A NOTE ON EDGE-CONNECTIVITY OF THE CARTESIAN PRODUCT OF GRAPHS

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(Received 25 January 2011)

Abstract

The main aim of this paper is to establish conditions that are necessary and sufficient for the edge-connectivity of the Cartesian product of two graphs to equal the sum of the edge-connectivities of the factors. The paper also clarifies an issue that has arisen in the literature on Cartesian products of graphs.

2010 Mathematics subject classification: primary 05C40; secondary 05C76.
Keywords and phrases: graph theory, Cartesian products, edge-connectivity.

1. Introduction

The concept of Cartesian products in graph theory can be traced back to a fundamental paper of Sabidussi [8]; we note that an interesting monograph on the topic by Imrich et al. [4] was published recently. In the present paper, we deal with the edge-connectivity of the Cartesian product of two graphs. We tend to use notation and terminology from Diestel [3] and Imrich et al. [4], giving preference to the former should they differ.

For a graph $G$, $V(G)$ denotes the set of vertices of $G$, $E(G)$ denotes the set of edges of $G$, and $\delta(G)$ denotes the minimum of the set of degrees of the vertices of $G$. Any element of $E(G)$ is identified with a 2-subset of $V(G)$; so all graphs in this paper are simple. Also in this paper all graphs are finite; that is, $|G| := |V(G)| < \infty$. The complete graph on $p$ vertices is denoted by $K^p$; thus $K^1$ denotes the trivial graph with only one vertex. A graph is connected if, for any two distinct vertices of the graph, there is a path of edges connecting them.

The edge-connectivity of a graph $G$ is the minimum number of edges that can be deleted from $G$ so that the resulting graph is disconnected or equal to $K^1$. (When we delete an edge, we do not delete the vertices with which the edge is incident.) Let $\lambda(G)$ denote the edge-connectivity of $G$. Obviously, if $G$ is not connected then $\lambda(G) = 0$, and, by definition, $\lambda(K^1) = 0$. 

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We now turn to the concept of the Cartesian product of two graphs. Let \( G = (V(G), E(G)) \) and \( H = (V(H), E(H)) \) be two graphs. The Cartesian product of \( G \) and \( H \) is a graph, which we denote by \( G \square H \), such that:
(i) \( V(G \square H) := V(G) \times V(H) \), the Cartesian product of the sets \( V(G) \) and \( V(H) \); and
(ii) \( \{(g_1, h_1), (g_2, h_2)\} \in E(G \square H) \) if and only if either:
   (a) \( g_1 = g_2 \) and \( \{h_1, h_2\} \in E(H) \); or
   (b) \( h_1 = h_2 \) and \( \{g_1, g_2\} \in E(G) \).

The Cartesian product is an operation that allows us to construct new graphs out of their factors, as in topology. Also, new insights may be obtained when a graph is represented as a Cartesian product; see Imrich et al. [4, Ch. 2].

It is known that \( G \square H \) is connected if and only if both \( G \) and \( H \) are connected [4, p. 6]. This prompts the following question. How is the edge-connectivity of \( G \square H \) related to the edge-connectivities of the factors \( G \) and \( H \)? The answer was provided by Xu and Yang in 2006 [9, p. 163, Theorem 3]. A shorter proof of their result has been given by Klavžar and Špacapan [5]. We summarize the answer as follows.

**Theorem 1.1 (Xu and Yang).** If \( G \) and \( H \) are connected graphs, and neither is isomorphic to \( K_1 \), then

\[
\lambda(G \square H) = \min\{\lambda(G)|H|, \lambda(H)|G|, \delta(G) + \delta(H)\}. \tag{1.1}
\]

Obviously (1.1) remains true if either \( G \) or \( H \) is not connected.

In a number of earlier publications ([1, p. 37], [7, Theorem 4], [6, p. 81]) it was claimed that if \( G \) and \( H \) are connected, then

\[
\lambda(G \square H) = \lambda(G) + \lambda(H). \tag{1.2}
\]

Although Klavžar and Špacapan [5] noted this error in the literature, they did not provide a counterexample; their focus was to provide a new proof of (1.1). In the next section, we provide such a counterexample. We then proceed to find necessary and sufficient conditions on \( G \) and \( H \) for (1.2) to be true.

2. **Counterexample**

In this section, we present an example of two graphs \( G \) and \( H \) such that

\[
\lambda(G \square H) > \lambda(G) + \lambda(H). \tag{2.1}
\]

Define the graph \( G \) by

\[ V(G) = \{1, 2, 3, 4, 5, 6\} \]

and

\[ E(G) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{4, 6\}, \{5, 6\}\}. \]

Observe that \( \lambda(G) = 1 \) and \( \delta(G) = 2 \).
Define the graph $H$ by
\[ V(H) = \{a, b, c\} \]
and
\[ E(H) = \{(a, b), (b, c)\}. \]
Observe that $\lambda(H) = 1$ and $\delta(H) = 1$.
Hence,
\[ \lambda(G) + \lambda(H) = 2. \]  \hfill (2.2)
We claim that
\[ \lambda(G \square H) = 3. \]  \hfill (2.3)
There are two different ways to establish this claim.
We can appeal to Theorem 1.1 and deduce that
\[ \lambda(G \square H) = \min\{(1)(3), (1)(6), 2 + 1\} = 3; \]
hence (2.3) is established.
Alternatively, if we do not want to use a result as recent as Theorem 1.1, we can proceed from first principles. If, in $G \square H$, we remove the three edges
\[ \{(3, a), (4, a)\}, \{(3, b), (4, b)\}, \{(3, c), (4, c)\} \]
then the resulting graph is disconnected; hence $\lambda(G \square H) \leq 3$. It remains to show that $\lambda(G \square H) > 2$. To this end, it is simple—but tedious—to check that, for any two distinct vertices of $G \square H$, there are three edge-disjoint paths joining them. So, if we remove any two edges from $G \square H$, then the resulting graph is connected; therefore $\lambda(G \square H) > 2$. Hence (2.3) is established.
From (2.2) and (2.3) it follows that $G$ and $H$ satisfy (2.1).

3. The main result

In contrast to the counterexample in the previous section, there are examples of nontrivial, connected graphs $G$ and $H$ for which (1.2) holds. In this section we establish conditions on $G$ and $H$ that are necessary and sufficient for (1.2) to be true.

THEOREM 3.1. Let $G$ and $H$ be finite, simple graphs. Then
\[ \lambda(G \square H) = \lambda(G) + \lambda(H) \]  \hfill (3.1)
if and only if one of the following conditions holds:
(C1) $|G| = 1$ or $|H| = 1$;
(C2) $G$ is disconnected and $H$ is disconnected;
(C3) $\lambda(G) = \delta(G) > 0$ and $\lambda(H) = \delta(H) > 0$;
(C4) $\lambda(G) = 1$ and $H$ is a complete graph ($K^p$, $p \geq 2$), or vice versa.
PROOF. (⇐) We begin by showing that each of the four conditions (C1)–(C4) implies (3.1). It is trivial to prove the implication (C1) ⇒ (3.1). It is also trivial to prove the implication (C2) ⇒ (3.1).

Assume (C3). In this case, both $G$ and $H$ are connected because $\lambda(G) > 0$ and $\lambda(H) > 0$. Chiue and Shieh\cite[Lemma 3]{2} have shown that

$$\lambda(G) + \lambda(H) \leq \lambda(G \sqcup H).$$

(3.2)

By Theorem 1.1, (C3) and (3.2),

$$\lambda(G \sqcup H) \leq \delta(G) + \delta(H) = \lambda(G) + \lambda(H) \leq \lambda(G \sqcup H).$$

This proves (3.1). Thus (C3) ⇒ (3.1).

Finally, assume (C4). Without loss of generality, assume that $\lambda(G) = 1$ and $H = K_p$, $p \geq 2$. Then

$$\lambda(G) + \lambda(H) = 1 + (p - 1) = p$$

and, by Theorem 1.1,

$$\lambda(G \sqcup H) = \min\{\lambda(G)|H|, \lambda(H)|G|, \delta(G) + \delta(H)\}$$

$$= \min\{p, (p - 1)|G|, \delta(G) + p - 1\} = p;$$

one can prove the last equality by showing that, under the assumptions, $p \leq (p - 1)|G|$ and $p \leq \delta(G) + p - 1$. Thus (3.1) is true. Hence (C4) ⇒ (3.1).

This completes the proof of the first implication.

(⇒) Assume that (3.1) holds. We consider various cases.

Case 1. If $|G| = 1$ or $|H| = 1$, then (C1) is true.

Case 2. Assume that $|G| > 1$ and $|H| > 1$.

Case 2.1. Assume that $G$ is disconnected and $H$ is connected. Then $G \sqcup H$ is disconnected. Hence, $\lambda(G) = \lambda(G \sqcup H) = 0$ and $\lambda(H) > 0$. This contradicts (3.1) and hence this case is not possible.

Case 2.2. Assume that $G$ is connected and $H$ is disconnected. Like Case 2.1, this case too is impossible.

Case 2.3. Assume that $G$ is disconnected and $H$ is disconnected. Then we have (C2).

Case 2.4. Assume that $G$ is connected and $H$ is connected. Then, using (3.1) and Theorem 1.1, we obtain

$$\lambda(G) + \lambda(H) = \lambda(G \sqcup H)$$

$$= \min\{\lambda(G)|H|, \lambda(H)|G|, \delta(G) + \delta(H)\}.$$
Case 2.4.1. Assume that $\lambda(G) + \lambda(H) = \delta(G) + \delta(H)$. For any finite, simple graph $X$, $\lambda(X) \leq \delta(X)$. Thus, $\lambda(G) = \delta(G)$ and $\lambda(H) = \delta(H)$. Since $G$ and $H$ are connected, $|G| \geq 2$, and $|H| \geq 2$, it follows that (C3) is true.

Case 2.4.2. Assume that $\lambda(G) + \lambda(H) = \lambda(G)|H|$. Then

$$\lambda(H) = \lambda(G)(|H| - 1).$$

Case 2.4.2.1. Assume that $\lambda(G) = 0$. This is impossible because $G$ is connected.

Case 2.4.2.2. Assume that $\lambda(G) = 1$. Then $\lambda(H) = |H| - 1$ and hence $H = K^p$ and $p \geq 2$ because $|H| \geq 2$. Thus (C4) is true.

Case 2.4.2.3. Assume that $\lambda(G) \geq 2$. Then

$$\lambda(H) \geq 2(|H| - 1) \geq 2\delta(H) \geq 2\lambda(H)$$

and therefore $\lambda(H) = 0$. This case is impossible since $H$ is connected.

Case 2.4.3. Assume that $\lambda(G) + \lambda(H) = \lambda(H)|G|$. This case is similar to Case 2.4.2.

This completes the proof of the second implication. □

Acknowledgements

Terry Mills is grateful to Divine Word University for assisting him to visit to DWU in 2009 and 2010. The authors thank a referee for pointing out an error in an earlier version of this paper.

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