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ON THE STRUCTURE OF AN ENDOMORPHISM NEAR-RING

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If G is an additive (but not necessarily abelian) group and S is a semigroup of endomorphisms of G, the endomorphism near-ring R of G generated by S consists of all the expressions of the form $\varepsilon_1 s_1 + \cdots + \varepsilon_n s_n$ where $\varepsilon_i = \pm 1$ and $s_i \in S$ for each i. When functions are written on the right, R forms a distributively generated left near-ring under pointwise addition and composition of functions. A basic reference on near-rings which has a substantial treatment of endomorphism near-rings is [**6**].

If the group of inner automorphisms of G is contained in S, the near-ring generated by S is said to be *tame*. Throughout this paper we shall restrict our attention to tame endomorphisms near-rings.

In Section 1, we shall obtain a more explicit description of the minimal R-subgroups of G than has previously appeared in the literature and will give a description of $R/J_2(R)$ when R satisfies the descending chain condition on right ideals. The second section is involved with local endomorphism near-rings satisfying the descending chain condition on right ideals. Here we obtain results about the structure of G and $R/J_2(R)$ when R is local which extend results obtained in [**2**] for finite groups. We also obtain a necessary condition for a tame endomorphism near-ring satisfying the descending chain condition on right ideals to be local.

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1. General structure results

If R is a tame endomorphism near-ring of G and H is a minimal R-subgroup of G, then it is known ([**6**, Corollary 10.18 and Theorem 10.30]) that H is either: (1) an elementary abelian p-group, (2) a divisible abelian group, or (3) a perfect group (i.e., $H' = H$). In fact, we can get an even sharper description in case (2). For suppose H is a minimal R-subgroup of G which is a divisible abelian group. By Theorem 9.13 of [**7**], we have that H is a direct sum of copies of the additive group of the rational numbers and copies of $\sigma(p^\infty)$ for various primes p where $\sigma(p^\infty)$ is the group:

$$\sigma(p^\infty) = \langle a_1, a_2, a_3, \ldots \mid pa_1 = 0,\ pa_2 = a_1, pa_3 = a_2, \ldots \rangle$$

However, were H to contain copies of $\sigma(p^\infty)$ for some prime p, the elements of order p in H would form a proper R-subgroup in H. Hence we have:

Proposition 1.1. *Let R be a tame endomorphism near-ring of G and H be a minimal R-subgroup of G. Then H is either an elementary abelian p-group, a direct sum of copies of the additive group of the rational numbers, or a perfect group.*

Now suppose that R satisfies the descending chain condition on right R-ideals. For the remainder of this paper we shall denote this chain condition by *dccr*. The *socle* of G, $\mathrm{Soc}(G)$, is the sum of the minimal R-subgroups of G and we have that $\mathrm{Soc}(G)=U\oplus V$ where U is the sum of the minimal R-subgroups of G that are abelian and V is the sum of those that are perfect. Moreover, because R satisfies *dccr*, V is finite ([**6**, Lemma 10.39]). The *socle series* of G is obtained by setting $\mathrm{Soc}_0(G)=0$ and $\mathrm{Soc}_k(G)$ to be the inverse image of $\mathrm{Soc}(G/\mathrm{Soc}_{k-1}(G))$ in G for $k>1$. We have that $\mathrm{Soc}_n(G)=G$ for some n ([**6**, Theorem 10.37]), and throughout we shall use n to denote the smallest such positive integer for which $\mathrm{Soc}_n(G)=G$.

For each $1\leqq k\leqq n$,

$$\mathrm{Soc}_k(G)/\mathrm{Soc}_{k-1}(G)=U_k\oplus V_k$$

where U_k is the sum of the minimal R-subgroups of $G/\mathrm{Soc}_{k-1}(G)$ that are abelian and V_k is the sum of those that are perfect. Further, we can write each of these summands as direct sums

$$U_k=\oplus\sum_i A_{ki} \text{ and } V_k=\oplus\sum_j B_{kj}$$

where A_{ki} and B_{kj} are minimal R-subgroups ([**6**, Theorem 10.20]). Indeed, both the sum and the R-subgroups B_{kj} for V_k are finite since V_k is finite.

Following Theorem 7.12 of [**6**], we can write $\bar{R}=R/J_2(R)$ as

$$\bar{R}=\oplus\sum_{i=1}^{m}\left(\oplus\sum_{j=1}^{r(i)}K_{ij}\right)$$

where each K_{ij} is a minimal right ideal of $\bar{R}$, $K_{11}, K_{21},\dots,K_{m1}$ are representatives of the isomorphism classes of minimal $\bar{R}$-modules, and $K_{i1}, K_{i2},\dots,K_{ir(i)}$ are isomorphic $\bar{R}$-modules.

We have that there are at most a finite number of $\bar{R}$-isomorphism (or R-isomorphism) classes of the $\bar{R}$-modules A_{ki} and B_{kj}. Let us denote these by $A_1,\dots,A_s$, $B_1,\dots,B_t$ where each A_i is abelian and each B_j is perfect. We will assume that the notation is arranged so that $A_i\simeq K_{i1}$ and $B_j\simeq K_{s+j,1}$. We must have $s+t=m$, for suppose $s+t<m$. Then

$$\bigcap_{k=1}^{s+t}\mathrm{Ann}_{\bar{R}}(K_{k1})\neq 0$$

and since

$$\mathrm{Ann}_{\bar{R}}(A_i)=\mathrm{Ann}_{\bar{R}}(K_{i1}) \text{ and } \mathrm{Ann}_{\bar{R}}(B_j)=\mathrm{Ann}_{\bar{R}}(K_{s+j,1})$$

we have that the annhilator of the socle series of G properly contains $J_2(R)$ contradicting Lemma 2.5 of [**1**].

We now give a description of the minimal $\bar{R}$-ideals

$$\bar{R}_i=\oplus\sum_{j=1}^{r(i)} K_{ij}.$$

Suppose $1\leqq i\leqq s$. Then $\bar{R}_i=R/\mathrm{Ann}_R(A_i)$ and A_i is a 2-primitive $\bar{R}_i$-module. As $\bar{R}_i$ is an endomorphism near-ring of an abelian group, we have that $\bar{R}_i$ is a ring. It then follows from ring theory $\bar{R}_i$ is a matrix ring over the division ring

$$D_i=\mathrm{End}_{\bar{R}_i}(A_i)=\mathrm{End}_R(A_i).$$

Let us denote this matrix ring as $\mathrm{Mat}_{k_i}(D_i)$ where k_i is the dimension of A_i over D_i. If $s<i\leqq m$, then $\bar{R}_i=R/\mathrm{Ann}_R(B_{i-s})$ and $\bar{R}_i=M_0(B_{i-s})$ by Theorem 10.21 of [**6**].

Let us summarize our observations as follows:

Theorem 1.2. *Let R be a tame endomorphism near-ring of a group G satisfying dccr and $A_1,\dots,A_s$, $B_1,\dots,B_t$ be a complete set of representatives of the isomorphism classes of minimal R-subgroups occurring in the factors of the socle series of G where each A_i is abelian and each B_j is perfect. Then $A_1,\dots,A_s$, $B_1,\dots,B_t$ form a set of representatives of all the isomorphism classes of R-modules of type 2 and*

$$R/J_2(R)=\mathrm{Mat}_{k_1}(D_1)\oplus\cdots\oplus\mathrm{Mat}_{k_s}(D_s)\oplus M_0(B_1)\oplus\cdots\oplus M_0(B_t)$$

where $D_i=\mathrm{End}_R(A_i)$ and k_i is the dimension of A_i over D_i.

2. Local endomorphism near-rings

Throughout this section R will be a tame endomorphism near-ring of G satisfying *dccr*,

$$0\subset\mathrm{Soc}_1(G)\subset\mathrm{Soc}_2(G)\subset\cdots\subset\mathrm{Soc}_n(G)=G$$

will be the socle series of G, and for each i we will write

$$\mathrm{Soc}_i(G)/\mathrm{Soc}_{i-1}(G)=\oplus\sum_j M_{ij}$$

where each M_{ij} is a minimal R-module.

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As defined in [4] (only modified to left near-rings), a near-ring R with identity is *local* if

$$L=\{r\in R\mid r \text{ does not have a right inverse}\}$$

is a right R-subgroup of R. Having R local is equivalent to having $L=J_2(R)$ by Theorem 2.10 of [4].

We first obtain a description of the structure of G and $R/J_2(R)$ when R is local. This result is essentially an extension of Theorem 2.2 of [2] which deals with the case when G is finite, only here we work with the minimal summands of the socle series instead of factors of a principal series.

Theorem 2.1. *Let R be a tame endomorphism near-ring of G satisfying dccr. If R is local, then:*

(i) *$M_{ij}\simeq R/J_2(R)$ as R-modules for all i and j.*

(ii) *Either $\mathrm{Soc}_i(G)/\mathrm{Soc}_{i-1}(G)$ is an elementary abelian p-group or is a direct sum of copies of the additive group of the rational numbers for every i. Moreover, G is a p-group of exponent p^n in the former case.*

(iii) *G is nilpotent.*

(iv) *The socle series is a central series.*

Proof.

(i) Let m be a nonzero element of M_{ij}. Then

$$\mathrm{Ann}_R(m)\supseteq \mathrm{Ann}_R(M_{ij})\supseteq J_2(R)$$

As $J_2(R)=L$ is the maximal right R-subgroup of R ([4, Theorem 2.2]), these containments are in fact equalities. Hence

$$M_{ij}=mR\simeq R/\mathrm{Ann}_R(m)=R/J_2(R).$$

(ii) If M_{ij} is perfect, $R/J_2(R)=M_0(M_{ij})$ by Theorem 1.2. But $R/J_2(R)$ must be a near-field ([4, Corollary 2.11]). As $M_0(M_{ij})$ is a near-field if and only if $M_{ij}\simeq\mathbb{Z}_2$, M_{ij} cannot be perfect. Hence M_{ij} is abelian and (ii) now follows from Proposition 1.1.

(iii) Let A be any one of the M_{ij}. It follows from the last corollary in [3] that G contains a nilpotent R-subgroup P such that $[G:P]$ is finite since every minimal factor of G is abelian as all such factors are isomorphic to A. Indeed, this subgroup P in [3] is the intersection of the near-ring centralizers of the minimal factors of G where the *near-ring centralizer* of a factor H of G is the R-submodule

$$NC_G(H)=\{g\in G\mid [gR,H]=0\}$$

of G. Note that the near-ring centralizer $NC_G(H)$ is a subgroup of the usual group centralizer $C_G(H)=\{g\in G \mid [g,H]=0\}$. Since all minimal factors of G are isomorphic to A, we have $P=NC_G(A)$.

If $P=G$, we are done, so suppose that this is not the case. Since any principal R-series of G/P has factors isomorphic to A, it follows from (ii) that G/P and A are finite p-groups. Let B be a minimal R-subgroup of G/P. Because $[gR,B]$ is an R-subgroup of G/P for any $g\in G$ and G/P is nilpotent, $[gR,B]$ is an R-subgroup of B properly contained in B. Hence $[gR,B]=0$ which gives us $NC_G(B)=G$. But then, as $A\simeq B$, $P=NC_G(A)=NC_G(B)=G$ so that the case $P\subset G$ does not occur.

(iv) Again let A denote any one of the M_{ij}. From the proof of (iii), we see that $NC_G(A)=G$ and consequently $C_G(A)=G$. Since each factor of the socle series is a direct sum of R-modules isomorphic to A, it follows that the socle series is central.

We will conclude this paper by showing that the converse to (i) of Theorem 2.1. holds. In doing so, we will need a result relating idempotents and local near-rings. In [4] (Theorem 4.2) it was shown that the only idempotents in a local near-ring R with identity are the trivial idempotents 0 and 1. Then it was shown in [5] (Theorem II.1) that the converse also holds when the near-ring R is a finite near-ring. However, as the interested reader can check, the proof of the converse given in [5] still applies if R satisfies the descending chain condition on right R-subgroups and hence we have:

Lemma 2.2. *Let R be a near-ring satisfying the descending chain condition on right R-subgroups. Then R is local if and only if R contains only the trivial idempotents* 0 *and* 1.

Theorem 2.3. *Let R be a tame endomorphism near-ring of G satisfying dccr. If $M_{ij}\simeq R/J_2(R)$ for all i and j, then R is local.*

Proof. As we have done before, let A denote one of the M_{ij}. Then $J_2(R)=\mathrm{Ann}_R(A)$. We will proceed by induction on the length n of the socle series. If $n=1$, G is a direct sum of copies of A and hence $J_2(R)=0$ since G is a faithful R-module. Thus $R\simeq A$ as R-modules and hence R is an R-module of type 2. By Theorem 1.2, R either is a matrix ring over a division ring D or is $M_0(A)$. Considering these possibilities, we see that the only way R can be of type 2 is for R to be a matrix ring of dimension 1, so $R=D$ and clearly R is local.

Now consider the induction step. Let $\bar{R}=R/\mathrm{Ann}_R(G/\mathrm{Soc}(G))$. Since

$$\mathrm{Ann}_R(G/\mathrm{Soc}(G))\subseteq \mathrm{Ann}_R(A)=J_2(R)$$

we have that $J_2(\bar{R})=J_2(R)/\mathrm{Ann}_R(G/\mathrm{Soc}(G))$. It then follows that $\bar{R}$ is an endomorphism near-ring on $G/\mathrm{Soc}(G)$ satisfying the hypothesis of the theorem. Consequently $\bar{R}$ is local. Applying Theorem 2.1, we get that $G/\mathrm{Soc}(G)$ is nilpotent and, if we view A as being a summand of $\mathrm{Soc}_2(G)/\mathrm{Soc}(G)$, $G/\mathrm{Soc}(G)$ acts trivially on A when we let $G/\mathrm{Soc}(G)$ act on

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A by conjugation. But this then gives us that G acts trivially on A under conjugation as well. Since Soc(G) is a direct sum of R-subgroups isomorphic to A and R is tame, we have that G acts trivially on Soc(G) under conjugation and hence Soc(G) is contained in the center of G.

Let e be a nonzero idempotent in R. We have from Theorem 5.7 of [**8**] that a tame endomorphism near-ring, satisfying *dccr* actually satisfies the stronger condition of having the descending chain condition on right R-subgroups. Hence it will follow from Lemma 2.2 that R is local if we can show that $e=1$. Since $\mathrm{Ann}_R(G/\mathrm{Soc}(G)) \subseteq J_2(R)$ and $J_2(R)$ is nilpotent ([**6**, Theorem 10.32]), the image of e in $\bar{R}$, $\bar{e}$, is a nonzero idempotent of $\bar{R}$. Since $\bar{R}$ is local, e is the identity on $G/\mathrm{Soc}(G)$ by Lemma 2.2. But this forces e to act as the identity on A. Hence e acts as the identity on Soc(G).

Now let $g \in G$. Since e acts as the identity on $G/\mathrm{Soc}(G)$, we have

$$ge = g + h$$

for some $h \in \mathrm{Soc}(G)$. Suppose that

$$e = \varepsilon_1 s_1 + \cdots + \varepsilon_k s_k$$

where each $\varepsilon_i = \pm 1$ and each s_i is an element of the semigroup of endomorphisms generating R. Then

$$\begin{aligned}(g+h)e &= (g+h)(\varepsilon_1 s_1 + \cdots + \varepsilon_k s_k)\\ &= \varepsilon_1(g+h)s_1 + \cdots + \varepsilon_k(g+h)s_k.\end{aligned}$$

Since $h \in \mathrm{Soc}(G)$ and Soc(G) is in the centre of G, we can rewrite this as

$$\begin{aligned}(g+h)e &= \varepsilon_1 g s_1 + \varepsilon_1 h s_1 + \cdots + \varepsilon_k g s_k + \varepsilon_k h s_k\\ &= \varepsilon_1 g s_1 + \cdots + \varepsilon_k g s_k + \varepsilon_1 h s_1 + \cdots + \varepsilon_k h s_k\\ &= ge + he = ge + h.\end{aligned}$$

Thus

$$\begin{aligned}ge^2 &= (g+h)e = ge + h\\ &= ge\end{aligned}$$

and $h=0$. But then $ge=g$ for all $g \in G$ so $e=1$ and the proof is complete.

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