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A NOTE ON EDGE-CONNECTIVITY OF THE CARTESIAN PRODUCT OF GRAPHS

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Abstract

The main aim of this paper is to establish conditions that are necessary and sufficient for the edge-connectivity of the Cartesian product of two graphs to equal the sum of the edge-connectivities of the factors. The paper also clarifies an issue that has arisen in the literature on Cartesian products of graphs.

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1. Introduction

The concept of Cartesian products in graph theory can be traced back to a fundamental paper of Sabidussi [8]; we note that an interesting monograph on the topic by Imrich *et al.* [4] was published recently. In the present paper, we deal with the edge-connectivity of the Cartesian product of two graphs. We tend to use notation and terminology from Diestel [3] and Imrich *et al.* [4], giving preference to the former should they differ.

For a graph G, $V(G)$ denotes the set of vertices of G, $E(G)$ denotes the set of edges of G, and $\delta(G)$ denotes the minimum of the set of degrees of the vertices of G. Any element of $E(G)$ is identified with a 2-subset of $V(G)$; so all graphs in this paper are simple. Also in this paper all graphs are finite; that is, $|G| := |V(G)| < \infty$. The complete graph on p vertices is denoted by K^p; thus K^1 denotes the trivial graph with only one vertex. A graph is connected if, for any two distinct vertices of the graph, there is a path of edges connecting them.

The edge-connectivity of a graph G is the minimum number of edges that can be deleted from G so that the resulting graph is disconnected or equal to K^1. (When we delete an edge, we do not delete the vertices with which the edge is incident.) Let $\lambda(G)$ denote the edge-connectivity of G. Obviously, if G is not connected then $\lambda(G) = 0$, and, by definition, $\lambda(K^1) = 0$.

We now turn to the concept of the Cartesian product of two graphs. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. The Cartesian product of G and H is a graph, which we denote by $G \square H$, such that:

(i) $V(G \square H) := V(G) \times V(H)$, the Cartesian product of the sets $V(G)$ and $V(H)$; and

(ii) $\{(g_1, h_1), (g_2, h_2)\} \in E(G \square H)$ if and only if either:

  (a) $g_1 = g_2$ and $\{h_1, h_2\} \in E(H)$; or

  (b) $h_1 = h_2$ and $\{g_1, g_2\} \in E(G)$.

The Cartesian product is an operation that allows us to construct new graphs out of their factors, as in topology. Also, new insights may be obtained when a graph is represented as a Cartesian product; see Imrich *et al.* [4, Ch. 2].

It is known that $G \square H$ is connected if and only if both G and H are connected [4, p. 6]. This prompts the following question. How is the edge-connectivity of $G \square H$ related to the edge-connectivities of the factors G and H? The answer was provided by Xu and Yang in 2006 [9, p. 163, Theorem 3]. A shorter proof of their result has been given by Klavžar and Špacapan [5]. We summarize the answer as follows.

THEOREM 1.1 (Xu and Yang). *If G and H are connected graphs, and neither is isomorphic to K^1, then*

$$\lambda(G \square H) = \min\{\lambda(G)|H|, \lambda(H)|G|, \delta(G) + \delta(H)\}. \tag{1.1}$$

Obviously (1.1) remains true if either G or H is not connected.

In a number of earlier publications ([1, p. 37], [7, Theorem 4], [6, p. 81]) it was claimed that if G and H are connected, then

$$\lambda(G \square H) = \lambda(G) + \lambda(H). \tag{1.2}$$

Although Klavžar and Špacapan [5] noted this error in the literature, they did not provide a counterexample; their focus was to provide a new proof of (1.1). In the next section, we provide such a counterexample. We then proceed to find necessary and sufficient conditions on G and H for (1.2) to be true.

2. Counterexample

In this section, we present an example of two graphs G and H such that

$$\lambda(G \square H) > \lambda(G) + \lambda(H). \tag{2.1}$$

Define the graph G by

$$V(G) = \{1, 2, 3, 4, 5, 6\}$$

and

$$E(G) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{4, 6\}, \{5, 6\}\}.$$

Observe that $\lambda(G) = 1$ and $\delta(G) = 2$.

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Define the graph H by

$$V(H) = \{a, b, c\}$$

and

$$E(H) = \{\{a, b\}, \{b, c\}\}.$$

Observe that $\lambda(H) = 1$ and $\delta(H) = 1$.

Hence,

$$\lambda(G) + \lambda(H) = 2. \tag{2.2}$$

We claim that

$$\lambda(G \,\Box\, H) = 3. \tag{2.3}$$

There are two different ways to establish this claim.

We can appeal to Theorem 1.1 and deduce that

$$\lambda(G \,\Box\, H) = \min\{(1)(3), (1)(6), 2 + 1\} = 3;$$

hence (2.3) is established.

Alternatively, if we do not want to use a result as recent as Theorem 1.1, we can proceed from first principles. If, in $G \,\Box\, H$, we remove the three edges

$$\{(3, a), (4, a)\}, \{(3, b), (4, b)\}, \{(3, c), (4, c)\}$$

then the resulting graph is disconnected; hence $\lambda(G \,\Box\, H) \leq 3$. It remains to show that $\lambda(G \,\Box\, H) > 2$. To this end, it is simple—but tedious—to check that, for any two distinct vertices of $G \,\Box\, H$, there are three edge-disjoint paths joining them. So, if we remove any two edges from $G \,\Box\, H$, then the resulting graph is connected; therefore $\lambda(G \,\Box\, H) > 2$. Hence (2.3) is established.

From (2.2) and (2.3) it follows that G and H satisfy (2.1).

3. The main result

In contrast to the counterexample in the previous section, there are examples of nontrivial, connected graphs G and H for which (1.2) holds. In this section we establish conditions on G and H that are necessary and sufficient for (1.2) to be true.

THEOREM 3.1. *Let G and H be finite, simple graphs. Then*

$$\lambda(G \,\Box\, H) = \lambda(G) + \lambda(H) \tag{3.1}$$

if and only if one of the following conditions holds:

(C1) $|G| = 1$ *or* $|H| = 1$*;*

(C2) *G is disconnected and H is disconnected;*

(C3) $\lambda(G) = \delta(G) > 0$ *and* $\lambda(H) = \delta(H) > 0$*;*

(C4) $\lambda(G) = 1$ *and H is a complete graph* (K^p, $p \geq 2$)*, or vice versa.*

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PROOF. ($\Leftarrow$) We begin by showing that each of the four conditions (C1)–(C4) implies (3.1). It is trivial to prove the implication (C1) $\Rightarrow$ (3.1). It is also trivial to prove the implication (C2) $\Rightarrow$ (3.1).

Assume (C3). In this case, both G and H are connected because $\lambda(G) > 0$ and $\lambda(H) > 0$. Chiue and Shieh [2, Lemma 3] have shown that

$$\lambda(G) + \lambda(H) \leq \lambda(G \square H). \tag{3.2}$$

By Theorem 1.1, (C3) and (3.2),

$$\lambda(G \square H) \leq \delta(G) + \delta(H) = \lambda(G) + \lambda(H) \leq \lambda(G \square H).$$

This proves (3.1). Thus (C3) $\Rightarrow$ (3.1).

Finally, assume (C4). Without loss of generality, assume that $\lambda(G) = 1$ and $H = K^p$, $p \geq 2$. Then

$$\lambda(G) + \lambda(H) = 1 + (p-1) = p$$

and, by Theorem 1.1,

$$\begin{aligned}\lambda(G \square H) &= \min\{\lambda(G)|H|, \lambda(H)|G|, \delta(G) + \delta(H)\}\\ &= \min\{p, (p-1)|G|, \delta(G) + p - 1\} = p;\end{aligned}$$

one can prove the last equality by showing that, under the assumptions, $p \leq (p-1)|G|$ and $p \leq \delta(G) + p - 1$. Thus (3.1) is true. Hence (C4) $\Rightarrow$ (3.1).

This completes the proof of the first implication.

($\Rightarrow$) Assume that (3.1) holds. We consider various cases.

Case 1. If $|G| = 1$ or $|H| = 1$, then (C1) is true.

Case 2. Assume that $|G| > 1$ and $|H| > 1$.

Case 2.1. Assume that G is disconnected and H is connected. Then $G \square H$ is disconnected. Hence, $\lambda(G) = \lambda(G \square H) = 0$ and $\lambda(H) > 0$. This contradicts (3.1) and hence this case is not possible.

Case 2.2. Assume that G is connected and H is disconnected. Like Case 2.1, this case too is impossible.

Case 2.3. Assume that G is disconnected and H is disconnected. Then we have (C2).

Case 2.4. Assume that G is connected and H is connected. Then, using (3.1) and Theorem 1.1, we obtain

$$\begin{aligned}\lambda(G) + \lambda(H) &= \lambda(G \square H)\\ &= \min\{\lambda(G)|H|, \lambda(H)|G|, \delta(G) + \delta(H)\}.\end{aligned}$$

Case *2.4.1*. Assume that $\lambda(G) + \lambda(H) = \delta(G) + \delta(H)$. For any finite, simple graph X, $\lambda(X) \le \delta(X)$. Thus, $\lambda(G) = \delta(G)$ and $\lambda(H) = \delta(H)$. Since G and H are connected, $|G| \ge 2$, and $|H| \ge 2$, it follows that (C3) is true.

Case 2.4.2. Assume that $\lambda(G) + \lambda(H) = \lambda(G)|H|$. Then

$$\lambda(H) = \lambda(G)(|H| - 1).$$

Case *2.4.2.1*. Assume that $\lambda(G) = 0$. This is impossible because G is connected.

Case *2.4.2.2*. Assume that $\lambda(G) = 1$. Then $\lambda(H) = |H| - 1$ and hence $H = K^p$ and $p \ge 2$ because $|H| \ge 2$. Thus (C4) is true.

Case *2.4.2.3*. Assume that $\lambda(G) \ge 2$. Then

$$\lambda(H) \ge 2(|H| - 1) \ge 2\delta(H) \ge 2\lambda(H)$$

and therefore $\lambda(H) = 0$. This case is impossible since H is connected.

Case *2.4.3*. Assume that $\lambda(G) + \lambda(H) = \lambda(H)|G|$. This case is similar to Case 2.4.2.

This completes the proof of the second implication. □

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