For a graph $G$, define the bisection width $bw(G)$ of $G$ as $\min \{ e_G(A, B) : \{A, B\} \text{ partitions } V(G) \text{ with } |A| - |B| \leq 1 \}$ where $e_G(A, B)$ denotes the number of edges in $G$ with one end in $A$ and one end in $B$. We show almost every cubic graph $G$ of order $n$ has $bw(G) \geq n/11$ while every such graph has $bw(G) \leq (n + 138)/3$. We also show that almost every $r$-regular graph $G$ of order $n$ has $bw(G) \geq c_r n$ where $c_r \to r/4$ as $r \to \infty$. Our last result is asymptotically correct.

1. INTRODUCTION

For a graph $G$, define the bisection width $bw(G)$ of $G$ by

$bw(G) = \min \{ e_G(A, B) : \{A, B\} \text{ partitions } V(G) \text{ with } |A| - |B| \leq 1 \}$

where $e_G(A, B)$ denotes the number of edges in $G$ with one end in $A$ and one end in $B$.

The problem of finding the bisection width of a graph is of fundamental importance in many divide-and-conquer stratagems and, as such, is the subject of an extensive literature. (See [4, 9, 10, 13, 15, 18] for general results and [6, 11] for results regarding VLSI design.)

Unfortunately, the bisection problem for graphs, in general, is NP-complete [12] and remains so for $r$-regular graphs [9]. Polynomial-time algorithms which give exact solutions are known only for trees and bounded-width planar graphs [9] while polynomial-time algorithms which give approximate solutions may give solutions which are far from exact [18]. Consequently, heuristic algorithms which hopefully give nearly exact solutions most of the time have been developed in [9, 13, 14, 16, 18].

In [9] a method was given for transforming a regular graph $G$ of order $n$ into a cubic graph $G^*$ of order $O(n^5)$ so that any minimum bisection of $G^*$ uses only edges of $G$. As a result, we content ourselves mainly with an examination of cubic graphs. As usual, we say that almost every graph has a property $Q$ provided the probability that a graph of order $n$ has property $Q$ tends to 1 as $n \to \infty$.

We show that almost every cubic graph $G$ of order $n$ has $bw(G) \geq n/11$ while every such graph has $bw(G) \leq (n + 138)/3$. We also show that almost every $r$-regular...
graph $G$ of order $n$ has $bw(G) \geq c_r n$ where $c_r \to r/4$ as $r \to \infty$. (Note that absolute lower bounds for the bisection width of a graph are not particularly informative, since they must be nearly zero.)

Our notation and terminology follows Bollobás [7].

2. An Upper Bound for the Bisection Width of a Cubic Graph

We give now an upper bound for the bisection width of a cubic graph.

**Theorem 1.** Every cubic graph $G$ of order $n$ has $bw(G) \leq (n + 138)/3$.

**Proof:** Let $\{A, B\}$ be an equisized partition of $V(G)$ with $bw(G) = e_G(A, B)$. Set $A_i = \{v \in A : e_G(v, B) = i\}$ for $0 \leq i \leq 3$ and $A_{i1} = \{v \in A : e_G(v, A - A_1) = i\}$ for $0 \leq i \leq 2$. (Define $B_i$ and $B_{i1}$ similarly.)

Suppose $x \in A_3$ and $y \in B_1 \cup B_2 \cup B_3$ with $xy \notin E(G)$; exchanging $x$ with $y$ shows $\{A, B\}$ is not an optimal partition, which is a contradiction. Consequently, $|B_1 \cup B_2 \cup B_3| \leq 3$ and $bw(G) \leq 9 \leq (n + 138)/3$. We assume $|A_3| = |B_3| = 0$.

Suppose $|B_2| \geq 4$. When $|A_2| \neq 0$, there exists $x \in A_2$ and $y \in B_2$ with $xy \notin E(G)$; exchanging $x$ with $y$ shows $\{A, B\}$ is not an optimal partition. Consequently, $|A_2| = 0$. When $G[A_{10} \cup A_{11}]$ is empty, we have $|A_{10} \cup A_{11}| \leq 1$. Then

$$3|A_0| \geq e_G(A_0, A_1) \geq 2|A_1| - 2$$

so that

$$n/2 = |A_0| + |A_1| \geq (5|A_1| - 2)/3$$

and

$$bw(G) \leq (3n + 4)/10 \leq (n + 138)/3.$$ 

When $G[A_{10} \cup A_{11}]$ is nonempty, there exist an edge $x_1x_2$ in $G[A_{10} \cup A_{11}]$ and $y_1, y_2 \in B_2$ with $e_G(\{x_1, x_2\}, \{y_1, y_2\}) = 0$; exchanging $\{x_1, x_2\}$ with $\{y_1, y_2\}$ shows $\{A, B\}$ is not an optimal partition. We assume $|A_2|, |B_2| \leq 3$.

Denote a path (cycle) of order $n$ by $P_n(C_n)$. Let

$$a = \max \{|E_1, \ldots, E_t|\}$$

where $\{E_1, \ldots, E_t\}$ is a set of vertex-disjoint subgraphs of $G[A]$ and each $E_i \cong P_3 \subseteq G[A_{10} \cup A_{11}]$ or

$n \leq 3 |G[A_0 \cup A_{11}]|$ with precisely one vertex in $A_0$ or

$n \leq 4 |G[A_0 \cup A_{10} \cup A_{11}]|$ with precisely one vertex in $A_0$ and precisely one vertex in $A_{10}$ or

$n \leq 5 |G[A_0 \cup A_{10} \cup A_{11}]|$ with only the centre vertex in $A_0$ and
let \( A_j^* = \bigcup \{ V(E_i) \cap A_j^* : 1 \leq i \leq a \} \) for \( 0 \leq j \leq 1 \). (Define \( b, \{ F_1, \ldots, F_t \} \), \( B_j^* \) for \( 0 \leq j \leq 1 \) similarly.)

**Claim.** \( \min\{a, b\} \leq 5 \).

Suppose \( a, b \geq 6 \). Choose \( e_G(E_i, F_j) = 0 \) with \( |E_i| = |F_j| \) as large as possible, say \( |E_i| \geq |F_j| \). When \( |E_i| = |F_j| \); exchanging \( E_i \) with \( F_j \) shows \( \{ A, B \} \) is not an optimal partition. When \( |E_i| = |F_j| + 1 \); exchanging \( E_i \) with \( F_j \), where \( E_i' \) is the subgraph of \( E_i \) contained in \( G[A_1] \), shows \( \{ A, B \} \) is not an optimal partition. When \( |E_i| = |F_j| + 2 \) then \( |E_i| = 5 \) and \( |F_j| = 3 \). Since \( b \geq 6 \), there exist \( F_k \neq F_j \) with \( e_G(E_i, F_k) = 0 \). By the above, \( |F_k| = 3 \); exchanging \( E_i \) with \( F_j \cup F_k' \), where \( F_k' \) is a subpath of order 2 contained in \( G[B_1] \), shows \( \{ A, B \} \) is not an optimal partition. \[ \]

We assume \( a \leq 5 \) so that \( |A_0^*| \leq 5 \) and \( |A_j^*| \leq 20 \).

**Claim.** \( |A_{10}| \leq 25 \).

Note that \( G[A_{10} \cup A_{11}] \) is a vertex-disjoint set of paths and cycles when \( |A_{10} \cup A_{11}| \neq 0 \), since \( \delta(G[A_{10} \cup A_{11}]) = 1 \) and \( \Delta(G[A_{10} \cup A_{11}]) = 2 \). Consequently, \( |A_{10}| \leq 25 \) since \( a \leq 5 \) (after breaking paths and cycles apart if necessary).

Let \( A_1' = \{ w \in A_1 - A_1^* : vw \in E(G) \text{ for some } v \in A_1^* \} \). Clearly, \( |A_1'| \leq 2 \cdot 5 = 10 \).

Set \( |A_{12}| = c|A_1| \) where \( c \in [0, 1] \).

Then

\[
|A_{11}| + |A_{12}| \geq |A_1| - 25
\]

so that

\[
|A_{11}| \geq (1 - c)|A_1| - 25.
\]

Now

\[
3|A_0| \geq e_G(A_0, A_1) \geq |A_{11}| + 2|A_{12}| - 3
\]

so that

\[
|A_0| \geq [(1 + c)|A_1| - 28]/3.
\]

Then

\[
n/2 \geq |A_0| + |A_1| \geq [(4 + c)|A_1| - 28]/3
\]

so that

\[
|A_1| \leq (3n + 56)/2(4 + c)
\]

and

\[
bw(G) \leq 6 + |A_1| \leq 6 + (3n + 56)/2(4 + c).
\]

Also

\[
|A_{11}| - |A_j^*| - |A_j'| - 5 \leq |A_{11} - (A_j^* \cup A_j')| - 5
\]

\[
\leq |A_0 - A_j^*| = |A_0| - |A_j^*|,
\]
by the maximality of $a$, so that
\[ |A_0| \geq |A_1| - 35 \geq (1 - c)|A_1| - 60. \]
Then
\[ n/2 \geq |A_0| + |A_1| \geq (2 - c)|A_1| - 60 \]
so that
\[ |A_1| \leq (n + 120)/2(2 - c) \]
and
\[ bw(G) \leq 6 + |A_1| \leq 6 + (n + 120)/2(2 - c). \]
Consequently,
\[ bw(G) \leq \min\{6 + (3n + 56)/2(4 + c), 6 + (n + 120)/2(2 - c)\} \]
\[ \leq (n + 138)/3, \]
since the above minimum is at most $(n + 138)/3$ for $n \geq 184$ and at most $6 + (3n + 56)/8 \leq (n + 138)/3$ for $n \leq 182$.

**Remark.** In general, if $\{A, B\}$ is a partition of the vertices of an $r$-regular graph $G$ of order $n$ with $bw(G) = e_G(A, B)$, one would hope that either $G[A]$ or $G[B]$ contains a small number of forbidden subgraphs (see definition of $a, b$ in Theorem 1) which, in turn, impose structure on $G[A]$ or $G[B]$ and give $bw(G) \leq c_r n + O(1)$ for some $c_r < r/4$. At present we have only the result of Goldberg and Gardner [13] that, for any such graph $G$, $bw(G) \leq r(n + \varepsilon n)/4$ where $\varepsilon_n = 1$ for odd $n$ and $\varepsilon_n = n/(n - 1)$ for even $n$. There are, however, limitations on how small the ratio $bw(G)/n$ can be made for $r$-regular graphs $G$ of order $n$.

An $r$-regular graph $G$ of order $n$ is an $(n, r, c)$-expander if $|N(X) - X| \geq c|X|$ for all $X \subseteq V(G)$ with $|X| \leq n/2$. (These and similar graphs have an extensive literature; see the references in [1].) Clearly, any $(n, r, c)$-expander $G$ has $bw(G) \geq c[n/2]$.

Let $\lambda_1(G)$ denote the second largest eigenvalue of the adjacency matrix of $G$ in absolute value. Note that $0 < \lambda_1(G) < r$ when $G$ is connected. Alon and Milman [3] have shown that any $r$-regular graph $G$ of order $n$ is an $(n, r, (r - \lambda_1(G))/2r)$-expander while Alon and Boppana [2] (see also [17]) have shown that $\lim_{n \to \infty} \lambda_1(G_n) \geq 2\sqrt{r - 1}$ for any sequence $\{G_n\}$ of such graphs. Lubotzky, Phillips and Sarnak [17] have shown this last result asymptotically correct by constructing infinite families of $r$-regular graphs $G$ with $\lambda_1(G) \leq 2\sqrt{r - 1}$ for all primes $r \equiv 1(\mod 4)$.

The above results imply that any $r$-regular graph $G$ of large order $n$ has $bw(G) \geq cn$ where $c$, unfortunately, is rather small. We improve this by showing that almost every $r$-regular graph $G$ of order $n$ has $bw(G) \geq c_r n$ where $c_r \to r/4$ as $r \to \infty$. 

Downloaded from https://www.cambridge.org/core. IP address: 54.70.40.11, on 26 Apr 2018 at 17:37:54, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0004972700003300
3. A LOWER BOUND FOR THE BISECTION WIDTH OF ALMOST EVERY CUBIC GRAPH

Bender and Canfield [5] gave the first formula for the asymptotic number of labelled \( r \)-regular graphs of order \( n \). Bollobás [8] gave a simpler proof of the same formula that, more importantly, contained a model for the set of regular graphs which can be used to study labelled random regular graphs. We describe now this model.

Let \( rn \) be even and \( q = rn/2 \). Let \( V = V_1 \cup \ldots \cup V_n \) be a disjoint union of \( rn \) labelled vertices where \( |V_i| = r \) for \( 1 \leq i \leq n \). A configuration is a 1-regular graph with vertex set \( V \). Denote the set of configurations by \( \Phi = \Phi(n, r) \).

\[ |\Phi| = (rn)!/2^q q! . \]

A configuration is good if when we shrink each set \( V_i \) to a vertex \( v_i \) we obtain a simple graph. Denote the set of good configurations by \( \Omega = \Omega(n, r) \) and the set of simple \( r \)-regular graphs with vertex set \( \{v_1, \ldots, v_n\} \) by \( \mathcal{G}^{(r)}_n \).

\[ |\Omega| = (r!)^n |\mathcal{G}^{(r)}_n| . \]

Now regard \( \Phi \) as a probability space where \( P(F) = |\Phi|^{-1} \) for any configuration \( F \). Bollobás [8] showed that

\[ P(\text{configuration } F \text{ is good}) \to e^{(1-r^2)/4} \quad (n \to \infty) \]

and, hence,

\[ |\mathcal{G}^{(r)}_n| \sim e^{(1-r^2)/4} |\Phi|/(r!)^n \quad (n \to \infty) . \]

Finally regard \( \mathcal{G}^{(r)}_n \) as a probability space where \( P(G) = |\mathcal{G}^{(r)}_n|^{-1} \) for any \( r \)-regular graph \( G \) with vertex set \( \{v_1, \ldots, v_n\} \). An immediate consequence of the preceding is that if the probability that a configuration has a certain property tends to 1 as \( n \to \infty \) then the probability that an \( r \)-regular graph has the corresponding property also tends to 1 as \( n \to \infty \).

For \( r \geq 3 \), let \( c = c_r \) be the unique real number in \((0, r/4)\) with \( 2(2-r)r^r = (2c)^2c(r-2c)^{(r-2c)} \). The constant exists since \( x^x(r-x)^{r-x} \) monotonically decreases on \([0, r/2]\). Note that \( c_3 = 0.0922357 \ldots \in (1/11, 1/10) \). We denote \( t(t-1)\ldots(t-k+1) \) by \( (t)_k \).

We give now

**Theorem 2.** Almost every cubic graph \( G \) of order \( n \) has \( bw(G) \geq n/11 \).

**Proof:** Let \( n = 2m \). Fix a partition \( \{A, B\} \) of \( \{1, \ldots, n\} \) with \( |A| = |B| = m \). Let \( V_A = \bigcup\{V_i : i \in A\} \) (Define \( V_B \) similarly). Note that the event \( e_F(V_A, V_B) = j \)
is a nonempty subset of $\Phi$ if and only if $3m$ and $j$ have the same parity. Put $p_j = (3m - j)/2$. Then

$$P(e_F(V_A, V_B) = j) = \frac{(3m)_j^2}{j!} \left[ \frac{(3m - j)!}{2^j p_j!} \right]^2 |\Phi|^{-1}$$

$$= \frac{[(3m)!]^2 (2j)!}{j! p_j! (2^{3n})!},$$

where the left factor of (1) is the number of ways of labelling the ends of the $j$ edges between $V_A$ and $V_B$ and the middle factor of (1) is the number of ways of completing the 1-factor in both $V_A$ and $V_B$.

For $j \geq 2$, we have

$$P(e_F(V_A, V_B) = j - 2) = \frac{j(j-1)}{(3m-j+2)^2} P(e_F(V_A, V_B) = j)$$

where $j(j-1)/(3m-j+2)^2$ increases with $j$. For even $3m$ and $2k \leq [c_3 n]$, we have

$$P(e_F(V_A, V_B) \leq 2k) = \sum_{\text{even } j \leq 2k} P(e_F(V_A, V_B) = j) \leq P(e_F(V_A, V_B) = 2k)(1 + \alpha + \cdots + \alpha^k),$$

where $\alpha = 2k(2k-1)/(3m-2k+2)^2$. Since $\alpha \leq (2c_3/3 - 2c_3)^2 \leq 1/2$, we have

$$P(e_F(V_A, V_B) \leq 2k) \leq 2P(e_F(V_A, V_B) = 2k).$$

Then

$$P(bw(F) \leq 2k) = P(e_F(V_A, V_B) \leq 2k \text{ for some } \{A, B\}) \leq \sum_{\{A, B\}} P(e_F(V_A, V_B) \leq 2k) \leq \binom{n}{m} \frac{[(3m)!]^2 (2k)! p_{2k}! (2^{3n})!}{(2k)! [p_{2k}!]^2 (3n)!}.$$

From $\binom{n}{m} = O(2^m m^{-1/2})$ and Stirling’s Formula, we obtain

$$P(bw(F) \leq 2k) = O\left( \frac{2^{-m} 3^m m^{3m+1/2}}{(2k)^{2k+1/2} (3m-2k)^{3m-2k+1}} \right).$$

Now write $2k = 2cm \leq [c_3 n]$ and we have

$$P(bw(F) \leq cn) = O\left( \frac{1}{n} \right).$$
For odd $3m$, a similar calculation with $2k$ replaced by $2k + 1$ gives the same result. Then
\[ P\left( bw\left( F\right) \leq \left\lfloor c_3n \right\rfloor \right) \to 0 \quad (n \to \infty) \]
and, consequently,
\[ P\left( bw\left( G \in G_n^{(3)} \right) \geq c_3n \right) \to 1 \quad (n \to \infty). \]

**Remark.** In general, a similar calculation shows that
\[ P\left( bw\left( G \in G_n^{(r)} \right) \geq c_r n \right) \to 1 \quad (n \to \infty). \]

Since $(2d)^2d(1-2d)^{1-2d}$ monotonically decreases to $1/2$ on $[0,1/4]$, we have
\[ (2d)^2d(1-2d)^{1-2d} \geq 2^2/r/2 \]
for fixed $d \in (0,1/4)$ and all sufficiently large $r$. Consequently, $c_r \geq rd$ so that $c_r \to r/4$ as $r \to \infty$. We summarize this now.

**Theorem 3.** Almost every $r$-regular graph $G$ of order $n$ has $bw(G) \geq c_r n$. Moreover, $c_r \to r/4$ as $r \to \infty$.

In view of the upper bound for the bisection width given by Goldberg and Gardner [13], this last result is asymptotically correct.

**References**


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