A NOTE ON EDGE-CONNECTIVITY OF THE CARTESIAN PRODUCT OF GRAPHS

LAKOA FITINA, C. T. LENARD and T. M. MILLS

(Received 25 January 2011)

Abstract

The main aim of this paper is to establish conditions that are necessary and sufficient for the edge-connectivity of the Cartesian product of two graphs to equal the sum of the edge-connectivities of the factors. The paper also clarifies an issue that has arisen in the literature on Cartesian products of graphs.

2010 Mathematics subject classification: primary 05C40; secondary 05C76.

Keywords and phrases: graph theory, Cartesian products, edge-connectivity.

1. Introduction

The concept of Cartesian products in graph theory can be traced back to a fundamental paper of Sabidussi [8]; we note that an interesting monograph on the topic by Imrich et al. [4] was published recently. In the present paper, we deal with the edge-connectivity of the Cartesian product of two graphs. We tend to use notation and terminology from Diestel [3] and Imrich et al. [4], giving preference to the former should they differ.

For a graph $G$, $V(G)$ denotes the set of vertices of $G$, $E(G)$ denotes the set of edges of $G$, and $\delta(G)$ denotes the minimum of the set of degrees of the vertices of $G$. Any element of $E(G)$ is identified with a 2-subset of $V(G)$; so all graphs in this paper are simple. Also in this paper all graphs are finite; that is, $|G| := |V(G)| < \infty$. The complete graph on $p$ vertices is denoted by $K^p$; thus $K^1$ denotes the trivial graph with only one vertex. A graph is connected if, for any two distinct vertices of the graph, there is a path of edges connecting them.

The edge-connectivity of a graph $G$ is the minimum number of edges that can be deleted from $G$ so that the resulting graph is disconnected or equal to $K^1$. (When we delete an edge, we do not delete the vertices with which the edge is incident.) Let $\lambda(G)$ denote the edge-connectivity of $G$. Obviously, if $G$ is not connected then $\lambda(G) = 0$, and, by definition, $\lambda(K^1) = 0$. 

© 2011 Australian Mathematical Publishing Association Inc. 0004-9727/2011 $16.00
We now turn to the concept of the Cartesian product of two graphs. Let \( G = (V(G), E(G)) \) and \( H = (V(H), E(H)) \) be two graphs. The Cartesian product of \( G \) and \( H \) is a graph, which we denote by \( G \Box H \), such that:

(i) \( V(G \Box H) := V(G) \times V(H) \), the Cartesian product of the sets \( V(G) \) and \( V(H) \); and

(ii) \( \{(g_1, h_1), (g_2, h_2)\} \in E(G \Box H) \) if and only if either:

(a) \( g_1 = g_2 \) and \( \{h_1, h_2\} \in E(H) \); or

(b) \( h_1 = h_2 \) and \( \{g_1, g_2\} \in E(G) \).

The Cartesian product is an operation that allows us to construct new graphs out of their factors, as in topology. Also, new insights may be obtained when a graph is represented as a Cartesian product; see Imrich et al. [4, Ch. 2].

It is known that \( G \Box H \) is connected if and only if both \( G \) and \( H \) are connected [4, p. 6]. This prompts the following question. How is the edge-connectivity of \( G \Box H \) related to the edge-connectivities of the factors \( G \) and \( H \)? The answer was provided by Xu and Yang in 2006 [9, p. 163, Theorem 3]. A shorter proof of their result has been given by Klavžar and Špacapan [5]. We summarize the answer as follows.

**Theorem 1.1 (Xu and Yang).** If \( G \) and \( H \) are connected graphs, and neither is isomorphic to \( K_1 \), then

\[
\lambda(G \Box H) = \min\{\lambda(G)|H|, \lambda(H)|G|, \delta(G) + \delta(H)\}. \tag{1.1}
\]

Obviously (1.1) remains true if either \( G \) or \( H \) is not connected.

In a number of earlier publications ([1, p. 37], [7, Theorem 4], [6, p. 81]) it was claimed that if \( G \) and \( H \) are connected, then

\[
\lambda(G \Box H) = \lambda(G) + \lambda(H). \tag{1.2}
\]

Although Klavžar and Špacapan [5] noted this error in the literature, they did not provide a counterexample; their focus was to provide a new proof of (1.1). In the next section, we provide such a counterexample. We then proceed to find necessary and sufficient conditions on \( G \) and \( H \) for (1.2) to be true.

**2. Counterexample**

In this section, we present an example of two graphs \( G \) and \( H \) such that

\[
\lambda(G \Box H) > \lambda(G) + \lambda(H). \tag{2.1}
\]

Define the graph \( G \) by

\[ V(G) = \{1, 2, 3, 4, 5, 6\} \]

and

\[ E(G) = \{(1, 2), (1, 3), (2, 3), (3, 4), (4, 5), (4, 6), (5, 6)\}. \]

Observe that \( \lambda(G) = 1 \) and \( \delta(G) = 2 \).
Define the graph $H$ by

$$V(H) = \{a, b, c\}$$

and

$$E(H) = \{\{a, b\}, \{b, c\}\}.$$ 

Observe that $\lambda(H) = 1$ and $\delta(H) = 1$.

Hence,

$$\lambda(G) + \lambda(H) = 2. \quad (2.2)$$

We claim that

$$\lambda(G \square H) = 3. \quad (2.3)$$

There are two different ways to establish this claim.

We can appeal to Theorem 1.1 and deduce that

$$\lambda(G \square H) = \min\{(1)(3), (1)(6), 2 + 1\} = 3;$$

hence (2.3) is established.

Alternatively, if we do not want to use a result as recent as Theorem 1.1, we can proceed from first principles. If, in $G \square H$, we remove the three edges

$$\{(3, a), (4, a)\}, \{(3, b), (4, b)\}, \{(3, c), (4, c)\}$$

then the resulting graph is disconnected; hence $\lambda(G \square H) \leq 3$. It remains to show that $\lambda(G \square H) > 2$. To this end, it is simple—but tedious—to check that, for any two distinct vertices of $G \square H$, there are three edge-disjoint paths joining them. So, if we remove any two edges from $G \square H$, then the resulting graph is connected; therefore $\lambda(G \square H) > 2$. Hence (2.3) is established.

From (2.2) and (2.3) it follows that $G$ and $H$ satisfy (2.1).

3. The main result

In contrast to the counterexample in the previous section, there are examples of nontrivial, connected graphs $G$ and $H$ for which (1.2) holds. In this section we establish conditions on $G$ and $H$ that are necessary and sufficient for (1.2) to be true.

**Theorem 3.1.** Let $G$ and $H$ be finite, simple graphs. Then

$$\lambda(G \square H) = \lambda(G) + \lambda(H) \quad (3.1)$$

if and only if one of the following conditions holds:

(C1) $|G| = 1$ or $|H| = 1$;

(C2) $G$ is disconnected and $H$ is disconnected;

(C3) $\lambda(G) = \delta(G) > 0$ and $\lambda(H) = \delta(H) > 0$;

(C4) $\lambda(G) = 1$ and $H$ is a complete graph ($K_p$, $p \geq 2$), or vice versa.
\textbf{Proof.} (⇐) We begin by showing that each of the four conditions (C1)–(C4) implies (3.1). It is trivial to prove the implication (C1) ⇒ (3.1). It is also trivial to prove the implication (C2) ⇒ (3.1).

Assume (C3). In this case, both \(G\) and \(H\) are connected because \(\lambda(G) > 0\) and \(\lambda(H) > 0\). Chiue and Shieh [2, Lemma 3] have shown that

\[
\lambda(G) + \lambda(H) \leq \lambda(G \Box H). \tag{3.2}
\]

By Theorem 1.1, (C3) and (3.2),

\[
\lambda(G \Box H) \leq \delta(G) + \delta(H) = \lambda(G) + \lambda(H) \leq \lambda(G \Box H).
\]

This proves (3.1). Thus (C3) ⇒ (3.1).

Finally, assume (C4). Without loss of generality, assume that \(\lambda(G) = 1\) and \(H = K^p, p \geq 2\). Then

\[
\lambda(G) + \lambda(H) = 1 + (p - 1) = p
\]

and, by Theorem 1.1,

\[
\lambda(G \Box H) = \min\{\lambda(G)|H|, \lambda(H)|G|, \delta(G) + \delta(H)\} = \min\{p, (p - 1)|G|, \delta(G) + p - 1\} = p;
\]

one can prove the last equality by showing that, under the assumptions, \(p \leq (p - 1)|G|\) and \(p \leq \delta(G) + p - 1\). Thus (3.1) is true. Hence (C4) ⇒ (3.1).

This completes the proof of the first implication.

(⇒) Assume that (3.1) holds. We consider various cases.

\textit{Case 1.} If \(|G| = 1\) or \(|H| = 1\), then (C1) is true.

\textit{Case 2.} Assume that \(|G| > 1\) and \(|H| > 1\).

\textit{Case 2.1.} Assume that \(G\) is disconnected and \(H\) is connected. Then \(G \Box H\) is disconnected. Hence, \(\lambda(G) = \lambda(G \Box H) = 0\) and \(\lambda(H) > 0\). This contradicts (3.1) and hence this case is not possible.

\textit{Case 2.2.} Assume that \(G\) is connected and \(H\) is disconnected. Like Case 2.1, this case too is impossible.

\textit{Case 2.3.} Assume that \(G\) is disconnected and \(H\) is disconnected. Then we have (C2).

\textit{Case 2.4.} Assume that \(G\) is connected and \(H\) is connected. Then, using (3.1) and Theorem 1.1, we obtain

\[
\lambda(G) + \lambda(H) = \lambda(G \Box H)
\]

\[
= \min\{\lambda(G)|H|, \lambda(H)|G|, \delta(G) + \delta(H)\}.
\]
Case 2.4.1. Assume that $\lambda(G) + \lambda(H) = \delta(G) + \delta(H)$. For any finite, simple graph $X$, $\lambda(X) \leq \delta(X)$. Thus, $\lambda(G) = \delta(G)$ and $\lambda(H) = \delta(H)$. Since $G$ and $H$ are connected, $|G| \geq 2$, and $|H| \geq 2$, it follows that (C3) is true.

Case 2.4.2. Assume that $\lambda(G) + \lambda(H) = \lambda(G)|H|$. Then

$$\lambda(H) = \lambda(G)(|H| - 1).$$

Case 2.4.2.1. Assume that $\lambda(G) = 0$. This is impossible because $G$ is connected.

Case 2.4.2.2. Assume that $\lambda(G) = 1$. Then $\lambda(H) = |H| - 1$ and hence $H = K^p$ and $p \geq 2$ because $|H| \geq 2$. Thus (C4) is true.

Case 2.4.2.3. Assume that $\lambda(G) \geq 2$. Then

$$\lambda(H) \geq 2(|H| - 1) \geq 2\delta(H) \geq 2\lambda(H)$$

and therefore $\lambda(H) = 0$. This case is impossible since $H$ is connected.

Case 2.4.3. Assume that $\lambda(G) + \lambda(H) = \lambda(H)|G|$. This case is similar to Case 2.4.2.

This completes the proof of the second implication. □

Acknowledgements

Terry Mills is grateful to Divine Word University for assisting him to visit to DWU in 2009 and 2010. The authors thank a referee for pointing out an error in an earlier version of this paper.

References

LAKOA FITINA, Department of Mathematics and Computing Science, Divine Word University, PO Box 483, Madang, Papua New Guinea
e-mail: lfitina@dwu.ac.pg

C. T. LENARD, Department of Mathematics and Statistics, La Trobe University, PO Box 199, Bendigo, Victoria 3552, Australia
e-mail: c.lenard@latrobe.edu.au

T. M. MILLS, Department of Mathematics and Statistics, La Trobe University, PO Box 199, Bendigo, Victoria 3552, Australia
e-mail: t.mills@latrobe.edu.au