Can. J. Math. Vol. **46** (3), 1994 pp. 574–585

ISOMETRIC CHARACTERIZATIONS OF ℓ_p^n SPACES

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ABSTRACT. The paper establishes some characterizations of ℓ_p^n spaces in terms of p-summing or p-nuclear norms of the identity operator on the given space E.

In particular, for an n-dimensional Banach space E and $1 \leq p < 2$, E is isometric to ℓ_p^n if and only if $\pi_p(E^*) \geq n^{1/p}$ and E^* has cotype p' with the constant one.

Furthermore, ℓ_p^n spaces are characterized by inequalities for p-summing norms of operators related to the John's ellipsoid of maximal volume contained in the unit ball of E.

Introduction. In this paper we establish characterizations of spaces ℓ_p^n in terms of ideal norms of certain natural operators related to an n-dimensional Banach space E. Some characterizations are given by conditions on p-summing and p'-nuclear norms of the identity operator on E, combined with the assumption on the cotype of the space. Other involve operators related to the John's ellipsoid of maximal volume contained in the unit ball of E. These characterizations generalize several known results for ℓ_∞^n and ℓ_1^n ([2], [3], [4], [7]). We also study similar problems also in the more concrete setting of subspaces of L_p-spaces.

Let us describe the content of the paper in more detail. Sections 1 and 2 contain notations and preliminaries on p-summing norms. In particular we observe, in Proposition 2.1, an upper estimate of the p-summing norm of an operator by the p-th moment of a related vector valued Gaussian random variable. This estimate appears several times in further arguments.

In Section 3 we prove that if $1 \leq p < 2$ and E is an n-dimensional Banach space such that $\pi_p(\mathrm{id}\colon E \longrightarrow E) \geq n^{1/p}$, then there exist $e_1, \ldots, e_n$ in E such that for every sequence of scalars $a_1, \ldots, a_n$ one has

$$\max_{i=1,\ldots,n} |a_i| \leq \Big\| \sum_{i=1}^n a_i e_i \Big\| \leq \Big(\sum_{i=1}^n |a_i|^{p'} \Big)^{1/p'}.$$

(Here $1/p + 1/p' = 1$.) It turns out that the vectors $e_1, \ldots, e_n$ are the contact points of the unit ball B_E of E with the John's ellipsoid of maximal volume contained in B_E.

Section 4 is devoted to study of p-summing norms of operators related to the ellipsoid of maximal volume. It is shown that some inequalities for these norms characterize ℓ_p^n. Finally in Section 5 we present some consequences of our results for subspaces of L_p. We also prove that if $2 < p < \infty$, then an n-dimensional subspace of L_p with the maximal Euclidean distance is isometric to ℓ_p^n. This complements a result obtained in [1] for $1 < p < 2$.

Received by the editors February 11, 1991.
AMS subject classification: 46B04, 46B15.

The paper is a part of the author's Ph. D. thesis being prepared under the supervision of Dr. N. Tomczak-Jaegermann. The author wishes to thank her for her helpful advice.

1. **Notation.** Let $(E, \|\cdot\|)$ be a finite dimensional Banach space over either $\mathbb{R}$ or $\mathbb{C}$ and let $\|\cdot\|_2$ denote the Euclidean norm on E induced by the ellipsoid of maximal volume contained in the unit ball of E. Let $\langle\cdot,\cdot\rangle$ denote the induced inner product, let $\|\cdot\|_*$ be the norm on E, dual to the original norm $\|\cdot\|$, and let $i_{2E}: (E, \|\cdot\|_2) \longrightarrow (E, \|\cdot\|)$ and $i_{E2} = (i_{2E})^{-1}$ be the formal identity operators.

Let $1 \leq p < \infty$ and let X and Y be Banach spaces. For an operator $S: X \longrightarrow Y$ set $\pi_p(S) = \inf c$ where the infimum is taken over all constants c such that

$$(1.1) \qquad \Big(\sum_j \|Sx_i\|^p\Big)^{1/p} \leq c \sup_{\substack{x^* \in X^* \\ \|x^*\| \leq 1}} \Big(\sum_j |\langle x_j, x^*\rangle|^p\Big)^{1/p}$$

for all finite sequences (x_j) in X; if no such c exists then $\pi_p(S) = \infty$. If $\pi_p(S) < \infty$ then S is said to be *p-summing* and $\pi_p(S)$ is called the *p-summing norm* of S.

For a real valued random variable ξ on a probability space (Ω, P) we denote by $\mathbb{E}\xi$ the expected value of ξ.

Finally, let $\gamma_1, \ldots, \gamma_n$ denote real or complex Gaussian random variables on (Ω, P). For $s \geq 1$ set $A_s = (\mathbb{E}|\gamma_1|^s)^{1/s}$. For any orthonormal basis (e_i) in ℓ_2^n, let $\mathbb{X}$ denote the ℓ_2^n-valued random variable defined by

$$(1.2) \qquad \mathbb{X} = \sum_{i=1}^n \gamma_i e_i.$$

Notice that the distribution of $\mathbb{X}$ does not depend on a choice of the basis (e_i).

2. **Preliminaries on p-summing norms.** We start by stating a simple observation which will be often used throughout the paper. It follows direction from the definition (1.1) of the p-summing norms (*cf. e.g.*, [9]).

PROPOSITION 2.1. *Let $1 \leq p < \infty$ and let T be an operator between two Banach spaces X and Y.*

(i) Suppose that there are functionals $x_1^*, x_2^*, \ldots, \in X$ such that

$$\|Tx\|^p \leq \sum_j |\langle x_j^*, x\rangle|^p \quad \textit{for all } x \in X.$$

Then $\pi_p(T) \leq (\sum_j \|x_j^*\|^p)^{1/p}$.

(ii) Let ξ be a random variable on a probability space (Ω, P) with values in $\big(X^*, \sigma(x^*, x)\big)$ and suppose that $\|Tx\|^p \leq \mathbb{E}|\langle \xi, x\rangle|^p$ for all $x \in X$. Then $\pi_p(T) \leq (\mathbb{E}\|\xi\|^p)^{1/p}$.

Recall that $\mathbb{X}$ is the ℓ_2^n-valued random variable defined in (1.2). It is easy to calculate that

$$(2.1) \qquad \|x\|_2 = A_s^{-1}(\mathbb{E}|\langle \mathbb{X}, x\rangle|^s)^{1/s} \quad \text{for } s \geq 1.$$

Now, let us give some simple conclusions from Proposition 2.1 which we will need further.

PROPOSITION 2.2. *The following equalities are true.*
(i) $\pi_p(\mathrm{id}\colon \ell_2^n \longrightarrow \ell_2^n) = A_p^{-1}(\mathbb{E}\|\mathbb{X}\|_2^p)^{1/p}$,
(ii) $\pi_p(\mathrm{id}\colon \ell_{p'}^n \longrightarrow \ell_p^n) = n^{1/p}$ *for* $1 \leq p \leq 2$,
(iii) $\pi_p(\mathrm{id}\colon \ell_2^n \longrightarrow \ell_p^n) = n^{1/p}$ *for* $1 \leq p \leq \infty$,
(iv) $\pi_{p'}(\mathrm{id}\colon \ell_p^n \longrightarrow \ell_2^n) = n^{1/p'}$ *for* $1 \leq p \leq 2$,
(v) $\pi_p(\mathrm{id}\colon \ell_{p'}^n \longrightarrow \ell_\infty^n) = n^{1/p}$ *for* $1 \leq p \leq \infty$.

Equality (i) was proved in a slightly different formulation by D. J. H. Garling [5] (*cf.* also, [9], p. 60).

Other equalities are well-known to specialists. For sake of the completeness we give a sketch of the proof.

PROOF. (i) The upper estimate follows from (2.1) and Proposition 2.1(ii). For the lower estimate observe that

$$(\mathbb{E}\|\mathbb{X}\|_2^p)^{1/p} = A_p^{-1}(\mathbb{E}\|\mathbb{X}\|_2^p)^{1/p} \cdot \sup_{\|x^*\|_2=1} (\mathbb{E}|\langle x^*, \mathbb{X}\rangle|^p)^{1/p}.$$

(iv) For $x \in \ell_p^n$ one has

$$\|\mathrm{id}(x)\|_2 = \Big(\underbrace{\sum_{\bar{\epsilon}_i=(\pm 1,\ldots,\pm 1)}}_{n\ \text{times}} \frac{1}{2^n}|\langle x, \bar{\epsilon}_i\rangle|^2\Big)^{1/2}$$
$$\leq \Big(\sum_{\bar{\epsilon}_i=(\pm 1,\ldots,\pm 1)} \frac{1}{2^n}|\langle x, \bar{\epsilon}_i\rangle|^{p'}\Big)^{1/p'}.$$

Again, by Proposition 2.1(i) we obtain

$$\pi_{p'}(\mathrm{id}\colon \ell_p \longrightarrow \ell_2) \leq \Big(\frac{1}{2^n}\sum_{\bar{\epsilon}_i=(\pm 1,\ldots,\pm 1)} \|\bar{\epsilon}_i\|_{p'}^{p'}\Big)^{1/p'} = n^{1/p'}.$$

Conversely,

$$\Big(\sum_{i=1}^n \|e_i\|_2^{p'}\Big)^{1/p'} = n^{1/p'} \sup_{\|y\|_{p'}=1} \Big(\sum_{i=1}^n |\langle y, e_i\rangle|^{p'}\Big)^{1/p'}.$$

We omit the proof of (ii), (iii) and (v). ■

As an interesting consequence we get an isometric characterization of ℓ_2^n as follows.

COROLLARY 2.3. *Let* $1 \leq p < \infty$. *An n-dimensional Banach space E is isometric to* ℓ_2^n *if and only if*

$$\pi_p\big((i_{2E})^*\big) = \pi_p(\mathrm{id}\colon \ell_2^n \longrightarrow \ell_2^n).$$

PROOF. By (2.1) and Proposition 2.1(ii) we obtain

$$\pi_p\big((i_{2E})^*\big) \leq A_p^{-1}(\mathbb{E}\|\mathbb{X}\|^p)^{1/p}.$$

Since $\|x\| \leq \|x\|_2$ for every $x \in E$, by Proposition 2.2(i) we get

$$A_p^{-1}(\mathbb{E}\|\mathbb{X}\|^p)^{1/p} \leq A_p^{-1}(E\|\mathbb{X}\|_2^p)^{1/p}$$
$$= \pi_p(\mathrm{id}\colon \ell_2^n \longrightarrow \ell_2^n) = \pi_p\big((i_{2E})^*\big).$$

Combining the two estimates we have $\|\mathbb{X}(\omega)\| = \|\mathbb{X}(\omega)\|_2$ almost everywhere. Hence, by the continuity, $\|\mathbb{X}(\omega)\| = \|\mathbb{X}(\omega)\|_2$ for every $\omega \in \Omega$ completing the proof. ∎

REMARK. For an n-dimensional Banach space E one has $\pi_2\big((i_{2E})^*\big) \leq \sqrt{n}\|(i_{2E})^*\| = \sqrt{n}$. Corollary 2.3 says in particular that if the 2-summing norm of the operator $(i_{2E})^*$ is maximal, then E is isometric to ℓ_2^n.

3. **Characterizations of ℓ_p^n in terms of ideal norms of the identity operator.** In this section we present characterizations of ℓ_p^n in terms of p'-summing and p-nuclear norms of the identity operator on the space.

We refer the reader to [9] for the standard definition of the p'-nuclear norm.

The definition of type p and cotype q constants, T_p' and C_q', respectively, used here, differ from the usual ones by replacing the L_2-Rademacher averages by the L_p- and L_q-averages respectively (*cf. e.g.* [9] p. 14). The main result of the section states:

THEOREM 3.1. *Let E be an n-dimensional Banach space. Let $1 \leq p < 2$. The following are equivalent:*

(i) $\pi_p(E) \geq n^{1/p}$,

(ii) There exist vectors $e_1, \ldots, e_n \in E$ such that for every choice of scalars $a_1, \ldots, a_n$ one has

$$\max_{i=1,\ldots,n} |a_i| \leq \Big\|\sum_{i=1}^n a_i e_i\Big\| \leq \Big(\sum_{i=1}^n |a_i|^{p'}\Big)^{1/p'},$$

(iii) $\nu_{p'}(E) \leq n^{1/p'}$.

Furthermore, E is isometric to $\ell_{p'}^n$ if and only if E satisfies one of the above conditions, and $C_{p'}'(E) = 1$.

For $p = 1$, implication (i) $\Rightarrow$ (ii) was proved in [2] and [4]; implication (iii) $\Rightarrow$ (ii) is the isometric version of a classical P_λ problem, proved by Nachbin [7].

The proof of the theorem is based on several results of independent interest. Proposition 3.2 below is crucial for further investigation. It involves the operator i_{E2} associated to the ellipsoid of maximal volume. The case $p = 1$ was proved in [3] (*cf.* also, [9], p. 266).

PROPOSITION 3.2. *Let $1 \leq p < 2$ and E be an n-dimensional Banach space such that*

$$\pi_p(i_{E2}) \geq n^{1/p}.$$

Then there exists an orthonormal basis $(e_j)_{j=1}^n$ in $(E, \|\cdot\|_2)$ such that $\|e_j\| = \|e_j\|_* = \|e_j\|_2 = 1$ for $j = 1, \ldots, n$.

We only give a sketch of the proof of Proposition 3.2 since it is similar to the one in the case $p = 1$.

PROOF OF PROPOSITION 3.2. By the well-known John's result (*cf.*, *e.g.*, [9], p. 118), there exist a positive integer N, vectors $x_1, \ldots, n_N$ in E and positive scalars $c_1, \ldots, c_N$ such that $\|x_j\| = \|x_j\|_* = 1$ ($j = 1, \ldots, N$), $\sum_{j=1}^N c_j = n$ and $x = \sum_{i=1}^N c_i \langle x, x_j \rangle x_j$ for $x \in E$.

We need the following lemma.

LEMMA 3.3. *Assume that $x_1, \ldots, x_N$ and $c_1, \ldots, c_N$ are as above. Let $M \subset \{1, \ldots, N\}$ be a subset such that $\sum_{j \in M} c_j = m$, for some positive integer, and that*

$$\langle x_s, x_j \rangle = 0 \quad \textit{for } x \notin M,\ j \in M.$$

Let $F_M = \operatorname{span}(x_j)_{j \in M}$ and let $P\colon (E, \|\cdot\|_2) \longrightarrow (E, \|\cdot\|_2)$ be the orthogonal projection onto F_M. If $\pi_p(Pi_{E2}) \geq m^{1/p}$, then there is a subset $J \subset M$ with $|J| = m$ such that $\langle x_i, x_j \rangle = 0$ for $i \neq j$, $i, j \in J$.

Obviously, Proposition 3.2 follows from Lemma 3.3 applied for $M = \{1, \ldots, N\}$ and $m = n$.

PROOF OF LEMMA 3.3. Proceeding by induction, assume that $m > 1$ and that the lemma is true for $m - 1$. Pick a vector $y \in F_M$ such that $a = \sum_{j \in M} c_j |\langle y, x_j \rangle|^2$ is maximal subject to $\sum_{j \in M} c_j |\langle y, x_j \rangle|^p = 1$.

Since $\|y\|_2^2 \leq \sum_{j=1}^N c_j |\langle y, x_j \rangle|^p \|y\|_2^{2-p}$ we get $a \leq 1$. On the other hand, for every $x \in E$, $\|Px\|_2 \leq a^{1/2} (\sum_{j \in M} c_j |\langle x, x_j \rangle|^p)^{1/p}$ which gives

$$m^{1/p} \leq \pi_p(Pi_{E2}) \leq a^{1/2} \Big(\sum_{j \in M} c_j \Big)^{1/p} = a^{1/2} m^{1/p} \quad \text{and} \quad a = 1.$$

Next, since $(\sum_{j \in M} c_j |\langle y, x_j \rangle|^2)^{1/2} = (\sum_{j \in M} |\langle y, x_j \rangle|^p)^{1/p}$ and $|\langle y, x_j \rangle| \leq 1$ it follows that there exists a subset $K \subset M$ such that

$$|\langle y, x_j \rangle| = \begin{cases} 1 & \text{for } s \in K \\ 0 & \text{for } s \in (1, \ldots, N) \setminus K. \end{cases}$$

Let $k_0 \in K$. Then for every $k \in K$, $x_k = \epsilon_k x_{k_0}$ with $|\epsilon_k| = 1$ and therefore we may assume that $y = x_{k_0}$.

Put $M_1 = M \setminus K$. Then $\langle y, x_i \rangle = 0$ for $i \in M_1$. In addition, $\langle x_s, x_k \rangle = 0$ for $s \in M_1$, $k \notin M_1$ and $\sum_{i \in M_1} c_i = m - 1$.

Finally, if $Q\colon (E, \|\ \|_2) \longrightarrow (E, \|\ \|_2)$ is the orthogonal projection onto $F_{M_1} = \operatorname{span}(x_i)_{i \in M_1}$ then

$$\pi_p(Qi_{E2}) \geq (m-1)^{1/p}.$$

Indeed, for every $x \in E$ one has

$$\|(P - Q)x\|_2 \leq \Big(\sum_{j \in K} c_j |\langle x, x_j \rangle|^p \Big)^{1/p}$$

and

$$\pi_p(Pi_{E2}) \leq \Big(\pi_p\big((P - Q) i_{E2} \big)^p + \pi_p(Qi_{E2})^p \Big)^{1/p}.$$

https://doi.org/10.4153/CJM-1994-030-x Published online by Cambridge University Press

The last inequality can be checked using definition (1.1). By applying Proposition 2.1(i) we obtain the required inequality.

The inductive hypothesis applied to the subset M_1 and the projection Q yields that there is a subset $J_0 \subset M_1$ with $|J_0| = m - 1$ such that $\langle x_j, x_i \rangle = \delta_{ij}$, $i,j \in J_0$. Then $J_0 \cup \{k_0\}$ obviously satisfies the condition of Lemma 3.3. ∎

In order to prove the next proposition we require the following lemma.

LEMMA 3.4. *Let $1 \leq p \leq q \leq \infty$ and let $(E, \|\cdot\|)$ be a normed space. Then for every choice of vectors $x_1, \ldots, x_n \in X$ the following inequality holds*

$$\left(\sum_{i=1}^{n} \|x_i\|^q\right)^{1/q} \leq \left(\alpha \sum_{i=1}^{n} \|x_i\|^p + \beta \left\|\sum_{i=1}^{n} x_i\right\|^p\right)^{1/p}, \tag{3.1}$$

where $\alpha = 2^{p/q-1}$ and $\beta = 1 - \alpha$.

PROOF. The lemma is obvious for $n = 1$. Proceeding by induction, assume that the lemma is true for $n - 1$. Without loss of generality, we may assume that $1 < p < q < \infty$, $\sum_{i=2}^{n} \|x_i\|^q = 1$ and $0 < \|x_1\| \leq \cdots \leq \|x_n\|$. It is easy to see that to prove (3.1) it is enough to check the following stronger inequality:

$$\left(\sum_{i=1}^{n} \|x_i\|^q\right)^{1/q} \leq \left(\alpha \sum_{i=1}^{n} \|x_i\|^p + \beta \left| \|x_1\| - \left\|\sum_{i=2}^{n} x_i\right\| \right|^p\right)^{1/p}. \tag{3.2}$$

Next, let us introduce the following notation:

$$(w)^s = \operatorname{sign}(w) \cdot |w|^s \quad \text{for } s > 1,\ w \in \mathbb{R};$$
$$A = \sum_{i=2}^{n} \|x_i\|^p;$$
$$a = \left\|\sum_{i=2}^{n} x_i\right\|;$$
$$\|x_1\| = t \in [0, 1].$$

Observe that in the above terms the following formulas are true:

$$\frac{d}{dt}|w|^p = p(w)^{p-1} \quad \text{and} \quad w(w)^{p-1} = |w|^p.$$

Finally, we can rewrite the inequality (3.2) in the following way:

$$0 \leq f(t) = \alpha(t^p + A) + \beta|t - a|^p - (t^q + 1)^{p/q} \quad \text{where } t \in [0, 1]. \tag{3.3}$$

To prove (3.3) observe that $f(0) \geq 0$ (by inductive hypothesis) and

$$\begin{aligned} f(1) &= \alpha(1 + A) + \beta|1 - a|^p - (1 + 1)^{p/q} \geq 2\alpha - 2^{p/q} \\ &= 2 \cdot 2^{p/q-1} - 2^{p/q} = 0 \end{aligned}$$

(since $A \geq 1$).

Now, let us suppose the contrary. There exists $t \in (0,1)$ such that $f'(t) = 0$ and $f(t) < 0$. Then

$$\begin{aligned} 0 &= p^{-1}(t-a)\cdot f'(t) \\ &= (t-a)[\alpha t^{p-1} + \beta(t-a)^{p-1} - t^{q-1}(t^q+1)^{p/q-1}] \\ &= \alpha(t-a)t^{p-1} - \alpha(t^p + A) + f(t) + (t^q+1)^{p/q} - t^{q-1}(t-a)(t^q+1)^{p/q-1} \\ &< -\alpha(at^{p-1} + A) + (t^q+1)^{p/q-1}[1 + t^{q-1}a] \\ &\le (t^q+1)^{p/q-1}[-at^{p-1} - A + 1 + t^{q-1}a] \quad (\text{since } (t^q+1)^{p/q-1} < \alpha) \\ &\le (t^q+1)^{p/q-1}a(t^{q-1} - t^{p-1}). \end{aligned}$$

To summarize, $0 < (t^q+1)^{p/q-1}a(t^{q-1} - t^{p-1})$ which gives $t^{q-1} > t^{p-1}$ and $p > q$. This is contradictory to the assumption and completes the proof of the lemma. ■

PROPOSITION 3.5. *Let $(E, \|\cdot\|)$ be an n-dimensional Banach space and $1 \le p < q \le \infty$. Suppose that there exists vectors $e_1, \dots, e_n \in E$ and $e_1^*, \dots, e_n^* \in E^*$ such that $\langle e_j^*, e_i\rangle = \delta_{ij}$ and $\|e_i\| = \|e_i^*\| = 1$ for $i = 1, \dots, n$. Consider on E the ℓ_q^n norm, say $\|\cdot\|_q$, induced by the basis $(e_i)_{i=1}^n$. Let i_{Eq} denote the formal identity operator from $(E, \|\cdot\|)$ to $(E, \|\cdot\|_q)$.*

If $\pi_p(i_{Eq}) \ge n^{1/p}$, then for every $a_1, \dots, a_n \in \mathbb{C}$ one has

$$\Big(\sum_{i=1}^n |a_i|^p\Big)^{1/p} \le \Big\|\sum_{i=1}^n a_i e_i^*\Big\|_*.$$

PROOF. We will suppose that $q < \infty$. In the case $q = \infty$ the proof is similar. First, we will show that

$$\pi_p(Ti_{Eq}) = \Big(\sum_{i=1}^n |a_i|^p\Big)^{1/p} \text{ where } T = \sum_{i=1}^n a_i e_i^* \otimes e_i. \tag{3.4}$$

Fix $x \in E$. Then $\|Ti_{Eq}x\|_q \le (\sum_{i=1}^n |a_i|^p |\langle x, e_i^*\rangle|^p)^{1/p}$ and so,

$$\pi_p(Ti_{Eq}) \le \Big(\sum_{i=1}^n |a_i|^p\Big)^{1/p}. \tag{3.5}$$

To see opposite inequality, choose $g_i \in \mathbb{C}$ $(i = 1, \dots, n)$ such that

$$\max |a_i| = (|a_i|^p + |g_i|^p)^{1/p}.$$

Define an operator

$$S \colon E \longrightarrow E, \quad S = \sum_{i=1}^n g_i e_i^* \otimes e_i.$$

Then for every $x \in \ell_q^n$ one has

$$\max |a_i| \, \|x\|_q \le (\|Tx\|_q^p + \|Sx\|_q^p)^{1/p}. \tag{3.6}$$

https://doi.org/10.4153/CJM-1994-030-x Published online by Cambridge University Press

Next, using definition (1.1) and (3.6) we obtain

$$\max |a_i|\pi_p(i_{Eq}) \leq \left(\pi_p^p(Ti_{Eq}) + \pi_p^p(Si_{Eq})\right)^{1/p}.$$

Hence, from (3.5) and above it follows

$$\begin{aligned} n^{1/p} \max |a_i| &\leq \max |a_i|\pi_p(i_{Eq}) \leq [\pi_p^p(Ti_{Eq}) + \pi_p^p(Si_{Eq})]^{1/p} \\ &\leq \Big[\sum_{i=1}^n |a_i|^p + \sum_{i=1}^n |g_i|^p\Big]^{1/p} = n^{1/p} \max |a_i| \end{aligned}$$

and (3.4) holds as required.

Finally, using Lemma 3.4, one has

$$\begin{aligned} \|Tx\|_q &= \Big(\sum_{i=1}^n |\langle Tx, e_i^*\rangle|^q\Big)^{1/q} \\ &\leq \Big(\alpha \sum_{i=1}^n |\langle Tx, e_i^*\rangle|^p + \beta \Big|\sum_{i=1}^n \langle Tx, e_i^*\rangle\Big|^p\Big)^{1/p} \\ &= \Big(\alpha \sum_{i=1}^n |a_i|^p \langle x, e_i^*\rangle|^p + \beta \Big|\Big\langle x, \sum_{i=1}^n a_i e_i^*\Big\rangle\Big|^p\Big)^{1/p}. \end{aligned}$$

Hence, the condition (3.4) and Proposition 2.1(i) give

$$\Big(\alpha \sum_{i=1}^n |a_i|^p + \beta \Big\|\sum_{i=1}^n a_i e_i^*\Big\|_*^p\Big)^{1/p} \geq \pi_p(Ti_{Eq}) = \Big(\sum_{i=1}^n |a_i|^p\Big)^{1/p}$$

which completes the proof. ∎

Now we are able to prove Theorem 3.1.

PROOF OF THEOREM 3.1. The fact that (i) implies (ii) follows from Proposition 3.5 for $q = 2$, Proposition 3.2 and the inequality

$$\pi_p(i_{E2}) \geq \pi_p(E) \geq n^{1/p}.$$

Next, condition (ii) implies that the following factorization holds

$$E \xrightarrow[V_1]{} \ell_\infty^n \xrightarrow[\Delta]{} \ell_{p'}^n \xrightarrow[V_2]{} E$$

with $\nu_{p'}(E) \leq \|V_1\| \, \|\Delta\| \, \|V_2\| \leq n^{1/p}$. Finally, (iii) implies (i) since

$$n = \text{trace}(\text{id}: E \longrightarrow E) \leq \pi_p(E)\nu_{p'}(E).$$

Before we pass to the second part of the theorem, observe that

$$C'_{p'}(E) = 1 \quad \text{iff} \quad T'_p(E^*) = 1. \tag{3.7}$$

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This can be checked directly for two vectors, and by induction for more vectors. Suppose that $\pi_p(E) \geq n^{1/p}$; so, $\pi_p(i_{E2}) \geq n^{1/p}$. Using Proposition 2.1(ii) and (2.1) we obtain

$$\begin{aligned} n^{1/p} &\leq \pi_p(i_{E2}) \leq A_p^{-1}(\mathbb{E}\|\mathbb{X}\|_*^p)^{1/p} \\ &= A_p^{-1}\Big(\mathbb{E}\int_0^1 \Big\|\sum_{i=1}^n r_i(t)\gamma_i e_i\Big\|_*^p dt\Big)^{1/p} \\ &\leq A_p^{-1}\Big(\mathbb{E}\sum_{i=1}^n |\gamma_i|^p \|e_i\|_*^p\Big)^{1/p} = n^{1/p}. \end{aligned}$$

Therefore, $A_p^{-1}(\mathbb{E}\|\mathbb{X}\|_p^p)^{1/p} = n^{1/p} = A_p^{-1}(\mathbb{E}\|\mathbb{X}\|_*^p)^{1/p}$. Since $\|x\|_p \leq \|x\|_*$ for every $x \in E$ we conclude that $\|\cdot\|_p = \|\cdot\|_*$ as in the proof of Corollary 2.3. ■

4. **The ellipsoid of maximal volume and other characterizations of ℓ_p^n spaces.** In this section we give some characterizations of ℓ_p^n space in terms of p-summing norms of an operator associated with the ellipsoid of maximal volume contained in the unit ball of E.

Before we start, let us introduce some new notation. Let $i_{E\infty}\colon (E, \|\cdot\|) \longrightarrow (E, \|\cdot\|_\infty)$ denote the formal identity operator where the norm $\|\cdot\|_\infty$ is given by a fixed Auerbach system on E. Similarly, we define $i_{E^*\infty}\colon (E^*, \|\cdot\|_*) \longrightarrow (E, \|\cdot\|_\infty)$. Finally, let

$$i_{E^*2} = (i_{2E})^* \quad \text{and} \quad i_{2E^*} = (i_{E2})^*.$$

THEOREM 4.1. *Let E be an n-dimensional linear space. Then for $1 \leq p < 2$ the following are equivalent.*

(i) E^* is isometric to ℓ_p^n,

(ii) $\pi_p(i_{E2}) \geq n^{1/p}$ and $\pi_{p'}(i_{E^*2}) \geq n^{1/p'}$,

(iii) $\pi_p(i_{E2}) \geq n^{1/p}$ and $\pi_p(i_{2E^*}) \leq n^{1/p}$,

Moreover, for $1 \leq p < \infty$ condition (i) is equivalent to

(iv) $\pi_p(i_{E\infty}) \geq n^{1/p}$ and $\pi_{p'}(i_{E^*\infty}) \geq n^{1/p'}$.

PROOF. By Proposition 2.2 we see that the condition (i) implies (ii), (iii) and (iv). First, suppose that

$$\pi_p(i_{E2}) \geq n^{1/p}.$$

By Proposition 3.2 and Proposition 3.5, we conclude that

$$(4.1) \qquad \|x\| \leq \|x\|_{p'} \quad \text{for } x \in E.$$

Now, let us suppose that

$$\pi_{p'}(i_{E^*2}) \geq n^{1/p'}.$$

Applying (4.1) and Proposition 2.1(ii) to (2.1) for $s = p'$ it follows that

$$\begin{aligned} n^{1/p'} &\leq \pi_{p'}(i_{E^*2}) \leq A_{p'}^{-1}(\mathbb{E}\|\mathbb{X}\|^{p'})^{1/p'} \\ &\leq A_{p'}^{-1}(\mathbb{E}\|\mathbb{X}\|_{p'}^{p'})^{1/p'} = n^{1/p'}. \end{aligned}$$

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Hence, $\|\cdot\|_{p'} = \|\cdot\|$. Next, let us suppose that (iii) holds. Again, by (2.1) for $s = p$, we obtain

$$n^{1/p} \le \pi_p(i_{E2}) \le A_p^{-1}(\mathbb{E}\|\mathbb{X}\|_*^p)^{1/p}. \tag{4.2}$$

Using the Pietsch Factorization Theorem [8] (*cf.* also, [9], p. 47) one can find a probability measure μ on $S_2^{n-1} = \{x : \|x\|_2 = 1\}$ such that

$$\|x^*\|_* \le n^{1/p}\Big(\int_{S_2^{n-1}} |\langle y, x^*\rangle|^p \, d\mu(y)\Big)^{1/p} \quad \text{for } x^* \in E^*.$$

By (4.2) and the above inequality one has

$$n^{1/p} \le A_p^{-1}(\mathbb{E}\|\mathbb{X}\|_*^p)^{1/p} \le n^{1/p} A_p^{-1}\Big(\mathbb{E}\int_{S_2^{n-1}} |\langle y, \mathbb{X}\rangle|^p \, d\mu(y)\Big)^{1/p} = n^{1/p}.$$

Therefore, $\mathbb{E}\|\mathbb{X}\|_*^p = \mathbb{E}\|\mathbb{X}\|_p^p$ and $\|\cdot\|_* = \|\cdot\|_p$, as before.

Finally, using Proposition 3.5 for $q = \infty$ we conclude from (iv) that

$$\|x^*\|_p \le \|x^*\|_* \quad \text{for } x^* \in E^*$$

and

$$\|x\|_{p'} \le \|x\| \quad \text{for } x \in E.$$

This implies that E is isometric to $\ell_{p'}$ completing the proof. ∎

5. **Finite dimensional subspaces of L_p.** In the last section of the paper we apply Theorem 3.1 to subspaces of L_p. We also get a characterization of n-dimensional subspaces of L_p with the maximal Euclidean distance.

COROLLARY 5.1. *Let E be an n-dimensional subspace of $L_p(\Omega, \mu)$. Then E is isometric to ℓ_p^n if and only if $\pi_{p'}(E) \ge n^{1/p'}$ for $2 < p < \infty$ or $\pi_p(E^*) \ge n^{1/p}$ for $1 \le p < 2$.*

The corollary follows immediately from Theorem 3.1 and the fact that $T_p'\big(L_p(\Omega, \mu)\big) = 1$.

PROPOSITION 5.2. *Fix n and $2 < p < \infty$. Then any n-dimensional subspace E of $L_p(\Omega, \mu)$ whose Euclidean distance is maximal, i.e., $d(E, \ell_2^n) = n^{1/2-1/p}$, is isometric to ℓ_p^n.*

For $1 < p < 2$, an analogous result was proved in [1].

The proof of Proposition 5.2 is based on well-known result of D.R. Lewis [6] which states:

PROPOSITION 5.3. *Fix n and $1 < p < \infty$. Then for any n-dimensional subspace E of $L_p(\Omega, \mu)$ there exists $f_1, \dots, f_n \in E$ such that*

$$\int f_i \bar{f_j} F^{p-2} \, d\mu = \delta_{ij}, \quad \textit{where } F = \Big(\sum_{i=1}^n |f_i|^2\Big)^{1/2}. \tag{5.1}$$

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PROOF OF PROPOSITION 5.2. Fix an arbitrary n-dimensional subspace of E of $L_p(\Omega,\mu)$. Denote $d(E,\ell_2^n)$ by d_E. First we follow Lewis' argument from [6]. Observe that (5.1) implies

$$\int\Big|\sum_{i=1}^n a_if_i\Big|^2 F^{p-2}\,d\mu=\sum_{i=1}^n |a_i|^2, \tag{5.2}$$

$$\|F\|_p=n^{1/p}. \tag{5.3}$$

Define an operator $T\colon E\longrightarrow L_2(\Omega,\mu)$ by $Tf=fF^{\frac{p-2}{2}}$, for $f\in E$. Using Hölder's inequality it is easy to see that

$$\|Tf\|_2^p\le\|f\|_p^2\|F\|_p^{p-2}.$$

Thus by (5.3), $\|T\|\le n^{1/2-1/p}$.

On the other hand, by (5.2) and Cauchy-Schwarz inequality we get, for every $h=\sum_{i=1}^n a_if_i\in E$

$$\begin{aligned}\|h\|_p^p&=\int\Big|\sum_{i=1}^n a_if_i\Big|^2\cdot\Big|\sum_{i=1}^n a_if_i\Big|^{p-2}d\mu\\ &\le\Big(\sum_{i=1}^n|a_i|^2\Big)^{\frac{p-2}{2}}\int\Big|\sum_{i=1}^n a_if_i\Big|^2F^{p-2}\,d\mu\\ &=\Big(\sum_{i=1}^n|a_i|^2\Big)^{p/2}=\Big[\int\Big|\sum_{i=1}^n a_if_i\Big|^2F^{p-2}\,d\mu\Big]^{p/2}=\|Th\|_2^p.\end{aligned} \tag{5.4}$$

Thus $\|T^{-1}\|\le 1$, and so,

$$d_E\le n^{1/2-1/p}. \tag{5.5}$$

Now, we proceed by induction in n. Assume that the proposition is valid for $(n-1)$-dimensional subspaces.

Let $E\subset L_p(\Omega,\mu)$, $\dim E=n$, $d_E=n^{1/2-1/p}$. Then $\|T^{-1}\|=1$. Fix $h\in E$ such that $\|h\|_p=\|Th\|_2=1$ and $h=\sum_{i=1}^n a_if_i$ for some scalars $a_i,\ldots,a_n$ where $f_1,\ldots,f_n$ are as in Proposition 5.4. Since all the inequalities in (5.5) become equalities, it follows that $|h|=F$ a.e. in the support A of h.

Moreover, there exists a functional ϕ such that $f_i=\phi a_i$ a.e.

Since the f_i's are linearly independent, we conclude that there exists $i_0\in\{1,\ldots,n\}$ such that $a_i=\delta_{ii_0}$. Without loss of generality assume that $i_0=1$. Therefore, $|h|=|f_i|$ a.e. and $f_2=f_3=\cdots=f_n=0$ a.e. on A. Next, observe that for any $f\in E$, the restriction $f\cdot\chi_A$ of f to A belongs to the one-dimensional subspace $[h]$ of $L_p(\Omega,\mu)$ generated by h.

Summarizing, $E=[h]\oplus_p E_1$ where

$$E_1=\{f\in E: f(w)=0\text{ a.e. on }A\}.$$

It is not difficult to show that

$$d_E\le(1+d_{E_1}^{\frac{1}{1/2-1/p}})^{1/2-1/p}.$$

By (5.5) for the space E_1 and above we obtain that $d_{E_1}=(n-1)^{1/2-1/p}$.

Finally, using the inductive hypothesis, we conclude the proof. ∎

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