A TENSOR EQUATION OF ELLIPTIC TYPE

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The theory of the systems of partial differential equations which arise in connection with the invariant differential operators on a Riemannian manifold may be developed by methods based on those of potential theory. It is therefore natural to consider in the same context the theory of elliptic differential equations, in particular those which are self-adjoint. Some results for a tensor equation in which appears, in addition to the operator $\triangle$ of tensor theory, a matrix or double tensor field defined on the manifold, are here presented. The equation may be written

$$\triangle\phi + A\phi = 0,$$

in a notation explained below.

A maximum principle holds for solutions of this equation, under certain conditions on the coefficient tensor A, as is shown in §2. In the following section the construction of de Rham for the Green's form of a closed manifold is extended to this equation, and the solvability of the Poisson equation corresponding is discussed. We then consider Dirichlet and Neumann problems on compact manifolds with boundary, treating first the case when A is positive definite, and then the general case. The necessary integral equation techniques have been developed in [**3**; **4**; **6**a; **6**b; **8**].

1. Definitions. We consider orientable Riemannian manifolds of dimension n and class C^∞. F will denote a closed, compact manifold, M a compact sub-manifold with $(n-1)$-dimensional boundary B also of class C^∞. If a manifold with boundary is given alone, we can define a closed C^∞ manifold (the double) of which the given manifold is a sub-manifold [**4**]. A positive definite metric tensor g_{ij} of class C^∞ is assumed given, and we assume that the curvature of the manifold under consideration is uniformly bounded.

Skew symmetric covariant tensors

$$\phi_{i_1 \ldots i_p}$$

of order p on F are associated with exterior differential forms of degree p $(0 \leqslant p \leqslant n)$;

$$\phi = \phi_{(i_1 \ldots i_p)}\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}. \tag{1.1}$$

We have the differential operator d, the adjoint $*$, and the co-differential operator $\delta = (-1)^{np+p+1} * d*$. The Laplacian $\triangle$ is an elliptic operator defined by

$$\triangle = d\delta + \delta d,$$

that is,

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(1.2) $$(\Delta\phi)_{i_1\ldots i_p} = - D^i D_i\, \phi_{i_1\ldots i_p} + \sum_{n=1}^{p} \Gamma_{j i_1 \ldots i_p}{}^{i(j_1\ldots j_p)} R^h{}_{j_n}{}^j{}_i \phi_{j_1\ldots j_{n-1} h j_{n+1}\ldots j_p}.$$

Here D_i denotes covariant differentiation,

$$\Gamma_{i_1\ldots i_p}{}^{j_1\ldots j_p}$$

is the skew symmetrized Kronecker symbol of order p, and R_{hijk} is the Riemann curvature tensor. Brackets enclosing a group of indices shall mean that summation is to be effected only over those combinations which are in increasing order.

Suppose that we are given on F a tensor field of order $2p$,

$$A = \{A_{i_1\ldots i_p, j_1\ldots j_p}\},$$

skew-symmetric in each set of p indices, and symmetric in the two sets of indices:

$$A_{i_1\ldots i_p, j_1\ldots j_p} = A_{j_1\ldots j_p, i_1\ldots i_p}.$$

The Ricci tensor R_{ij}, for $p = 1$, and the Riemann tensor $R_{i_1 i_2 j_1 j_2}$, for $p = 2$, are examples of such tensor fields. The components of A are assumed to be of class C^1 in each admissible coordinate system on the manifold. The differential form corresponding to the p-tensor

$$A_{i_1\ldots i_p (j_1\ldots j_p)}\, \phi^{(j_1\ldots j_p)}$$

will be denoted by $A\phi$. The tensor A will be *non-singular* if there exists a second tensor B such that

$$A_{i_1\ldots i_p (j_1\ldots j_p)}\, B^{(j_1\ldots j_p) k_1\ldots k_p} = \Gamma_{i_1\ldots i_p}{}^{k_1\ldots k_p}.$$

Again, A will be termed *positive definite* if the condition

$$A_{(i_1\ldots i_p)(j_1\ldots j_p)}\, \phi^{(i_1\ldots i_p)}\, \phi^{(j_1\ldots j_p)} = 0$$

implies that $\phi^{i_1\ldots i_p}$ is zero. The generalized Kronecker delta is positive definite in this sense. These properties may be interpreted as conditions on the symmetric square matrix of order $\binom{n}{p}$ of independent components of A in any coordinate system. If A is positive definite, A is non-singular. We have

$$*(A\phi) = *A\,(*\phi),$$

where $*A$ is a double skew symmetric tensor density of rank $n - p$:

$$*A_{j_1\ldots j_{n-p}, k_1\ldots k_{n-p}} = e_{(i_1\ldots i_p) j_1\ldots j_{n-p}}\, e_{(h_1\ldots h_p) k_1\ldots k_{n-p}}\, A^{(i_1\ldots i_p)(h_1\ldots h_p)}.$$

Here $e_{i_1 \ldots i_n}$ is the volume n-tensor density.

We introduce the scalar product of two p-tensors

$$(\phi, \psi)_F = \int_F \phi \wedge *\psi = \int_F \psi \wedge *\phi = \int_F \phi_{(i_1\ldots i_p)}\, \psi^{(i_1\ldots i_p)} *1,$$

and the positive definite norm $N(\phi) = (\phi, \phi)_F$. Corresponding to our differential equation

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(1.3) $$\Delta_A\phi = \Delta\phi + A\phi = 0$$

is the Dirichlet integral

(1.4) $$D(\phi, \psi) = (d\phi, d\psi) + (\delta\phi, \delta\psi) + (\phi, A\psi).$$

If A is positive definite, then $D(\phi, \phi)$ is also positive definite. The Green's formula is

(1.5) $$D(\phi, \psi) - (\phi, \Delta\psi + A\psi) = \int_B [\phi \wedge *d\psi - \delta\psi \wedge *\phi],$$

where B is the bounadry of the domain under consideration. On a closed manifold F, if A is positive definite, zero is the only solution of (1.3).

The boundary operators t and n are defined as in [**3**; **4**]; $t\phi$ is the p-form induced on B by the p-form ϕ defined on M, and $n\phi = \phi - t\phi$. The relations $*t = n*$, $*n = t*$ hold.

2. Maximum principle. When $p = 0$, so that (1.3) is a scalar equation and A is a scalar invariant, there holds the following maximum principle: if $A > 0$, no solution has a proper positive maximum, or negative minimum. That is, the square of the solution has no proper maximum. Under analogous conditions, a similar result holds for $p > 0$. The square of ϕ is the invariant

(2.1) $$\phi^2 = \phi_{(i_1\dots i_p)}\,\phi^{(i_1\dots i_p)} \geqslant 0.$$

We have

(2.2) $$\begin{aligned}\tfrac{1}{2}D^iD_i\,\phi^2 &= \tfrac{1}{2}D^iD_i\,\phi^{(i_1\dots i_p)}\,\phi_{(i_1\dots i_p)}\\ &= (D^i\,\phi^{(i_1\dots i_p)})(D_i\,\phi_{(i_1\dots i_p)}) + \phi^{(i_1\dots i_p)}\,D^iD_i\,\phi_{(i_1\dots i_p)}.\end{aligned}$$

The first term on the right is non-negative. In view of (1.2) and (1.3), if ϕ is a solution of our differential equation, we may write the second term in the form

$$[A_{(i_1\dots i_p)(j_1\dots j_p)} + C_{(i_1\dots i_p)(j_1\dots j_p)}]\,\phi^{(i_1\dots i_p)}\,\phi^{(j_1\dots j_p)},$$

where

(2.3) $$\begin{aligned}&C_{i_1\dots i_p, j_1\dots j_p}\\ &\quad= \sum_{n=1}^{p} \Gamma^{i(k_1\dots k_p)}{}_{j(i_1\dots i_p)}\,\Gamma_{k_1\dots k_{n-1}hk_{n+1}\dots k_p, j_1\dots j_p}\,R^h{}_{j_n}{}^j{}_i\,.\end{aligned}$$

If therefore the double tensor $A + C$ is positive definite, the quantity $D^iD_i\phi^2$ which appears in (2.2) is positive unless $\phi^2 = 0$. On the other hand, this quantity is non-positive at a maximum of ϕ^2. Consequently ϕ^2 has no maximum value in the interior of any domain in which ϕ is a solution of (1.3).

When a maximum principle holds in this form, (1.3) has no non-zero solutions regular on a closed manifold F, and furthermore the solution of the Dirichlet problem on a manifold with boundary is unique. If C is positive definite, this leads to an improvement of the results which can be obtained by use of the Dirichlet integral. Sets of conditions under which C is positive definite have been formulated by Lichnerowicz [7]. We remark that in a space of constant curvature K, $C = nK\Gamma$ is positive definite if K is positive. When $p = 1$, $C_{ij} = R_{ij}$.

To compare the limitations obtained in a closed manifold by the Dirichlet integral and the maximum principle, consider the following example with $p = 1$:

$$(\triangle\phi)_i + \lambda R_i^{\ j}\phi_j = 0, \qquad \lambda \text{ real};$$

or,

$$D^k D_k \phi_i + (1 - \lambda) R_i^{\ j}\phi_j = 0.$$

The Dirichlet integral shows that if $-R_{ij}$ is positive definite (positive mean curvature) there are no solutions for $\lambda < 0$ and if R_{ij} is positive definite, no solutions for $\lambda > 0$. The maximum principle shows in the first case ($-R_{ij}$ positive definite) there are no solutions if $\lambda < +1$, which is an improvement; but in the other case shows only that there are no solutions if $\lambda > +1$. We remark that a Killing vector ξ_i, with $D_j\xi_i + D_i\xi_j = 0$, satisfies the above equation with $\lambda = +2$. If R_{ij} is positive definite, there are no Killing vectors, as is shown [2] by both methods.

3. The Green's form. The method used by de Rham to construct the Green's form for Laplace's equation on a closed manifold carries over with but minor alterations to the equation (1.3). For completeness, however, we will describe the construction [8].

We assume that there exists a positive number η such that if x and y are any two points at a distance less than η, a unique geodesic can be drawn from x to y. Let $s(x, y)$ be the geodesic distance so defined, and set

$$a_{i,j} = -\frac{1}{2}\frac{\partial^2 s(x, y)}{\partial x^i \partial y^j}.$$

Let

$$a_{i_1 \ldots i_p, j_1 \ldots j_p}$$

be the determinant

$$|a_{i_\rho j_\sigma}|, \qquad 1 \leqslant \rho, \sigma \leqslant p.$$

Let $\rho(x, y)$ be a function of class C^∞, $\rho = \rho(s(x, y)) = 1$ for $s(x, y) < \frac{1}{2}\eta$, $\rho = 0$ for $s(x, y) \geqslant \eta$. If ω_n denotes the area of the unit sphere in E_n, we define the parametrix

$$(3.1)\quad \omega(x, y) = \omega_p(x, y) = \frac{s^{-n+2}(x, y)\,\rho(x, y)}{(n-2)\,\omega_n} a_{(i_1 \ldots i_p)(j_1 \ldots j_p)}\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}\, dy^{i_1} \wedge \ldots \wedge dy^{i_p}.$$

For $n = 2$, $s^{-n+2}/(n-2)$ is to be replaced by $(-\log s)$.

The integral operator Ω with kernel $\omega = \omega_p$ has the following properties [8]:

(1) $\Omega\phi(x) = (\omega(x, y), \phi(y))_F$ is of class C^∞ if ϕ is of class C^∞.

(2) Ω is self-adjoint since ω is symmetric.

(3) $q(x, y) = \triangle_x(x, y)$ is $O(s^{-n+2})$ as $s \to 0$.

We set

$$Q_A\phi(x) = (q(x, y) + A(x)\,\omega(x, y), \phi(y))_F \tag{3.2}$$

and

$$Q'_A\phi(x) = (q(y, x) + A(y)\,\omega(y, x), \phi(y))_F. \tag{3.3}$$

It then follows that

$$\Omega\triangle_A\phi = \Omega\triangle\phi + \Omega A\phi = \phi + Q'_A\phi \tag{3.4}$$

and

$$\triangle_A\Omega\phi = \triangle\Omega\phi + A\Omega\phi = \phi + Q_A\phi. \tag{3.5}$$

Consider the non-homogeneous equation

$$\triangle\psi + A\psi = \rho, \tag{3.6}$$

where ρ is a given p-form of class C'. Let E denote the linear space of solution of (1.3) regular in F. If $\phi \in E$, we have from (3.4),

$$\phi + Q'_A\,\phi = 0, \tag{3.7}$$

and this equation has only a finite number of linearly independent solutions, so that E is finite dimensional. For $\phi \in E$ we also have

$$(\rho, \phi) = (\triangle\psi + A\psi, \phi) = (\psi, \triangle\phi + A\phi) = 0.$$

A necessary condition for the solvability of (3.6) is therefore that ρ be orthogonal to E.

To solve (3.6) set $\psi = \Omega\xi$, so that (3.5) yields the integral equation

$$\xi + Q_A\xi = \rho. \tag{3.8}$$

A solution ξ exists if and only if ρ is orthogonal to all solutions of the homogeneous transposed equation (3.7). We must necessarily assume that ρ is orthogonal to E; let E_1 denote the orthogonal complement of E in the space of solutions of (3.7). The following device of de Rham shows that (3.6) may still be solved even if ρ is not orthogonal to E_1 but only to E. If $\phi_1 \in E_1$, $\phi_1 \not\equiv 0$, then

$$((\triangle + A)^2\phi, \rho_1) \not\equiv 0$$

for every $\phi \in E_1$, since

$$((\triangle + A)^2\phi_1, \phi_1) = N((\triangle + A)\,\rho_1) > 0$$

since ϕ_1 is orthogonal to E and is not zero. To each ϕ_1 in E_1 corresponds a linear functional $((\triangle + A)^2\phi_1, \phi)$ defined for $\phi \in E_1$, which functional does not vanish identically if $\phi_1 \not\equiv 0$. Conversely therefore, each linear functional on the finite-dimensional space E_1 can be represented in the form $((\triangle + A)^2\phi_1, \phi)$ for a suitable ϕ_1. Thus there exists a ϕ_1 with

$$(\rho, \phi) = ((\triangle + A)^2\phi_1, \phi), \qquad \rho \in E.$$

It follows that $\rho - (\triangle + A)^2\phi_1$ is orthogonal to E_1, and also to E since each term separately is orthogonal to E. The integral equation

$$\xi - Q_A\xi = \rho - (\triangle + A)^2\phi_1$$

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therefore has a solution ξ, and now ψ, defined by

$$\psi = \Omega\xi + (\triangle + A)\phi_1$$

is a solution of (3.6). We therefore have

THEOREM 1. *On a closed Riemannian space there are at most a finite number of linearly independent solutions of* $\triangle\phi + A\phi = 0$. *If* A, *or* $A + C$, *is positive definite, zero is the only solution. The non-homogeneous equation* $\triangle\phi + A\phi = \rho$ *has a solution if and only if* ρ *is orthogonal to all solutions of the homogeneous equation.*

Let $\alpha_p(x, y)$ be the reproducing kernel of the space E, and let

$$E\phi = (\alpha, \phi)_F \tag{3.9}$$

be the projection of an arbitrary form ϕ upon E. By Theorem I the equation

$$\triangle\phi + A\phi = \rho - E\rho \tag{3.10}$$

has a solution ϕ. Furthermore it has a unique solution orthogonal to E. Let this solution be denoted by $G\rho = G_A\rho$. It follows as in (8) that G_A is an integral operator whose kernel, which we denote by $g_A(x, y)$ is symmetric, has the singularity of a local fundamental singularity for (1.3), and satisfies

$$(\triangle + A)_x g_A(x, y) = -\alpha(x, y), \qquad x \neq y. \tag{3.11}$$

If A is positive definite, $\alpha \equiv 0$ since E is zero, and $g_A(x, y)$ is a fundamental solution in the large for equation (1.3).

4. Boundary value problems. We shall now assume that the double tensor A is defined and positive definite on a compact manifold M with boundary B. Let us enlarge M by adjoining to M a neighbourhood $B \times I$ of the boundary B with a fixed closed interval I, B being identified with an endpoint of I. If we call this enlarged finite manifold M_1, the double F_1 of M_1 consists of M_1 and an oppositely oriented replica of M_1, with corresponding boundary points identified. Clearly A is defined in a natural way on F_1 except in the replica of the boundary neighbourhood. By a suitable averaging or interpolation, A can now be defined in the combined boundary neighbourhoods so as to be positive definite and of the same degree of regularity (up to and including C^∞) as in M. We may therefore regard A as defined in F, a closed manifold of which M is a sub-manifold. Similarly the metric tensor g_{ij} can be extended to F, remaining positive definite and of class C^∞. Since A is positive definite on F, the Green's form $g_A(x, y)$ of F is a fundamental singularity in the large for (1.3) on M.

The Dirichlet problem consists of finding a solution ϕ of (1.3) with $t\phi$, $t{*}\phi$ taking assigned continuous boundary values on B. For the Neumann problem the assigned data are $t{*}d\phi$, $t{*}d{*}\phi$. Since the Dirichlet integral for M is positive definite, it follows at once that solutions of these problems are unique.

We construct double and single layer potentials [**3**]:

$$\mu(x) = \int_B [\rho(y) \wedge *dg_A(x, y) + \rho(y) \wedge *d*g_A(x, y)], \tag{4.1}$$

$$\nu(x) = \int_B [g_A(x, y) \wedge *d\sigma(y) + *g_A(x, y) \wedge *d*\sigma(y)] \tag{4.2}$$

with densities ρ and σ respectively, on B. The singularity of $g_A(x, y)$ is, in its principal term, the same as the singularity of the de Rham form $g(x, y)$. Consequently the regularity behaviour and discontinuities of (4.1) and (4.2) are the same as for the corresponding potentials in [**3**]. As the argument point passes from M across B, the quantities

$$t\mu, \quad t*\mu, \quad t*d\nu, \quad t*d*\nu$$

have the respective discontinuities

$$t\rho, \quad t*\rho, \quad -t*d\sigma, \quad -t*d*\sigma.$$

On the boundary B we have, as limiting values from the interior of M,

$$\begin{aligned} t_-\mu &= \tfrac{1}{2}t\rho + t\int_B (\rho \wedge *dg_A + *\rho \wedge *d*g_A), \\ t_-*\mu &= \tfrac{1}{2}t*\rho + t*\int_B (\rho \wedge *dg_A + *\rho \wedge *d*g_A), \\ t_-*d\nu &= -\tfrac{1}{2}t*d\sigma + t*d\int_B (g_A \wedge *d\sigma + *g_A \wedge *d*\sigma), \\ t_-*d*\nu &= -\tfrac{1}{2}t*d*\sigma + t*d*\int_B (g_A \wedge *d\sigma + *g_A \wedge *d*\sigma). \end{aligned} \tag{4.3}$$

The integrals on the right are to be interpreted as principal values. To obtain limiting values on B from the complement CM of M in F, the signs of the leading terms on the right should be reversed. Limiting values from CM will be indicated by a subscript $+$ sign.

The solution of the Dirichlet problem is therefore to be obtained by solving the integral equations

$$t_-\mu = t\phi, \quad t_-*\mu = t*\phi, \tag{4.4}$$

where $t\phi$, $t*\phi$ are the assigned continuous boundary values. In analogous fashion, the Neumann problem may be solved by means of the system.

$$t_-*d\nu = t*d\phi, \quad t_-*d*\nu = t*d*\phi, \tag{4.5}$$

for given continuous $t*d\phi$, $t*d*\phi$ on B. These are systems of $\binom{n}{p}$ singular integral equations. Since $g_A(x, y) = g_A(y, x)$, it can be shown as in [**3**] that the kernels of the systems (4.4) and (4.5) are transposes of each other.

The condition for the existence of a solution is that the assigned non-homogeneous term be orthogonal to every solution of the homogeneous transposed equation [**5**]. In each case the homogeneous transposed equation is obtained when we attempt to solve the boundary value problem of the complementary type for the complementary domain CM.

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For the Dirichlet problem we must therefore show that

$$\int_B [\phi \wedge *d\sigma + *\phi \wedge *d*\sigma] = 0 \tag{4.6}$$

for all solutions σ of the equations

$$t_+*d\nu = 0, \quad t_+*d*\nu = 0.$$

Now (4.7) imply $\nu \equiv 0$ in CM. Since ν is continuous across B, we have $t_-\nu = 0$, $t_-*\nu = 0$, and therefore $\nu \equiv 0$ in M. Therefore the discontinuity conditions, which read

$$t_-*d\nu = t*d\sigma, \qquad t_-*d*\nu = t*d*\sigma,$$

imply that $t*d\sigma$ and $t*d*\sigma$ vanish. Thus (4.6) holds for all continuous $t\phi$, $t*\phi$.

THEOREM II. *If A is positive definite in M, there exists a unique solution of the Dirichlet problem for $\triangle\phi + A\phi = 0$, with $t\phi$, $t*\phi$ having given continuous values on B.*

The condition of solvability for the Neumann problem is

$$\int_B [\rho \wedge *d\phi + *\rho \wedge *d*\phi] = 0, \tag{4.8}$$

for all solutions ρ of the equations

$$t_+\mu = 0, \quad t_+*\mu = 0. \tag{4.9}$$

These conditions imply that μ vanishes identically in CM It follows as in [**3**] that $t*d\mu$ and $t*d*\mu$ are continuous across B, and therefore

$$t_-*d\mu = 0, \quad t_-*d*\mu = 0.$$

It follows now that $\mu = 0$ in M. The discontinuity conditions show that

$$t_-\mu = -t\rho, \quad t_-*\mu = -t*\rho,$$

and therefore $t\rho$ and $t*\rho$ are zero. Thus (4.8) is always satisfied.

THEOREM III. *If A is positive definite in M, there exists a unique solution of the Neumann problem for $\triangle\phi + A\phi = 0$, with $t*d\phi$, $t*d*\phi$ having given continuous values on B.*

The Green's and Neumann's forms for M corresponding to the Dirichlet and Neumann problems are easily defined by subtracting a regular solution, suitably determined, from the fundamental solution $g_A(x, y)$. Representation formulae for the solutions of the Dirichlet and Neumann problems follow in the usual way from Green's formula (1.5). We note that the boundary conditions are

$$t\, G_A(x, y) = 0, \quad t*G_A(x, y) = 0, \quad t*dN_A(x, y) = 0, \quad t*d*N_A(x, y) = 0, \tag{4.10}$$

and that $G_A(x, y)$ and $N_A(x, y)$ are symmetric in their two arguments. As in the scalar theory, the difference

$$K_A(x, y) = N_A(x, y) - G_A(x, y) \tag{4.11}$$

is the reproducing kernel for solutions of (1.3) in the D-metric (1.4) over M. The kernel (4.11) has many of the formal properties developed in [**1**].

5. Boundary value problems (continued). We now consider the Dirichlet and Neumann problems for M when the coefficient tensor A is not restricted to be positive definite. It is then possible that the solutions of the boundary value problems will, if they exist, not be unique. We require A to be of class C^1 in $M + B$. Consider the equation (1.3) in the form

$$\Delta\phi + A_0\phi = (A_0 - A)\phi = B\phi, \tag{5.1}$$

say, where A_0 is yet to be chosen. We shall need to choose A_0 positive definite and such that $B = A_0 - A$ is non-singular. Let A_1 be any positive definite double tensor of the type of A.

The characteristic roots of the matrix of independent components of A_1 in any coordinate system are real, since A_1 is assumed symmetric, and positive, since A_1 is positive definite. Also the characteristic roots of $A_1 - \epsilon A$ are continuous functions of ϵ, and of the point where $A_1 - \epsilon A$ is evaluated. For ϵ sufficiently small, and positive, the roots are positive at any point of M. Since M is compact, it follows that there is an $\epsilon_1 > 0$ such that the roots of $A_1 - \epsilon_1 A$ are positive everywhere in $M + B$. We now choose $A_0 = \epsilon_1^{-1}A_1$; $B = \epsilon_1^{-1}A_1 - A$ is positive definite and therefore non-singular.

Denote by $G_0(x, y)$ and $N_0(x, y)$ the Green's and Neumann's tensors of M for the equation

$$\Delta\phi + A_0\phi = 0.$$

We see that any solution ϕ of the Dirichlet problem for (1.3) must satisfy the integral equation

$$\phi(x) + (B\phi(y),\ G_0(x, y))_M = \psi(x), \tag{5.2}$$

where $\Delta\psi + A_0\psi = 0$, and $t\psi$, $t{*}\psi$ assume the given boundary values of the problem. If we apply the operator $\Delta + A_0$ to (5.2), we find, conversely,

$$\Delta\phi + A_0\phi - B\phi = 0,$$

so that ϕ is a solution of (5.1), that is to say, of (1.3). Also, on taking boundary values we note that the integrated term yields zero because of the boundary behaviour of the Green's function, and therefore that $t\phi = t\psi$, $t{*}\phi = t{*}\psi$. If $t\psi$, $t{*}\psi$ are assigned continuously, ψ is uniquely determined, by Theorem II, and any solution of (5.2) is a solution of the Dirichlet problem for (1.3).

A solution of (5.2) exists if and only if ψ is orthogonal to all solutions χ of the homogeneous transposed equation

$$\chi(x) + B(x)\ (\chi(y),\ G_0(x, y))_M = 0. \tag{5.3}$$

Since B is non-singular, B^{-1} exists, and we have

$$\begin{aligned} 0 &= B^{-1}\chi(x) + (BB^{-1}\chi(y),\ G_0(x, y))_M \\ &= B^{-1}\chi(x) + (B^{-1}\chi(y),\ BG_0(x, y))_M. \end{aligned}$$

Therefore the tensor $B^{-1}\chi$ is a solution of the homogeneous equation corresponding to (5.2):

$$\phi(x) + (B\phi(y),\ G_0(x, y))_M = 0. \tag{5.4}$$

This equation has continuous iterated kernels of sufficiently high order, and therefore a finite number of linearly independent solutions $\phi_r (r = 1, \ldots, N)$. Thus any solution χ of (5.3) is of the form $\chi = B\phi$ for some solution $\phi = \sum a_r \phi_r$ of (5.4). The condition of solvability of (5.2) is therefore $(\chi, \psi) = (B\phi, \psi) = 0$ for every solution p of (5.4). We have from (1.5) and (5.4)

$$\begin{aligned}(\chi, \psi) = (B\phi, \psi) &= (\triangle\phi + A_0\,\phi, \psi)\\ &= (\phi, \triangle\psi + A_0\psi) + \int_B [\phi \wedge *d\psi - \delta\psi \wedge *\phi\\ &\qquad - \psi \wedge *d\phi + \delta\phi \wedge *\psi].\end{aligned}$$

The volume integral vanishes since $\triangle\psi + A\psi = 0$, and $t\phi = 0$, $t{*}\phi = 0$ since ϕ is a solution of (5.4). We therefore have

THEOREM IV. *There exists a solution of the Dirichlet problem for* $\triangle\phi + A\phi = 0$, $t\phi = t\psi$, $t{*}\phi = t{*}\psi$ *if and only if*

$$\int_B [\psi \wedge *d\phi_r - \delta\phi_r \wedge *\psi] = 0 \tag{5.5}$$

for every solution ϕ_r *of* $\triangle\phi + A\phi = 0, \quad t\phi = 0, \quad t{*}\phi = 0.$

The condition (5.5) involves only the given data $t\psi$, $t{*}\psi$, and the eigentensors ϕ_r. The most general solution of the problem is of course of the form

$$\phi + \sum a_r \phi_r$$

where ϕ is any particular solution.

The Neumann problem may also be treated this way. Any solution of the problem must satisfy the integral equation

$$\phi(x) + (B\phi(y),\ N_0(x, y))_M = \psi(x), \tag{5.6}$$

where $\psi(x)$ is that solution of the equation $\triangle\psi + A_0\psi = 0$ with the assigned values of $t{*}d\psi$, $t{*}d{*}\psi$. Conversely, a solution of (5.6) provides a solution of the Neumann problem. A solution of (5.6) exists if and only if ψ is orthogonal to all solutions of the homogeneous transposed equation

$$\chi(x) + B(x)(\chi(y),\ N_0(x, y))_M = 0. \tag{5.7}$$

As before, $\chi = B\phi_r$ where ϕ_r is a solution of (5.6) with $\psi = 0$. The orthogonality condition can be transformed to read

$$\begin{aligned}0 = (\psi, \chi) = (\psi, B\phi) &= (\triangle\phi + A_0\phi, \psi)\\ &= (\phi, \triangle\psi + A_0\psi) + \int_B (\phi \wedge *d\psi - \delta\psi \wedge *\phi\\ &\qquad - \psi \wedge *d\phi - \delta\phi \wedge *\psi).\end{aligned}$$

The volume integral and two terms of the surface integral vanish, as before.

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THEOREM V. *There exists a solution of the Neumann problem for* $\triangle\phi + A\phi = 0$, $t{*}d\phi = t{*}d\psi$, $t{*}d{*}\phi = t{*}d{*}\psi$ *if and only if*

$$\int_B (\phi_r \wedge *d\psi - \delta\psi \wedge *\phi_r) = 0 \tag{5.8}$$

for every solution ϕ_r *of* $\triangle\phi + A\phi = 0, \quad t{*}d\phi = 0, \quad t{*}d{*}\phi = 0.$

The form of the general solution is obvious.

We remark that the third, or mixed, boundary value problem of potential theory for the equation (1.3) may also be treated this way. The indirect method used here circumvents some of the characteristic difficulties of the corresponding proofs for the Laplace equation, such as the lack of a fundamental singularity in the large.

6. The Laplace equation. The methods used in the foregoing work enable us to give a quite short proof of the following result on the existence of a fundamental singularity in the large for the tensor Laplace equation on M:

THEOREM VI. *There exists a fundamental singularity in the large on* M *for* $\triangle\phi = 0$ *if and only if the Dirichlet problem for* $\triangle\phi = 0$ *on* M *has at most one solution.*

We first show that, if the Dirichlet problem has at most one solution, the fundamental singularity exists. Let F be the double of M, and let us extend the differential equation to F in the form

$$\triangle\phi + A\phi = 0, \tag{6.1}$$

where A is, as before, a matrix or double p-tensor such that A is C^∞ in F, positive definite in $F - M$, and zero in M itself. For instance, the Kronecker tensor Γ multiplied by a suitable scalar factor provides such a tensor. Then we construct the Green's form of F for (6.1); from (3.11) we see that this Green's form will itself be a fundamental singularity in the large for (6.1), provided only that there exist no everywhere regular solutions of (6.1) in F, except zero. If such a regular solution did exist, it would have a zero Dirichlet integral over F. Since A is positive definite in $F - M$, the solution must be zero there, and by continuity it must be zero on B. Hence it is harmonic in M, and has zero boundary value; by hypothesis it must be identically zero. Thus the Green's form does provide the desired fundamental singularity.

The converse part of the theorem may be established as follows. Assume that a fundamental singularity exists in M for the Laplace operator, then we may use it to solve the Poisson equation $\triangle\phi = \beta$ for arbitrary $\beta \in C^3$ in M. From Green's formulae (1.4) and (1.5) with $A = 0$, we see that the conditions $\triangle\phi = 0$ in M, with $t\phi = 0$, $t{*}\phi = 0$ on B, imply that $d\phi = 0$, $\delta\phi = 0$ in M. From Theorem IV we see that there exists a solution in M of the Laplace equation having arbitrary continuous boundary values. Suppose now that ϕ is a solution of $\triangle\phi = 0$ with $t\phi = 0$, $t{*}\phi = 0$. In view of the remarks just made, there exists a p-tensor ψ with $\triangle\psi = \phi$ in M, $t\psi = 0$, $t{*}\psi = 0$ on B. Then

$$N(\phi) = (\phi, \triangle\psi)$$

$$= (\psi, \triangle\phi) + \int_B [\phi \wedge *d\psi - \delta\psi \wedge *\phi - \psi \wedge *d\psi + \delta\phi \wedge *\psi] = 0$$

since all terms on the right vanish. Hence $\phi \equiv 0$, which shows that a solution of the Dirichlet problem for M is unique. This completes the proof.

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