap A^{\varepsilon}_{N_0} \neq \varnothing \text{ on } G_K\} \leq (2dK)^n 2^n p^{(\alpha-\varepsilon)n}(1-p)^{n-(\alpha+\varepsilon)n}.$$

Thus, taking $K < (4dp^{\alpha-\varepsilon})^{-1}$ and observing that $(1-p)^{1-\alpha-\varepsilon} < 1$, we have $4dKp^{\alpha-\varepsilon}(1-p)^{1-\alpha-\varepsilon} < 1$, and so

$$\mathrm{P}_p\{\omega \in \Omega;\; (A^{\varepsilon}_{N_0} \cap S_0(\omega)) = \varnothing \text{ on } G_K\} = 1 \quad \text{for all } N_0 \in \mathbb{N}.$$

Using standard arguments, we can conclude that

$$\mathrm{P}_p\{\omega \in \Omega;\; (A^{\varepsilon}_{\infty} \cap S_0(\omega)) = \varnothing \text{ on } G_K\} = 1.$$

As $\mu_\alpha(A^{\varepsilon}_{\infty}) = 1$, we show that

$$\mathrm{P}_p\{\omega \in \Omega;\; \mu_\alpha(S_0(\omega)) = 1 \text{ on } G_K\} = 0,$$

or, equivalently, using translation invariance, we can conclude that

$$\mathrm{P}_p\biggl(\bigcup_{v\in\mathbb{V}}\{\omega \in \Omega;\, \mu_\alpha(S_v(\omega)) = 1 \text{ on } G_K\}\biggr) = 0.$$

Moreover, by the proof above we can conclude a stronger statement:

$$\mathrm{P}_p\biggl(\bigcup_{v\in\mathbb{V}}\{\omega \in \Omega;\, \mu_\alpha(S_v(\omega)) = 0 \text{ on } G_K\}\biggr) = 1.$$

4. Final remarks

As a straightforward corollary of Lemmas 1 and 2, we obtain the precise scaling behavior of the truncation constant $K(p)$ as p goes to 0 for ordinary percolation.

Corollary 1. *There exists a constant $\lambda_0 \in (1/2d, -2\ln(1-p_c(\mathbb{Z}^d)))$ such that if $K(p) = \lfloor \lambda/p \rfloor$, it holds that*

$$\lim_{p\to 0} \mathrm{P}_p\{\omega \in \Omega;\ (1,1,\dots,1,\dots) \textit{ is seen in } \omega \textit{ on } G_K\} = \begin{cases} 0 & \textit{if } \lambda < \lambda_0, \\ 1 & \textit{if } \lambda > \lambda_0. \end{cases}$$

Proof. It is enough to observe that

$$\mathrm{P}_p\{\omega \in \Omega;\ (1,1,\dots,1,\dots) \text{ is seen in } \omega \text{ on } G_{\lfloor \lambda/p \rfloor}\}$$

is increasing in λ and must be 0 or 1, by translation invariance. Lemma 2 says that $\lambda_0 > 1/2d$, and with a simple modification of the proof of Lemma 1 we can show that

$$\lambda_0 < -2\ln(1-p_c(\mathbb{Z}^d)).$$

In [8] it was shown that $\lim_{d\to\infty} 2dp_c(\mathbb{Z}^d) = 1$. Therefore, this constant λ_0 must be such that $\limsup_{d\to\infty} d\lambda_0 \le 1$.

Related to the comment above, a natural question to ask is:

- What is the asymptotic behavior, in the dimension d, of λ_0 in Theorem 2?

As already mentioned, it is an open question whether the events $\{\mu_\alpha(S_\infty) = 1\}$ and $\bigcup_{v\in\mathbb{V}}\{\mu_\alpha(S_v) = 1\}$ have a sharp threshold behavior on the scaling as the one given in Theorem 2. In many cases, the counting-paths reasoning of Theorem 4 becomes sharper as the parameter becomes extreme, whence we obtain the following result.

Conjecture 1. *For any $\varepsilon > 0$, let $K(p) = \lfloor 1/p^{\alpha+\varepsilon} \rfloor$. Then*

$$\lim_{p\to 0} \mathrm{P}_p\bigg(\bigcup_{v\in\mathbb{V}}\{\omega \in \Omega;\ \mu_\alpha(S_v(\omega)) = 1 \textit{ on } G_K\}\bigg) = 1.$$

This would not, however, determine completely the precise scaling behavior of the event above. Indeed, we could ask if there is a $\lambda_0 \in (0,\infty)$ such that, if $K(p) = \lfloor \lambda/p^\alpha \rfloor$, the following limit holds:

$$\lim_{p\to 0} \mathrm{P}_p\bigg(\bigcup_{v\in\mathbb{V}}\{\omega \in \Omega;\ \mu_\alpha(S_v(\omega)) = 1 \text{ on } G_K\}\bigg) = \begin{cases} 0 & \text{if } \lambda < \lambda_0, \\ 1 & \text{if } \lambda > \lambda_0. \end{cases}$$

Related to the discussion above, we could ask:

- Is the threshold scaling for the event $\{\omega \in \Omega;\ \mu_\alpha(S_\infty(\omega)) = 1 \text{ on } G_K\}$ the same as for the event $\{\omega \in \Omega;\ \text{there exists } v \in \mathbb{V} \text{ with } \mu_\alpha(S_v(\omega)) = 1 \text{ on } G_K\}$ or is it strictly smaller?

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