MULTIPLICATIVE SUB-HODGE STRUCTURES OF CONJUGATE VARIETIES

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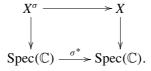email: schreied@math.uni-bonn.de

Abstract

For any subfield $K \subseteq \mathbb{C}$, not contained in an imaginary quadratic extension of $\mathbb{Q}$, we construct conjugate varieties whose algebras of K-rational (p, p)-classes are not isomorphic. This compares to the Hodge conjecture which predicts isomorphisms when K is contained in an imaginary quadratic extension of $\mathbb{Q}$; additionally, it shows that the complex Hodge structure on the complex cohomology algebra is not invariant under the $\mathrm{Aut}(\mathbb{C})$-action on varieties. In our proofs, we find simply connected conjugate varieties whose multilinear intersection forms on $H^2(-, \mathbb{R})$ are not (weakly) isomorphic. Using these, we detect nonhomeomorphic conjugate varieties for any fundamental group and in any birational equivalence class of dimension ≥ 10.

1. Introduction

For a smooth complex projective variety X and an automorphism σ of $\mathbb{C}$, the conjugate variety X^σ is defined via the fiber product diagram

$$\begin{array}{ccc} X^\sigma & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \xrightarrow{\ \sigma^*\ } & \mathrm{Spec}(\mathbb{C}). \end{array}$$

To put it another way, X^σ is the smooth variety whose defining equations in some projective space are given by applying σ to the coefficients of the equations of X. As abstract schemes – but in general not as schemes over $\mathrm{Spec}(\mathbb{C})$ – X and

X^σ are isomorphic. This has several important consequences for the singular cohomology of conjugate varieties. For instance, pullback of forms induces a σ-linear isomorphism between the algebraic de Rham complexes of X and X^σ. This induces an isomorphism of complex Hodge structures

$$H^*(X, \mathbb{C}) \otimes_\sigma \mathbb{C} \xrightarrow{\sim} H^*(X^\sigma, \mathbb{C}), \tag{1}$$

where $\otimes_\sigma \mathbb{C}$ means that the tensor product is taken where $\mathbb{C}$ maps to $\mathbb{C}$ via σ; see [4]. In particular, Hodge and Betti numbers of conjugate varieties coincide.

The singular cohomology with $\mathbb{Q}_\ell$-coefficients coincides on smooth complex projective varieties with ℓ-adic étale cohomology. Since étale cohomology does not depend on the structure morphism to $\mathrm{Spec}(\mathbb{C})$, we obtain isomorphisms of graded $\mathbb{Q}_\ell$-algebras and $\mathbb{C}$-algebras, respectively:

$$H^*(X, \mathbb{Q}_\ell) \xrightarrow{\sim} H^*(X^\sigma, \mathbb{Q}_\ell) \quad \text{and} \quad H^*(X, \mathbb{C}) \xrightarrow{\sim} H^*(X^\sigma, \mathbb{C}), \tag{2}$$

depending on an embedding $\mathbb{Q}_\ell \subseteq \mathbb{C}$. Since the latter isomorphism is $\mathbb{C}$-linear, it is not induced by (1).

Only recently, Charles discovered that there are however aspects of singular cohomology which are not invariant under conjugation:

THEOREM 1 (Charles [3]). *There exist conjugate smooth complex projective varieties with distinct real cohomology algebras.*

It is the aim of this paper to further investigate to what extent cohomological data are invariant under the $\mathrm{Aut}(\mathbb{C})$-action on varieties.

1.1. Algebras of K-rational (p, p)-classes. For any subfield $K \subseteq \mathbb{C}$, we denote the space of K-rational (p, p)-classes on X by

$$H^{p,p}(X, K) := H^{p,p}(X) \cap H^{2p}(X, K);$$

the corresponding graded K-algebra is denoted by $H^{*,*}(X, K)$. The Hodge conjecture predicts that $H^{*,*}(X, \mathbb{Q})$ is generated by algebraic cycles. Since each algebraic cycle $Z \subseteq X$ induces a canonical cycle $Z^\sigma \subseteq X^\sigma$ and vice versa, the Hodge conjecture implies:

CONJECTURE 2. *The graded $\mathbb{Q}$-algebra $H^{*,*}(-, \mathbb{Q})$ is conjugation invariant.*

Apart from the (few) cases where the Hodge conjecture is known, and apart from Deligne's result [5] which settles Conjecture 2 for abelian varieties, the above conjecture remains wide open; see [4, 19].

The above consequence of the Hodge conjecture motivates the investigation of potential conjugation invariance of $H^{*,*}(-, K)$ for an arbitrary field of

coefficients $K \subseteq \mathbb{C}$. If $K = \mathbb{Q}(iw)$ with $w^2 \in \mathbb{N}$ is an imaginary quadratic extension of $\mathbb{Q}$, then the real part, as well as $1/w$ times the imaginary part, of a $\mathbb{Q}(iw)$-rational (p, p)-class is $\mathbb{Q}$-rational. Hence,

$$H^{*,*}(-, \mathbb{Q}(iw)) \cong H^{*,*}(-, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}(iw).$$

It follows that the Hodge conjecture predicts the conjugation invariance of $H^{*,*}(-, K)$, when K is contained in an imaginary quadratic extension of $\mathbb{Q}$. In this paper, we are able to settle all remaining cases:

THEOREM 3. *Let $K \subseteq \mathbb{C}$ be a subfield, not contained in an imaginary quadratic extension of $\mathbb{Q}$. Then there exist conjugate smooth complex projective varieties whose graded algebras of K-rational (p, p)-classes are not isomorphic.*

By Theorem 3, there are conjugate smooth complex projective varieties X, X^σ with

$$H^{*,*}(X, \mathbb{C}) \not\cong H^{*,*}(X^\sigma, \mathbb{C}).$$

This shows the following:

COROLLARY 4. *The complex Hodge structure on the complex cohomology algebra of smooth complex projective varieties is not invariant under the $\mathrm{Aut}(\mathbb{C})$-action on varieties.*

Corollary 4 is in contrast to (1) and (2) which show that the complex Hodge structure in each degree, as well as the $\mathbb{C}$-algebra structure of $H^*(-, \mathbb{C})$, is $\mathrm{Aut}(\mathbb{C})$-invariant. The above corollary also shows that there is no embedding $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$ which guarantees that the isomorphism (2), induced by isomorphisms between ℓ-adic étale cohomologies, respects the complex Hodge structures.

Theorem 3 will follow from Theorems 5 and 6 below. Firstly, if K is different from $\mathbb{R}$ and $\mathbb{C}$, then Theorem 3 follows from:

THEOREM 5. *Let $K \subseteq \mathbb{C}$ be a subfield, not contained in an imaginary quadratic extension of $\mathbb{Q}$. If K is different from $\mathbb{R}$ and $\mathbb{C}$, then there exist for any $p \geq 1$ and in any dimension $\geq p + 1$ conjugate smooth complex projective varieties X, X^σ with*

$$H^{p,p}(X, K) \not\cong H^{p,p}(X^\sigma, K).$$

It is worth noting that Theorem 5 does not remain true if one restricts to smooth complex projective varieties that can be defined over $\overline{\mathbb{Q}}$; see Remark 18.

Next, the case $K = \mathbb{R}$ in Theorem 3 follows from the case where $K = \mathbb{C}$ since

$$H^{*,*}(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong H^{*,*}(X, \mathbb{C})$$

holds; so it remains to deal with $K = \mathbb{C}$. As the isomorphism type of the $\mathbb{C}$-vector space $H^{p,p}(-, \mathbb{C})$ coincides on conjugate varieties, we now really need to make use of the algebra structure of $H^{*,*}(-, \mathbb{C})$. Remarkably, it turns out that it suffices to use only a very little amount of the latter, namely the symmetric multilinear intersection form

$$H^{1,1}(X, \mathbb{C})^{\otimes n} \longrightarrow H^{2n}(X, \mathbb{C}),$$

where $n = \dim(X)$. We explain our result, Theorem 6 below, in the next subsection.

1.2. Multilinear intersection forms on $H^{1,1}(-, K)$ and $H^2(-, K)$.

We say that two symmetric K-multilinear forms $V^{\otimes n} \to K$ and $W^{\otimes n} \to K$ on two given K-vector spaces V and W are (weakly) isomorphic if there exists a K-linear isomorphism $V \cong W$ which respects the given multilinear forms (up to a multiplicative constant). If K is closed under taking nth roots, then weakly isomorphic intersection forms are already isomorphic.

For a smooth complex projective variety X of dimension n, the cup product defines symmetric multilinear forms

$$H^{1,1}(X, K)^{\otimes n} \longrightarrow H^{2n}(X, K) \cong K \quad \text{and} \quad H^2(X, K)^{\otimes n} \longrightarrow H^{2n}(X, K) \cong K,$$

where $H^{2n}(X, K) \cong K$ is the canonical isomorphism that is induced by integrating de Rham classes over X. The weak isomorphism types of the above multilinear forms are determined by the isomorphism types of the graded K-algebras $H^{*,*}(X, K)$ and $H^{2*}(X, K)$ respectively.

By the Lefschetz theorem, the Hodge conjecture is true for $(1, 1)$-classes and so it is known that the isomorphism type of the intersection form on $H^{1,1}(-, \mathbb{Q})$ is conjugation invariant. Additionally, it follows from (2) that the isomorphism types of the intersection forms on $H^2(-, \mathbb{Q}_\ell)$ and $H^2(-, \mathbb{C})$ are invariant under conjugation. Our result, which settles the case $K = \mathbb{C}$ in Theorem 3, contrasts with these positive results:

THEOREM 6. *There exist in any dimension ≥ 4 simply connected conjugate smooth complex projective varieties whose $\mathbb{R}$-multilinear intersection forms on $H^2(-, \mathbb{R})$, as well as $\mathbb{C}$-multilinear intersection forms on $H^{1,1}(-, \mathbb{C})$, are not weakly isomorphic.*

The examples that we will construct in the proof of Theorem 6 in Section 6 are defined over cyclotomic number fields. For instance, one series of examples is defined over $\mathbb{Q}[\zeta_{12}]$; their complex $(1, 1)$-classes are spanned by $\mathbb{Q}[\sqrt{3}]$-rational ones. This yields examples X, X^σ such that the intersection forms on the

equidimensional vector spaces $H^{1,1}(X, \mathbb{Q}[\sqrt{3}])$ and $H^{1,1}(X^\sigma, \mathbb{Q}[\sqrt{3}])$ are not weakly isomorphic; see Corollary 29.

It follows from Theorem 6 that the even-degree real cohomology algebra $H^{2*}(-, \mathbb{R})$, as well as the subalgebra $SH^2(-, \mathbb{R})$ which is generated by $H^2(-, \mathbb{R})$, is not invariant under conjugation. Since Charles's examples have dimension ≥ 12 and fundamental group $\mathbb{Z}^8$, Theorem 6 generalizes Theorem 1 in several different directions. Another generalization of Theorem 1, namely Theorem 7 below, is explained in the following subsection.

1.3. Applications to conjugate varieties with given fundamental group.
Conjugate varieties are homeomorphic in the Zariski topology but in general not in the analytic one. Historically, this was first observed by Serre in [13], who constructed conjugate varieties whose fundamental groups are infinite but nonisomorphic. The first nonhomeomorphic conjugate varieties with finite fundamental group were constructed by Abelson [1]. His construction however only works for nonabelian finite groups which satisfy some strong cohomological condition.

Other examples of conjugate varieties which are not homeomorphic (or, weaker: not deformation equivalent) are constructed in [2, 3, 6, 10, 15]. Again, the fundamental groups of these examples are of special shapes. In particular, our conjugate varieties in Theorem 6 are the first known nonhomeomorphic examples which are simply connected. This answers a question, posed more than 15 years ago by Reed in [11]. Reed's question was our initial motivation for studying conjugate varieties and leads us to the more general problem of determining those fundamental groups for which nonhomeomorphic conjugate varieties exist. Since the fundamental group of smooth varieties is a birational invariant, the problem of detecting nonhomeomorphic conjugate varieties in a given birational equivalence class refines this problem. Building upon the examples that we will construct in the proof of Theorem 6, we will be able to prove the following:

THEOREM 7. *Any birational equivalence class of complex projective varieties in dimension ≥ 10 contains conjugate smooth complex projective varieties whose even-degree real cohomology algebras are nonisomorphic.*

Theorem 7 implies immediately:

COROLLARY 8. *Let G be the fundamental group of a smooth complex projective variety. Then there exist conjugate smooth complex projective varieties with fundamental group G, but nonisomorphic even-degree real cohomology algebras.*

In Theorem 34 in Section 8 we show that the examples in Theorem 7 can be chosen to have nonisotrivial deformations. This is in contrast to the observation that the previously known nonhomeomorphic conjugate varieties tend to be rather rigid; see Remark 35.

1.4. Constructions and methods of proof. Using products of special surfaces with projective space, we will prove Theorem 5 in Section 3. The key idea is to construct real curves in the moduli space of abelian surfaces and Kummer K3 surfaces, on which $\dim(H^{1,1}(-, K))$ is constant. Using elementary facts about modular forms, we then prove that each of our curves contains a transcendental point, that is a point whose coordinates are algebraically independent over $\mathbb{Q}$. The action of $\mathrm{Aut}(\mathbb{C})$ being transitive on the transcendental points of our moduli spaces, Theorem 5 follows as soon as we have seen that our assumptions on K ensure the existence of two real curves as above on which $\dim(H^{p,p}(-, K))$ takes different (constant) values.

For the proof of Theorem 6 in Section 6 we use the Charles–Voisin method [3, 18]; see Section 4. We start with simply connected surfaces $Y \subseteq \mathbb{P}^N$ with special automorphisms, constructed in Section 5. Then we blow up five smooth subvarieties of $Y \times Y \times \mathbb{P}^N$, for example the graphs of automorphisms of Y. In order to keep the dimensions low, we then pass to a complete intersection subvariety T of this blow-up. If $\dim(T) \geq 4$, then the cohomology of T encodes the action of the automorphisms on $H^2(Y, \mathbb{R})$ and $H^{1,1}(Y, \mathbb{C})$. The latter can change under the $\mathrm{Aut}(\mathbb{C})$-action, which will be the key ingredient in our proofs.

In order to prove Theorem 7 in Section 7, we start with a smooth complex projective variety Z of dimension ≥ 10, representing a given birational equivalence class. From our previous results, we will be able to pick a four-dimensional variety T and an automorphism σ of $\mathbb{C}$ with $Z \cong Z^\sigma$ such that T and T^σ have nonisomorphic even-degree real cohomology algebras. Since T is four-dimensional, we can embed it into the exceptional divisor of the blow-up $\hat{Z}$ of Z at a point and define $W = Bl_T(\hat{Z})$. Then, $W^\sigma = Bl_{T^\sigma}(\hat{Z}^\sigma)$ is birational to $Z^\sigma \cong Z$. Moreover, we will be able to arrange that $b_2(T)$ is larger than $b_4(Z) + 4$. This will allow us to show that any isomorphism between $H^{2*}(W, \mathbb{R})$ and $H^{2*}(W^\sigma, \mathbb{R})$ induces an isomorphism between $H^{2*}(T, \mathbb{R})$ and $H^{2*}(T^\sigma, \mathbb{R})$. Theorem 7 will follow.

1.5. Conventions. Using Serre's GAGA principle [14], we usually identify a smooth complex projective variety X with its corresponding analytic space, which is a Kähler manifold. For a codimension p subvariety V in X, we denote the corresponding (p, p)-class in $H^*(X, \mathbb{Z})$ by $[V]$.

2. Preliminaries

2.1. Cohomology of blow-ups. In this subsection we recall important properties about the cohomology of blow-ups, which we will use (tacitly) throughout Sections 4, 6 and 7. Let $Y \subseteq X$ be Kähler manifolds and let $\tilde{X} = Bl_Y(X)$ be the blow-up of X in Y with exceptional divisor $D \subseteq \tilde{X}$. We then obtain a commutative diagram

$$
\begin{array}{ccc}
D & \xrightarrow{\;j\;} & \tilde{X} \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\
Y & \xrightarrow{\;i\;} & X,
\end{array}
$$

where i denotes the inclusion of Y into X and j denotes the inclusion of the exceptional divisor D into $\tilde{X}$. Letting r denote the codimension of Y in X, we then have the following; see [17, p. 180].

LEMMA 9. *There is an isomorphism of integral Hodge structures*

$$
H^k(X, \mathbb{Z}) \oplus \left(\bigoplus_{i=0}^{r-2} H^{k-2i-2}(Y, \mathbb{Z}) \right) \xrightarrow{\sim} H^k(\tilde{X}, \mathbb{Z}),
$$

where on $H^{k-2i-2}(Y, \mathbb{Z})$, the natural Hodge structure is shifted by $(i+1, i+1)$. On $H^k(X, \mathbb{Z})$, the above morphism is given by π^*. On $H^{k-2i-2}(Y, \mathbb{Z})$ it is given by $j_* \circ h^i \circ p^*$, where h denotes the cup product with $c_1(\mathcal{O}_D(1)) \in H^2(D, \mathbb{Z})$ and j_* is the Gysin morphism of the inclusion $j \colon D \hookrightarrow \tilde{X}$.

By the above lemma, each cohomology class of $\tilde{X}$ is a sum of pullback classes from X and push-forward classes from D. The ring structure on $H^*(\tilde{X}, \mathbb{Z})$ is therefore uncovered by the following lemma:

LEMMA 10. *Let $\alpha, \beta \in H^*(D, \mathbb{Z})$ and $\eta \in H^*(X, \mathbb{Z})$. Then,*

$$
\pi^*(\eta) \cup j_*(\alpha) = j_*(p^*(i^*\eta) \cup \alpha) \quad \text{and} \quad j_*(\alpha) \cup j_*(\beta) = -j_*(h \cup \alpha \cup \beta),
$$

where $h = c_1(\mathcal{O}_D(1)) \in H^2(D, \mathbb{Z})$.

Proof. Using $i \circ p = \pi \circ j$, the first assertion follows immediately from the projection formula. For the second assertion, one first proves

$$
j_*(\alpha) \cup j_*(\beta) = j_*(1) \cup j_*(\alpha \cup \beta) \tag{3}
$$

by realizing that the dual statement in homology holds. Next, note that $j_*(1) = c_1(\mathcal{O}_{\tilde{X}}(D))$. Moreover, the restriction of $\mathcal{O}_{\tilde{X}}(D)$ to D is isomorphic to $\mathcal{O}_D(-1)$.

This implies $-h = j^*(j_*(1))$ and so the projection formula yields

$$-j_*(h \cup \alpha \cup \beta) = j_*(1) \cup j_*(\alpha \cup \beta).$$

This concludes the proof by (3). □

2.2. Eigenvalues of conjugate endomorphisms.

Let X be a smooth complex projective variety with endomorphism f and let σ be an automorphism of $\mathbb{C}$. Via base change, f induces an endomorphism f^σ of X^σ. If an explicit embedding of X into some projective space $\mathbb{P}^N$ with homogeneous coordinates $z = [z_0 : \ldots : z_N]$ is given, then f^σ is determined by

$$f^\sigma(\sigma(z))) = \sigma(f(z))$$

for all $z \in X$, where σ acts on each homogeneous coordinate simultaneously. On cohomology, we obtain linear maps

$$f^* : H^{p,q}(X) \longrightarrow H^{p,q}(X) \quad \text{and} \quad (f^\sigma)^* : H^{p,q}(X^\sigma) \longrightarrow H^{p,q}(X^\sigma).$$

These maps commute with the σ-linear isomorphism

$$H^{p,q}(X) \xrightarrow{\sim} H^{p,q}(X^\sigma)$$

induced by (1). This observation proves:

LEMMA 11. *The set of eigenvalues of $(f^\sigma)^*$ on $H^{p,q}(X^\sigma)$ is given by the σ-conjugate of the set of eigenvalues of f^* on $H^{p,q}(X)$.*

2.3. The j-invariant of elliptic curves.

Recall that the j-invariant of an elliptic curve E with affine Weierstrass equation $y^2 = 4x^3 - g_2 x - g_3$ equals

$$j(E) = 1728 \cdot \frac{g_2^3}{g_2^3 - 27g_3^2}.$$

Two elliptic curves are isomorphic if and only if their j-invariants coincide. From the above formula, we deduce $j(E^\sigma) = \sigma(j(E))$ for all $\sigma \in \text{Aut}(\mathbb{C})$. For an element τ in the upper half-plane $\mathbb{H}$, we use the notation

$$E_\tau := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \quad \text{and} \quad j(\tau) := j(E_\tau). \tag{4}$$

Then, j induces an isomorphism between any fundamental domain of the action of the modular group $\text{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ and $\mathbb{C}$. Moreover, j is holomorphic on $\mathbb{H}$ with a cusp of order one at $i \cdot \infty$.

2.4. Kummer K3 surfaces and theta constants.

Let $M \in M_2(\mathbb{C})$ be a symmetric matrix whose imaginary part is positive definite. Then,

$$A_M := \mathbb{C}^2/(\mathbb{Z}^2 + M\mathbb{Z}^2)$$

is a principal polarized abelian surface. The associated Kummer K3 surface $K3(A_M)$ is the quotient of the blow-up of A_M at its 16 2-torsion points by the involution $\cdot(-1)$. Equivalently, $K3(A_M)$ is the blow-up of $A_M/(-1)$ at its 16 singular points.

Let L_M be a symmetric line bundle on A_M which induces the principal polarization on A_M. The linear series $|L_M^{\otimes 2}|$ then defines a morphism $A_M \longrightarrow \mathbb{P}^3$. This morphism induces an isomorphism of $A_M/(-1)$ with a degree-four hypersurface

$$\{F_M = 0\} \subseteq \mathbb{P}^3.$$

The coefficients of F_M are given by homogeneous degree-12 expressions in the coordinates of Riemann's second-order theta constant $\Theta_2(M) \in \mathbb{P}^3$; see [8] and also [12, Example 1.1]. This constant is defined as

$$\Theta_2(M) := [\Theta_2[0,0](M) : \Theta_2[1,0](M) : \Theta_2[0,1](M) : \Theta_2[1,1](M)]. \quad (5)$$

Here, for $\delta \in \{0,1\}^2$, the complex number $\Theta_2[\delta](M)$ denotes the Fourier series

$$\Theta_2[\delta](M) := \sum_{n \in \mathbb{Z}^2} e^{2\pi i \cdot Q_M(n+\delta/2)}, \quad (6)$$

where $Q_M(z)$ is the quadratic form $z^t M z$, associated with M.

The above discussion allows us to calculate conjugates of $K3(A_M)$ explicitly.

LEMMA 12. *If $\sigma(\Theta_2(M)) = \Theta_2(M')$ holds for some automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$, then*

$$K3(A_M)^\sigma \cong K3(A_{M'}).$$

Proof. As mentioned above, the coefficients of F_M and $F_{M'}$ are polynomial expressions in the coordinates of $\Theta_2(M)$ and $\Theta_2(M')$ respectively. The action of σ therefore maps the polynomial F_M to $F_{M'}$ and hence $\{F_M = 0\}$ to $\{F_{M'} = 0\}$. Moreover, this action maps the 16 singular points of $\{F_M = 0\}$ to the 16 singular points of $\{F_{M'} = 0\}$. The lemma follows from the above description of $K3(A_M)$ and $K3(A_{M'})$ as smooth models of $\{F_M = 0\}$ and $\{F_{M'} = 0\}$ respectively. $\square$

REMARK 13. The linear series $|L_M^{\otimes 3}|$ defines an embedding of A_M into $\mathbb{P}^8$. It is in principle possible to use this embedding in order to calculate conjugates A_M^σ of A_M. In the preceding section we only presented the analogous (easier) calculation for the associated Kummer K3 surface which will suffice for our purposes.

3. Proof of Theorem 5

In this section, we prove Theorem 5 from the introduction. For this purpose, let us fix a subfield $K \subseteq \mathbb{C}$, different from $\mathbb{R}$ and $\mathbb{C}$, which is not contained in an imaginary quadratic extension of $\mathbb{Q}$. We then need to construct for any $p \geq 1$ and in any dimension $n \geq p+1$ conjugate smooth complex projective varieties X, X^σ with $H^{p,p}(X, K) \not\cong H^{p,p}(X^\sigma, K)$. After taking products with $\mathbb{P}^{n-2}$, it clearly suffices to settle the case $p = 1$ and $n = 2$.

We denote by $K_\mathbb{R} := K \cap \mathbb{R}$ the real subfield of K. The proof of Theorem 5 for $p = 1$ and $n = 2$ is now divided into four different cases. Cases 1 and 2 deal with $K_\mathbb{R} \neq \mathbb{Q}$; in Cases 3 and 4 we settle $K_\mathbb{R} = \mathbb{Q}$.

In Cases 1–3 we will consider for $\tau \in \mathbb{H}$ the elliptic curve E_τ with associated j-invariant $j(\tau)$ from (4), and use the following:

LEMMA 14. *Let $L \subset \mathbb{C}$ be a subfield. Then we have, for any $a, b \in \mathbb{R}_{>0}$,*

$$
\dim(H^{1,1}(E_{ia} \times E_{ib}, L)) = \begin{cases} 2 & \text{if } a/b \notin L \text{ and } a \cdot b \notin L, \\ 3 & \text{if } a/b \in L \text{ and } a \cdot b \notin L, \\ & \quad \text{or if } a/b \notin L \text{ and } a \cdot b \in L, \\ 4 & \text{if } a/b \in L \text{ and } a \cdot b \in L. \end{cases}
$$

Proof. For $j = 1, 2$, we denote the holomorphic coordinate on the jth factor of $E_{ia} \times E_{ib}$ by $z_j = x_j + iy_j$. Then there are basis elements

$$\alpha_1, \beta_1 \in H^1(E_{ia}, \mathbb{Z}) \quad \text{and} \quad \alpha_2, \beta_2 \in H^1(E_{ib}, \mathbb{Z})$$

such that

$$dz_1 = \alpha_1 + ia \cdot \beta_1 \in H^{1,0}(E_{ia}) \quad \text{and} \quad dz_2 = \alpha_2 + ib \cdot \beta_2 \in H^{1,0}(E_{ib}).$$

We deduce that the following four $(1,1)$-classes form a basis of $H^{1,1}(E_{ia} \times E_{ib})$:

$$\alpha_1 \cup \beta_1, \quad \alpha_2 \cup \beta_2, \quad \alpha_1 \cup \alpha_2 + ab \cdot \beta_1 \cup \beta_2 \quad \text{and} \quad \alpha_1 \cup \beta_2 + (a/b) \cdot \alpha_2 \cup \beta_1.$$

The lemma follows. $\qquad\square$

Case 1. $K_\mathbb{R}$ *is uncountable.*

The restriction of the j-invariant to $i \cdot \mathbb{R}_{\geq 1}$ is injective. Since $K_\mathbb{R}$ is uncountable, it follows that there is some $\lambda \geq 1$ in $K_\mathbb{R}$ such that $j(i\lambda)$ is transcendental.

By the assumptions, $K_\mathbb{R}$ is different from $\mathbb{R}$. The additive action of $K_\mathbb{R}$ on $\mathbb{R}$ has therefore more than one orbit and so $\mathbb{R}_{\geq 1} \setminus K_\mathbb{R}$ is uncountable. As above, it follows that there is some $\mu \in \mathbb{R}_{\geq 1} \setminus K_\mathbb{R}$ such that $j(i\mu)$ is transcendental. Hence, there is some $\sigma \in \text{Aut}(\mathbb{C})$ with $\sigma(j(i\lambda)) = j(i\mu)$. Since $j(i) = 1$, it follows from

the discussion in Section 2.3 that

$$X := E_{i\lambda} \times E_i \quad \text{with} \quad X^\sigma \cong E_{i\mu} \times E_i.$$

Since $\lambda \in K$ and $\mu \notin K$, it follows from Lemma 14 that $H^{1,1}(X, K)$ and $H^{1,1}(X^\sigma, K)$ are not equidimensional. This concludes Case 1.

Case 2. $K_{\mathbb{R}}$ is countable and $K_{\mathbb{R}} \neq \mathbb{Q}$.
  Here we will need the following lemma.

LEMMA 15. *Let $\lambda \in \mathbb{R}_{>0}$ be irrational, and let $U \subseteq \mathbb{R}_{>0}$ be an uncountable subset. Then there is some $\mu \in U$ such that $j(\mu)$ and $j(\lambda\mu)$ are algebraically independent over $\mathbb{Q}$.*

Proof. For a contradiction, suppose that $j(\mu)$ and $j(\lambda\mu)$ are algebraically dependent over $\mathbb{Q}$ for all $\mu \in U$. Since the polynomial ring in two variables over $\mathbb{Q}$ is countable, whereas U is uncountable, we may assume that $j(\mu)$ and $j(\lambda\mu)$ satisfy the same polynomial relation for all $\mu \in U$. Any uncountable subset of $\mathbb{R}$ contains an accumulation point. Hence, the identity theorem yields a polynomial relation between the holomorphic functions $j(\tau)$ and $j(\lambda\tau)$ in the variable $\tau \in \mathbb{H}$. That is,

$$\sum_{l=0}^{n} c_l(j(\tau)) \cdot j(\lambda\tau)^l = 0,$$

where $c_l(j(\tau))$ is a polynomial in $j(\tau)$ which is nontrivial for $l = n$. We may assume that n is the minimal integer such that a polynomial relation as above exists. The modular form $j(\tau)$ does not satisfy any nontrivial polynomial relation since it has a cusp of order one at $i\infty$. Thus, $n \geq 1$. For $k \in \mathbb{Z}$, we have $j(\tau) = j(\tau + k)$ and so the above identity yields

$$\sum_{l=0}^{n} c_l(j(\tau)) \cdot (j(\lambda\tau)^l - j(\lambda\tau + \lambda k)^l) = 0,$$

for all $k \in \mathbb{Z}$. Since λ is irrational, $\lambda\tau$ and $\lambda\tau + \lambda k$ do not lie in the same $\mathrm{SL}_2(\mathbb{Z})$ orbit and so $j(\lambda\tau) - j(\lambda\tau + \lambda k)$ is nonzero for all $k \in \mathbb{Z}$. Thus,

$$\sum_{l=1}^{n} c_l(j(\tau)) \cdot \sum_{h=0}^{l-1} j(\lambda\tau)^h j(\lambda\tau + \lambda k)^{l-1-h} = 0.$$

If we now choose a sequence of integers $(k_m)_{m \geq 1}$ such that λk_m tends to zero modulo $\mathbb{Z}$, then the above identity tends to the identity

$$\sum_{l=1}^{n} c_l(j(\tau)) \cdot l \cdot j(\lambda\tau)^{l-1} = 0.$$

This contradicts the minimality of n. Lemma 15 follows. $\qquad\square$

Since $K_\mathbb{R}$ is countable, it follows that, for any $t > 0$,

$$U_t := \{\mu \in \mathbb{R}_{\geq 1} \mid t\mu^2 \notin K\}$$

is uncountable. By the assumptions in Case 2, $K_\mathbb{R}$ contains a positive irrational number λ. Additionally, we pick a positive irrational number $\lambda' \notin K$.

Then, by Lemma 15, there are elements $\mu \in U_\lambda$ and $\mu' \in U_{\lambda'}$ such that $j(i\mu)$ and $j(i\lambda\mu)$, as well as $j(i\mu')$ and $j(i\lambda'\mu')$, are algebraically independent over $\mathbb{Q}$. It follows that for some $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have

$$X := E_{i\lambda\mu} \times E_{i\mu} \quad \text{with} \quad X^\sigma \cong E_{i\lambda'\mu'} \times E_{i\mu'}.$$

Since $\lambda \in K$ and $\lambda\mu^2, \lambda', \lambda'\mu'^2 \notin K$, it follows from Lemma 14 that $H^{1,1}(X, K)$ and $H^{1,1}(X^\sigma, K)$ are not equidimensional. This concludes Case 2.

Case 3. K is uncountable and $K_\mathbb{R} = \mathbb{Q}$.

Since K is uncountable, there are elements $\tau, \tau' \in \mathbb{H}$ with $\tau, \overline{\tau'} \in K$ such that $j(\tau)$ and $j(\tau')$ are algebraically independent over $\mathbb{Q}$. Also, there are positive numbers $\mu, \mu' \in \mathbb{R}_{>0}$ with $\mu\mu', \mu/\mu' \notin K_\mathbb{R} = \mathbb{Q}$ such that $j(i\mu)$ and $j(i\mu')$ are algebraically independent over $\mathbb{Q}$. For some $\sigma \in \mathrm{Aut}(\mathbb{C})$, we then have

$$X := E_\tau \times E_{\tau'} \quad \text{with } X^\sigma \cong E_{i\mu} \times E_{i\mu'}.$$

Since $\tau, \overline{\tau'} \in K$, the space $H^{1,1}(X, K)$ is at least three-dimensional. Conversely, $H^{1,1}(X^\sigma, K)$ is two-dimensional by Lemma 14. This concludes Case 3.

Case 4. K is countable and $K_\mathbb{R} = \mathbb{Q}$.

This case is slightly more difficult; instead of using products of elliptic curves, we will use Kummer K3 surfaces and their theta constants; see Section 2.4. We begin with the definition of certain families of such surfaces. For $t = t_1 + it_2 \in \mathbb{C}$ with $t_1 \neq 0$ and $\mu \in \mathbb{R}_{>0}$, we consider the symmetric matrix

$$M(\mu, t) := i\frac{\mu}{2t_1} \cdot \begin{pmatrix} 2t_1 & 1 \\ 1 & |t|^2 \end{pmatrix}.$$

For a suitable choice of $t \in \mathbb{C}$, the matrix $-iM(\mu, t)$ is positive definite for all $\mu > 0$ and so the abelian surface $A_{M(\mu,t)}$, as well as its associated Kummer K3 surface, exists. For such t, we have the following lemma, where $\hat{A}$ denotes the dual of the abelian surface A.

LEMMA 16. *Let $L \subseteq \mathbb{C}$ be a subfield, let $\mu > 0$ and let $t = t_1 + it_2 \in \mathbb{C}$ such that $-i \cdot M(\mu, t)$ is positive definite. If t_1, $|t|^2$ and $\det(M(\mu, t))$ do not lie in L, then*

$$\dim(H^{1,1}(K3(\hat{A}_{M(\mu,t)}), L)) = \begin{cases} 17 & \text{if } (|t|^2 + 2t_1 \cdot L) \cap L = \emptyset, \\ 18 & \text{otherwise.} \end{cases}$$

Proof. Fix $t \in \mathbb{C}$ and $\mu > 0$ such that $-i \cdot M(\mu, t)$ is positive definite and assume that t_1, $|t|^2$ and $\det(M(\mu, t))$ do not lie in L. The rational degree-two Hodge structure of a Kummer surface $K3(A)$ is the direct sum of 16 divisor classes with the rational degree-two Hodge structure of A. It therefore remains to investigate the dimension of $H^{1,1}(\hat{A}_{M(\mu,t)}, L)$.

We denote the holomorphic coordinates on $\mathbb{C}^2$ by $z = (z_1, z_2)$, where $z_j = x_j + i y_j$. The cohomology of $\hat{A}_{M(\mu,t)}$ is given by the homology of $A_{M(\mu,t)}$ and so

$$\alpha_1 = dx_1, \quad \alpha_2 = dx_2, \quad \alpha_3 = \mu/(2t_1) \cdot (2t_1 dy_1 + dy_2),$$
$$\alpha_4 = \mu/(2t_1) \cdot (dy_1 + |t|^2 dy_2)$$

form a basis of $H^1(\hat{A}_{M(\mu,t)}, \mathbb{Q})$. Next, $H^{1,1}(\hat{A}_{M(\mu,t)})$ has basis $dz_1 \cup d\bar{z}_1$, $dz_1 \cup d\bar{z}_2$, $dz_2 \cup d\bar{z}_1$ and $dz_2 \cup d\bar{z}_2$. This basis can be expressed in terms of $\alpha_j \cup \alpha_k$, where $1 \le j < k \le 4$. Applying the Gauss algorithm then yields the following new basis of $H^{1,1}(\hat{A}_{M(\mu,t)})$:

$$\Omega_1 := \alpha_2 \cup \alpha_4 + \alpha_1 \cup \alpha_3,$$
$$\Omega_2 := \alpha_1 \cup \alpha_4 - |t|^2 \cdot \alpha_1 \cup \alpha_3,$$
$$\Omega_3 := \alpha_2 \cup \alpha_3 - 2t_1 \cdot \alpha_1 \cup \alpha_3,$$
$$\Omega_4 := \alpha_3 \cup \alpha_4 - \det(M(\mu, t)) \cdot \alpha_1 \cup \alpha_2.$$

From this description it follows that if a linear combination $\sum \lambda_i \Omega_i$ is L-rational, then all λ_i lie in L. Moreover, since $\det(M(\mu, t)) \notin L$, the coefficient λ_4 needs to vanish.

Since $t_1, |t|^2 \notin L$, neither Ω_2 nor Ω_3 is L-rational. We conclude that $H^{1,1}(\hat{A}_{M(\mu,t)}, L)$ is two-dimensional if $|t|^2 + 2t_1 \cdot l_1 = l_2$ has a solution $l_1, l_2 \in L$, and it is one-dimensional otherwise. The lemma follows. $\qquad\square$

In the following we will stick to parameters t that are contained in a sufficiently small neighborhood of $1/3 + 3i$. For such t, the matrix $-i \cdot M(\mu, t)$ is positive definite. The reason for the explicit choice of the base point $1/3 + 3i$ is that it slightly simplifies the proof of the subsequent lemma. In order to state it, we call a point in $\mathbb{P}^3$ transcendental if its coordinates in some standard affine chart are algebraically independent over $\mathbb{Q}$. Equivalently, $z \in \mathbb{P}^3$ is transcendental if and only if $P(z) \ne 0$ for all nontrivial homogeneous polynomials P with rational coefficients. That is, the transcendental points of $\mathbb{P}^3$ are those which lie in the complement of the (countable) union of hypersurfaces which can be defined over $\mathbb{Q}$. It is important to note that $\operatorname{Aut}(\mathbb{C})$ acts transitively on this set of points.

LEMMA 17. *There is a neighborhood $V \subseteq \mathbb{C}$ of $1/3 + 3i$, such that for all $t = t_1 + it_2 \in V$ with 1, t_1 and $|t|^2$ linearly independent over $\mathbb{Q}$, the following holds. Any uncountable subset $U \subseteq \mathbb{R}_{>0}$ contains a point $\mu \in U$ with:*

(1) *the matrix* $-i \cdot M(\mu, t)$ *is positive definite;*

(2) *the determinant of* $M(\mu, t)$ *is not rational;*

(3) *the theta constant* $\Theta_2(M(\mu, t))$ *is a transcendental point of* $\mathbb{P}^3$.

Proof. We define the quadratic form

$$Q(z) := 2t_1 z_1^2 + 2z_1 z_2 + |t|^2 z_2^2,$$

where $z = (z_1, z_2) \in \mathbb{R}^2$. For $\delta \in \{0, 1\}^2$, the homogeneous coordinate $\Theta_2[\delta](M(\mu, t))$ of the theta constant $\Theta_2(M(\mu, t))$ is then given by

$$\Theta_2[\delta](M(\mu, t)) = \sum_{n \in \mathbb{Z}^2} \exp\left(-\frac{\pi\mu}{t_1} \cdot Q(n + \delta/2)\right); \tag{7}$$

see (6). At the point $t = 1/3 + 3i$, we have

$$Q(z)|_{t=1/3+3i} = \frac{2}{3} \cdot (z_1 + 3z_2/2)^2 + \frac{137}{18} \cdot z_2^2.$$

This shows that there is a neighborhood V of $1/3 + 3i$ such that $-i \cdot M(\mu, t)$ is positive definite for all $t \in V$ and all $\mu > 0$. For such t, the function in (7) is a modular form in the variable $i \cdot \mu \in \mathbb{H}$; see [7].

Let us now pick some $t \in V$ with 1, t_1 and $|t|^2$ linearly independent over $\mathbb{Q}$. Then $-i \cdot M(\mu, t)$ is positive definite and so $\det(M(\mu, t))$ is a nonzero multiple of μ^2. After possibly removing countably many points of U, we may therefore assume

$$\det(M(\mu, t)) \notin \mathbb{Q}$$

for all $\mu \in U$.

For a contradiction, we now assume that there is no $\mu \in U$ such that $\Theta_2(M(\mu, t))$ is a transcendental point of $\mathbb{P}^3$. Since the polynomial ring in four variables over $\mathbb{Q}$ is countable, we may then assume that there is one homogeneous polynomial P with $P(\Theta_2(M(\mu, t))) = 0$ for all $\mu \in U$. Since $U \subseteq \mathbb{R}_{>0}$ is uncountable, it contains an accumulation point. Then the identity theorem yields

$$P(\Theta_2(M(-i\tau, t))) = 0, \tag{8}$$

where the left hand side is considered as a holomorphic function in $\tau \in \mathbb{H}$.

For $\tau \to i\infty$, the modular form $\Theta_2[\delta](M(-i\tau, t))$ from (7) is dominated by the summand where the exponent $Q(n)$ with $n \in \mathbb{N}^2 + \delta$ is minimal. After possibly shrinking V, these minima $n_\delta \in \mathbb{N}^2 + \delta$ of $Q(n)$ are given as follows:

$$n_{0,0} = (0, 0), \quad n_{1,0} = \pm(1/2, 0), \quad n_{0,1} = \pm(-1, 1/2)$$

and

$$n_{1,1} = \pm(-1/2, 1/2).$$

Noting that $Q(n_{0,0})$ vanishes, we conclude that for $\tau \to i\infty$, the monomial

$$\Theta_2[0, 0](M(-i\tau, t))^h \cdot \Theta_2[1, 0](M(-i\tau, t))^j$$
$$\cdot \Theta_2[0, 1](M(-i\tau, t))^k \cdot \Theta_2[1, 1](M(-i\tau, t))^l$$

is dominated by the summand

$$2 \cdot \exp\left(\frac{\pi i \tau}{t_1} \cdot (j \cdot Q(n_{1,0}) + k \cdot Q(n_{0,1}) + l \cdot Q(n_{1,1}))\right).$$

The left hand side in (8) is then dominated by those summands for which

$$j \cdot Q(n_{1,0}) + k \cdot Q(n_{0,1}) + l \cdot Q(n_{1,1})$$

is minimal. We will therefore arrive at a contradiction as soon as we have seen that this summand is unique. That is, it suffices to see that $Q(n_{1,0})$, $Q(n_{0,1})$ and $Q(n_{1,1})$ are linearly independent over $\mathbb{Q}$. In order to see the latter, we calculate

$$Q(n_{1,0}) = t_1/2, \quad Q(n_{0,1}) = |t|^2/4 + 2t_1 - 1$$

and

$$Q(n_{1,1}) = |t|^2/4 + t_1/2 - 1/2.$$

The claim is now obvious since 1, t_1 and $|t|^2$ are linearly independent over $\mathbb{Q}$ by the assumptions. This finishes the proof of the lemma. □

We are now able to conclude Case 4. Let V be the neighborhood of $1/3 + 3i$ from Lemma 17. Since $K_{\mathbb{R}} = \mathbb{Q}$ and since K is not contained in any imaginary quadratic extension of $\mathbb{Q}$, we may pick some $t = t_1 + it_2 \in K \cap V$ which is not quadratic over $\mathbb{Q}$. Then t_1 is not rational since otherwise $(t - t_1)^2$ would lie in $K_{\mathbb{R}} = \mathbb{Q}$, which yielded a quadratic relation for t over $\mathbb{Q}$. It follows that 1, $t + \bar{t} = 2t_1$ and $t \cdot \bar{t} = |t|^2$ are linearly independent over $\mathbb{Q}$, as otherwise $\bar{t}$ would lie in K and so $t + \bar{t} = 2t_1 \in K_{\mathbb{R}} = \mathbb{Q}$ would have been rational. Hence, the assumptions of Lemma 17 are satisfied and so there is some $\mu \in \mathbb{R}_{>0}$ such that the pair (μ, t) satisfies (1)–(3) in Lemma 17.

Next, we consider $t' = t'_1 + 3i \in V$ with 1, t'_1 and t'^2_1 linearly independent over $\mathbb{Q}$. Since V is a neighborhood of $1/3 + 3i$, there are uncountably many values for t'_1 such that t' has the above property. We claim that we can choose t'_1 within this uncountable set such that additionally

$$2t'_1 \lambda_1 = \lambda_2 + |t'|^2 \tag{9}$$

has no solution $\lambda_1, \lambda_2 \in K$. In order to prove this, suppose that t'_1 is a solution of (9) for some $\lambda_1, \lambda_2 \in K$. Since $|t'|^2$ is a real number, it follows that t'_1 lies

in the set of quotients x/y where x and y are imaginary parts of some elements of K. Since K is countable, so is the latter set. Our claim follows since we can choose t_1' within an uncountable set. That is, we have just shown that there is a point $t' = t_1' + 3i \in V$ with 1, t_1' and $|t'|^2$ linearly independent over $\mathbb{Q}$ such that additionally, (9) has no solution in K. Then again the assumptions of Lemma 17 are met and so there is some $\mu' \in \mathbb{R}_{>0}$ such that the pair (μ', t') satisfies (1)–(3) in Lemma 17.

Since (μ, t) and (μ', t') satisfy Lemma 17, $\Theta_2(M(\mu, t))$ and $\Theta_2(M(\mu', t'))$ are transcendental points of $\mathbb{P}^3$. Because $\mathrm{Aut}(\mathbb{C})$ acts transitively on such points it follows that there is some automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$ with

$$\sigma(\Theta_2(M(\mu, t))) = \Theta_2(M(\mu', t')).$$

As the functor $A \mapsto \hat{A}$ on the category of abelian varieties commutes with the $\mathrm{Aut}(\mathbb{C})$-action, it therefore follows from Lemma 12 that

$$X := K3(\hat{A}_{M(\mu, t)}) \quad \text{with} \quad X^\sigma \cong K3(\hat{A}_{M(\mu', t')}).$$

By our choices, t_1, $|t|$ and $\det(M(\mu, t))$ lie in $\mathbb{R} \setminus \mathbb{Q}$ and the same holds for the pair (μ', t'). Since $K_{\mathbb{R}} = \mathbb{Q}$, it follows that (μ, t), as well as (μ', t'), satisfies the assumptions of Lemma 16. Since (9) has no solution in K, whereas

$$2t_1\lambda_1 = \lambda_2 + |t|^2$$

has the solution $\lambda_1 = t$ and $\lambda_2 = t^2$ in K, it follows from Lemma 16 that $H^{1,1}(X, K)$ and $H^{1,1}(X^\sigma, K)$ are not equidimensional. This concludes Case 4 and hence finishes the proof of Theorem 5.

REMARK 18. Theorem 5 does not remain true if one restricts to smooth complex projective varieties which can be defined over $\overline{\mathbb{Q}}$. Indeed, for each smooth complex projective variety X there is a finitely generated extension K_X of $\mathbb{Q}$ such that for all $p \geq 0$ the group $H^{p,p}(X, \mathbb{C})$ is generated by K_X-rational classes. As there are only countably many varieties over $\overline{\mathbb{Q}}$, it follows that there is an extension K of $\mathbb{Q}$ which is generated by countably many elements such that for each smooth complex projective variety X over $\overline{\mathbb{Q}}$ and for each $p \geq 0$, the dimension of $H^{p,p}(X, K)$ equals $h^{p,p}(X)$. The above claim follows, since $h^{p,p}(X)$ is invariant under conjugation.

4. The Charles–Voisin construction

In this section we carry out a variant of a general construction method due to Charles and Voisin [3, 18]. The proofs of Propositions 19 and 20 below will then be the technical heart of the proof of Theorem 6 in Section 6.

We start with a smooth complex projective surface Y with $b_1(Y) = 0$ and automorphisms $f, f' \in \mathrm{Aut}(Y)$. Then we pick an embedding

$$i: Y \hookrightarrow \mathbb{P}^N$$

and assume that f^* and f'^* fix the pullback i^*h of the hyperplane class $h \in H^2(\mathbb{P}^N, \mathbb{Z})$.

For a general choice of points u, v, w and t of $\mathbb{P}^N$ and y of Y, the following smooth subvarieties of $Y \times Y \times \mathbb{P}^N$ are disjoint:

$$
\begin{aligned}
Z_1 &:= Y \times y \times u, \quad Z_2 := \Gamma_{\mathrm{id}_Y} \times v, \quad Z_3 := \Gamma_f \times w, \\
Z_4 &:= \Gamma_{f'} \times t, \quad Z_5 := y \times \Gamma_i,
\end{aligned}
\tag{10}
$$

where Γ denotes the graph of a morphism. The blow-up

$$X := Bl_{Z_1 \cup \cdots \cup Z_5}(Y \times Y \times \mathbb{P}^N)$$

of $Y \times Y \times \mathbb{P}^N$ along the union $Z_1 \cup \cdots \cup Z_5$ is a smooth complex projective variety. Since $b_1(Y) = 0$ and $\dim(Y) = 2$, it follows from the description of the cohomology of blow-ups (see Section 2.1) that the cohomology algebra of X is generated by degree-two classes.

Next, let σ be any automorphism of $\mathbb{C}$. Then the automorphisms f and f' of Y induce automorphisms f^σ and f'^σ of Y^σ. Since conjugation commutes with blow-ups, we have

$$X^\sigma = Bl_{Z_1^\sigma \cup \cdots \cup Z_5^\sigma}(Y^\sigma \times Y^\sigma \times \mathbb{P}^N),$$

where we identified $\mathbb{P}^N$ with its conjugate $\mathbb{P}^{N^\sigma}$, and where

$$
\begin{aligned}
Z_1^\sigma &= Y^\sigma \times y^\sigma \times u^\sigma, \quad Z_2^\sigma = \Gamma_{\mathrm{id}_{Y^\sigma}} \times v^\sigma, \quad Z_3^\sigma = \Gamma_{f^\sigma} \times w^\sigma, \\
Z_4^\sigma &= \Gamma_{f'^\sigma} \times t^\sigma, \quad Z_5^\sigma = y^\sigma \times \Gamma_{i^\sigma}.
\end{aligned}
$$

Here u^σ, v^σ, w^σ and t^σ are points on $\mathbb{P}^N$, $y^\sigma \in Y^\sigma$, and $i^\sigma : Y^\sigma \hookrightarrow \mathbb{P}^N$ is the inclusion induced by i. The pullback of the hyperplane class via i^σ is denoted by $i^{\sigma*}h^\sigma$.

In the next proposition, we will assume that the surface Y has the following properties:

(A1) There exist elements $\alpha, \beta \in H^{1,1}(Y, \mathbb{Q})$ with $\alpha^2 = \beta^2 = 0$ and $\alpha \cup \beta \neq 0$.

(A2) The sets of eigenvalues of f^* and f'^* on $H^2(Y, \mathbb{C})$ are distinct.

Then, for a smooth complete intersection subvariety

$$T \subseteq X,$$

with $\dim(T) \geq 4$, the following holds:

PROPOSITION 19. *Suppose that* (A1) *and* (A2) *hold, and let* $K \subseteq \mathbb{C}$ *be a subfield. Then any weak isomorphism between the K-multilinear intersection forms on* $H^2(T, K)$ *and* $H^2(T^\sigma, K)$ *induces an isomorphism of graded K-algebras*

$$\psi : H^*(Y, K) \xrightarrow{\sim} H^*(Y^\sigma, K),$$

with the following two properties:

(P1) *in degree two,* ψ *maps* i^*h *to a multiple of* $i^{\sigma*}h^\sigma$;

(P2) *the isomorphism* ψ *commutes with the induced actions of* f *and* f', *that is*

$$\psi \circ f^* = (f^\sigma)^* \circ \psi \quad and \quad \psi \circ (f')^* = (f'^\sigma)^* \circ \psi.$$

Proposition 19 has an analogue for isomorphisms between intersection forms on $H^{1,1}(-, K)$. In order to state it, we need the following variant of (A2):

(A3) the sets of eigenvalues of f^* and f'^* on $H^{1,1}(Y, \mathbb{C})$ are distinct and Aut($\mathbb{C}$)-invariant.

Note that f^* and f'^* are defined on integral cohomology and so their sets of eigenvalues on $H^2(Y, \mathbb{C})$ – but not on $H^{1,1}(Y, \mathbb{C})$ – are automatically Aut($\mathbb{C}$)-invariant. For this reason, we did not have to impose this additional condition in (A2).

PROPOSITION 20. *Suppose that* (A1) *and* (A3) *hold, and let* $K \subseteq \mathbb{C}$ *be a subfield which is stable under complex conjugation. Then any weak isomorphism between the K-multilinear intersection forms on* $H^{1,1}(T, K)$ *and* $H^{1,1}(T^\sigma, K)$ *induces an isomorphism of graded K-algebras*

$$\psi : H^{*,*}(Y, K) \xrightarrow{\sim} H^{*,*}(Y^\sigma, K),$$

which satisfies (P1) *and* (P2) *of Proposition* 19.

REMARK 21. The assumption (A1) in the above propositions is only needed if $\dim(T) = 4$.

In the following two subsections we prove Propositions 19 and 20 respectively; important steps will be similar to the Charles–Voisin arguments in [3, 18].

4.1. Proof of Proposition 19.   Suppose that there is a K-linear isomorphism

$$\phi' : H^2(T, K) \xrightarrow{\sim} H^2(T^\sigma, K), \tag{11}$$

which induces a weak isomorphism between the respective multilinear intersection forms.

By the Lefschetz hyperplane theorem, the natural maps

$$H^k(X, K) \longrightarrow H^k(T, K) \quad \text{and} \quad H^k(X^\sigma, K) \longrightarrow H^k(T^\sigma, K) \qquad (12)$$

are isomorphisms for $k < n$ and injective for $k = n$, where $n := \dim(T)$. Using this we will identify classes on X and X^σ of degree $\leq n$ with classes on T and T^σ respectively.

We denote by $SH^2(-, K)$ the subalgebra of $H^*(-, K)$ that is generated by $H^2(-, K)$. Its quotient by all elements of degree $\geq r + 1$ is denoted by $SH^2(-, K)^{\leq r}$. Since $\dim(T) \geq 4$, we obtain from (12) the canonical isomorphisms

$$SH^2(X, K)^{\leq 4} \xrightarrow{\sim} SH^2(T, K)^{\leq 4} \quad \text{and} \quad SH^2(X^\sigma, K)^{\leq 4} \xrightarrow{\sim} SH^2(T^\sigma, K)^{\leq 4}.$$

CLAIM 1. *The isomorphism ϕ' from* (11) *induces a unique isomorphism*

$$\phi \colon SH^2(X, K)^{\leq 4} \xrightarrow{\sim} SH^2(X^\sigma, K)^{\leq 4}$$

of graded K-algebras.

Proof. In degree two, we define ϕ to coincide with ϕ' from (11). Since the respective algebras are generated in degree two, this determines ϕ uniquely as a homomorphism of K-algebras; we have to check that it is well-defined though. In order to see the latter, let $\alpha_1, \ldots, \alpha_r$ and $\beta_1, \ldots, \beta_r$ be elements in $H^2(T, K)$. Then we have to prove

$$\sum_i \alpha_i \cup \beta_i = 0 \quad \Rightarrow \quad \sum_i \phi'(\alpha_i) \cup \phi'(\beta_i) = 0.$$

Let us assume that $\sum_i \alpha_i \cup \beta_i = 0$. Since ϕ' induces a weak isomorphism between the corresponding intersection forms, this implies

$$\sum_i \phi'(\alpha_i) \cup \phi'(\beta_i) \cup \eta = 0 \quad \text{in} \quad H^{2n}(T^\sigma, K),$$

for all $\eta \in SH^2(T^\sigma, K)^{2n-4}$. The class $\sum_i \phi'(\alpha_i) \cup \phi'(\beta_i) \cup \eta$ lies in $SH^2(T^\sigma, K)$ and hence it is a pullback of a class on X. Therefore, the above condition is equivalent to saying that

$$\sum_i \phi'(\alpha_i) \cup \phi'(\beta_i) \cup \eta \cup [T^\sigma] = 0 \quad \text{in} \quad H^{2N+8}(X^\sigma, K),$$

for all $\eta \in SH^2(X^\sigma, K)^{2n-4}$. Since the cohomology of X is generated by degree-two classes, Poincaré duality shows

$$\sum_i \phi'(\alpha_i) \cup \phi'(\beta_i) \cup [T^\sigma] = 0 \quad \text{in} \quad H^{2N-2n+12}(X^\sigma, K).$$

Since $[T^\sigma]$ is the $(N + 4 - n)$th power of some hyperplane class on X^σ, the hard Lefschetz theorem implies

$$\sum_i \phi'(\alpha_i) \cup \phi'(\beta_i) = 0 \quad \text{in} \quad H^4(X^\sigma, K),$$

as we wanted. Similarly, one proves that ϕ'^{-1} induces a well-defined inverse of ϕ. This finishes the proof of the claim. □

From now on, we will work with the isomorphism ϕ of K-algebras from Claim 1 instead of the weak isomorphism of intersection forms ϕ' from (11).

In order to describe the degree-two cohomology of X, we denote by $D_i \subseteq X$ the exceptional divisor above Z_i and we denote by h the pullback of the hyperplane class of $\mathbb{P}^N$ to X. Then, by Lemma 9:

$$H^2(X, K) = \left(\bigoplus_{i=1}^5 [D_i] \cdot K \right) \oplus H^2(Y \times Y, K) \oplus h \cdot K. \qquad (13)$$

Similarly, we denote by $D_i^\sigma \subseteq X^\sigma$ the conjugate of D_i by σ and we denote by h^σ the pullback of the hyperplane class of $\mathbb{P}^N$ to X^σ. This yields

$$H^2(X^\sigma, K) = \left(\bigoplus_{i=1}^5 [D_i^\sigma] \cdot K \right) \oplus H^2(Y^\sigma \times Y^\sigma, K) \oplus h^\sigma \cdot K. \qquad (14)$$

Next, we pick a base point $0 \in Y$ and consider the projections

$$Y \times Y \longrightarrow Y \times 0 \quad \text{and} \quad Y \times Y \longrightarrow 0 \times Y.$$

Using pullbacks, this allows us to view $H^*(Y \times 0, K)$ and $H^*(0 \times Y, K)$ as subspaces of $H^*(Y \times Y, K)$. By assumption, the first Betti number of Y vanishes and so we have a canonical identity

$$H^2(Y \times Y, K) = H^2(Y \times 0, K) \oplus H^2(0 \times Y, K), \qquad (15)$$

of subspaces of $H^2(X, K)$. A similar statement holds on X^σ.

CLAIM 2. *The isomorphism ϕ respects the decompositions in (13) and (14), that is,*

$$\phi(H^2(Y \times Y, K)) = H^2(Y^\sigma \times Y^\sigma, K), \qquad (16)$$

$$\phi([D_i] \cdot K) = [D_i^\sigma] \cdot K \quad \text{for all } i = 1, \dots, 5, \qquad (17)$$

$$\phi(h \cdot K) = h^\sigma \cdot K. \qquad (18)$$

Proof. In order to prove (16), we define S to be the linear subspace of $H^2(X, K)$ which is spanned by all classes whose square is zero. By the ring structure of

the cohomology of blow-ups (see Lemma 10), S is contained in $H^2(Y \times Y, K)$. Furthermore, let S^2 be the subspace of $H^4(X, K)$ which is given by products of elements in S. By assumption (A1), this subspace contains $H^4(Y \times 0, K)$ and $H^4(0 \times Y, K)$. By the ring structure of the cohomology of X, it then follows that $H^2(Y \times Y, K)$ in (13) is equal to the linear subspace of $H^2(X, K)$ that is spanned by those classes whose square lies in S^2.

By Lefschetz's theorem on $(1, 1)$-classes, the cohomology of Y^σ also satisfies (A1). Hence, $H^2(Y^\sigma \times Y^\sigma, K)$ inside $SH^2(X^\sigma, K)^{\leq 4}$ has an intrinsic description similar to the one that we have found for $H^2(Y \times Y, K)$ inside $SH^2(X, K)^{\leq 4}$. This proves (16).

In order to prove (17) and (18), we need the following lemma, also used in [3, 18]. In order to state it, we define for $i = 1, \ldots, 5$ the following kernels:

$$F_i := \ker(\cup[D_i] : H^2(Y \times Y, K) \longrightarrow H^4(X, K)). \tag{19}$$

Using Lemmas 9 and 10, we obtain the following lemma, which is the analogue of Charles's Lemma 7 in [3].

LEMMA 22. *Using the identification* (15)*, the kernels $F_i \subseteq H^2(Y \times Y, K)$ are given as follows:*

$$F_1 = \{(0, \beta) : \beta \in H^2(Y, K)\}, \tag{20}$$

$$F_2 = \{(\beta, -\beta) : \beta \in H^2(Y, K)\}, \tag{21}$$

$$F_3 = \{(f^*\beta, -\beta) : \beta \in H^2(Y, K)\}, \tag{22}$$

$$F_4 = \{(f'^*\beta, -\beta) : \beta \in H^2(Y, K)\}, \tag{23}$$

$$F_5 = \{(\beta, 0) : \beta \in H^2(Y, K)\}. \tag{24}$$

In addition to the above lemma, we have as in [3] the following:

LEMMA 23. *Let $\alpha \in H^2(Y \times Y, K)$ be a nonzero class. Then the images of*

$$\cup \alpha, \cup h, \cup[D_1], \ldots, \cup[D_5] : H^2(Y \times Y, K) \longrightarrow H^4(X, K)$$

are in direct sum, $\cup h$ is injective and

$$\dim(\ker \cup \alpha) < b_2(Y). \tag{25}$$

Proof. Apart from (25), the assertions in Lemma 23 are immediate consequences of the ring structure of the cohomology of blow-ups; see Lemmas 9 and 10.

In order to proof (25), we write

$$\alpha = \alpha_1 + \alpha_2$$

according to the decomposition (15). Without loss of generality, we assume that $\alpha_1 \neq 0$. Then, $\cup \alpha$ restricted to $H^2(0 \times Y, K)$ is injective. Moreover, by Poincaré

duality there is some $\beta_1 \in H^2(Y \times 0, K)$ with

$$\beta_1 \cup \alpha_1 \neq 0.$$

Then, $\beta_1 \cup \alpha$ is nontrivial and does not lie in the image of $\cup \alpha$ restricted to $H^2(0 \times Y, K)$. Thus, $\dim(\text{im}(\cup \alpha)) > b_2(Y)$ and (25) follows. $\qquad \square$

Of course, the obvious analogues of Lemmas 22 and 23 hold on X^σ.

Note the following elementary fact from linear algebra. If a finite number of linear maps $l_1, \ldots, l_r$ between two vector spaces have images in direct sum, then the kernel of a linear combination $\sum \lambda_i l_i$ is given by the intersection of all $\ker(l_i)$ with $\lambda_i \neq 0$.

By Lemma 22, each F_i has dimension $b_2(Y)$ and hence the above linear algebra fact together with Lemma 23 shows that there is a permutation $\rho \in \text{Sym}(5)$ with

$$\phi([D_i] \cdot K) = [D^\sigma_{\rho(i)}] \cdot K.$$

We are now able to prove (18). For some real numbers $a_0, \ldots, a_5$ and for some class $\beta^\sigma \in H^2(Y^\sigma \times Y^\sigma, K)$ we have

$$\phi(h) = a_0 h^\sigma + \sum_{j=1}^{5} a_j [D^\sigma_j] + \beta^\sigma.$$

For $i = 1, \ldots, 4$, the cup product $h \cup [D_i]$ vanishes and hence

$$a_0 h^\sigma \cup [D^\sigma_{\rho(i)}] + \sum_{j=1}^{5} a_j [D^\sigma_j] \cup [D^\sigma_{\rho(i)}] + \beta^\sigma \cup [D^\sigma_{\rho(i)}] = 0.$$

Since the cup product $[D^\sigma_j] \cup [D^\sigma_k]$ vanishes for $j \neq k$, we deduce

$$a_0 h^\sigma \cup [D^\sigma_{\rho(i)}] + a_{\rho(i)} [D^\sigma_{\rho(i)}]^2 + \beta^\sigma \cup [D^\sigma_{\rho(i)}] = 0$$

for all $i = 1, \ldots, 4$. From Lemma 9, it follows that $a_{\rho(i)}$ vanishes for all $i = 1, \ldots, 4$.

If i is such that $\rho(i) \in \{1, \ldots, 4\}$, then

$$h^\sigma \cup [D^\sigma_{\rho(i)}] = 0 \quad \text{and so} \quad \beta^\sigma \cup [D^\sigma_{\rho(i)}] = 0.$$

By Lemma 22, the intersection $\bigcap_{j \neq k} F_j$ is zero for each $k = 1, \ldots, 5$. Since the same holds on X^σ, we deduce that β^σ vanishes. Hence,

$$\phi(h) = a_0 h^\sigma + a_{\rho(5)} [D^\sigma_{\rho(5)}].$$

In $H^4(X, K)$ we have the identity

$$h \cup [D_5] = (i^* h) \cup [D_5] \in H^2(Y \times Y) \cup [D_5],$$

and similarly on X^σ. Since (16) is already proven, we deduce

$$a_0 h^\sigma \cup [D^\sigma_{\rho(5)}] + a_{\rho(5)}[D^\sigma_{\rho(5)}]^2 \in H^2(Y^\sigma \times Y^\sigma) \cup [D^\sigma_{\rho(5)}].$$

This implies $a_{\rho(5)} = 0$. Since ϕ is an isomorphism, $a_0 \neq 0$ follows, which proves (18).

It remains to prove (17). That is, we need to see that $\rho \in \mathrm{Sym}(5)$ is the identity. This will be achieved by an argument similar to that in [3, Lemma 11].

Note that $h \cup [D_i]$, as well as $h^\sigma \cup [D^\sigma_i]$, vanishes for $i \neq 5$ and is nontrivial for $i = 5$. Since (18) is already proven, $\rho(5) = 5$ follows.

By the assumption on Y, f^* and f'^* fix i^*h. Therefore, the intersection $F_2 \cap F_3 \cap F_4$ is nontrivial. Conversely, $F_1 \cap F_i = 0$ for all $i = 2, 3, 4$. Since analogue statements hold on X^σ, we obtain $\rho(1) = 1$.

Next, we use that $F_i \oplus F_j = H^2(Y \times Y, K)$ for all $i = 1, 5$ and $j = 2, 3, 4$. This allows us to define for $2 \leq j, k \leq 4$ endomorphisms $g_{j,k}$ of F_1 via the following composition:

$$g_{j,k} \colon F_1 \hookrightarrow F_5 \oplus F_j \xrightarrow{pr_1} F_5 \hookrightarrow F_1 \oplus F_k \xrightarrow{pr_1} F_1.$$

There is a canonical identification between F_1 and $H^2(Y, K)$. Using Lemma 22, a straightforward calculation then shows

$$\begin{aligned} g_{3,2} = f^*, \quad g_{4,2} = f'^*, \quad g_{4,3} = (f' \circ f^{-1})^*, \\ g_{j,j} = \mathrm{id} \quad \text{and} \quad g_{j,k} = g_{k,j}^{-1}, \end{aligned} \tag{26}$$

for all $2 \leq j, k \leq 4$.

We define, in analogy to (19),

$$F^\sigma_i := \ker(\cup[D^\sigma_i] \colon H^2(Y^\sigma \times Y^\sigma, K) \longrightarrow H^4(X^\sigma, K)).$$

These subspaces are described by the corresponding statements of Lemma 22. Thus, the above construction yields for any $2 \leq j, k \leq 4$ endomorphisms $g^\sigma_{j,k}$ of F^σ_1. Using the canonical identification of F^σ_1 with $H^2(Y^\sigma, K)$, these endomorphisms are given by

$$\begin{aligned} g^\sigma_{3,2} = (f^\sigma)^*, \quad g^\sigma_{4,2} = (f'^\sigma)^*, \quad g^\sigma_{4,3} = (f' \circ f^{-1})^{\sigma *}, \\ g^\sigma_{j,j} = \mathrm{id} \quad \text{and} \quad g^\sigma_{j,k} = (g^\sigma_{k,j})^{-1}, \end{aligned} \tag{27}$$

for all $2 \leq j, k \leq 4$.

Since ϕ maps $[D_1]$ to a multiple of $[D^\sigma_1]$, it follows that the restriction of ϕ to F_1 induces a K-linear isomorphism

$$\psi \colon F_1 = H^2(Y, K) \xrightarrow{\sim} H^2(Y^\sigma, K) = F^\sigma_1. \tag{28}$$

Since ϕ maps F_i isomorphically to $F^\sigma_{\rho(i)}$, the above isomorphism satisfies

$$\psi \circ g_{j,k} = g^\sigma_{\rho(j),\rho(k)} \circ \psi \tag{29}$$

for all $2 \leq j, k \leq 4$.

We now denote the eigenvalues of $g_{j,k}$ by $\mathrm{Eig}(g_{j,k})$, and similarly for $g^\sigma_{j,k}$. Since f and f' are automorphisms, it follows from (A2) and (26) that $\mathrm{Eig}(g_{3,2})$ and $\mathrm{Eig}(g_{4,2})$ are distinct $\mathrm{Aut}(\mathbb{C})$-invariant sets of roots of unity. By Lemma 11 and since $g_{j,k} = g^{-1}_{k,j}$, we deduce

$$\mathrm{Eig}(g_{3,2}) = \mathrm{Eig}(g_{2,3}) = \mathrm{Eig}(g^\sigma_{3,2}) = \mathrm{Eig}(g^\sigma_{2,3}),$$
$$\mathrm{Eig}(g_{4,2}) = \mathrm{Eig}(g_{2,4}) = \mathrm{Eig}(g^\sigma_{4,2}) = \mathrm{Eig}(g^\sigma_{2,4}).$$

Since $g_{4,3} = g_{2,3} \circ g_{4,2}$ and $g_{3,4} = g_{2,4} \circ g_{3,2}$, it also follows that each of the sets $\mathrm{Eig}(g_{3,4})$, $\mathrm{Eig}(g_{4,3})$, $\mathrm{Eig}(g^\sigma_{3,4})$ and $\mathrm{Eig}(g^\sigma_{4,3})$ is distinct from $\mathrm{Eig}(g_{2,3})$ and $\mathrm{Eig}(g_{4,2})$. Therefore, (29) implies that ρ respects the subsets $\{2, 3\}$ and $\{2, 4\}$. Hence, $\rho = \mathrm{id}$, as we wanted. This finishes the proof of Claim 2. $\qquad\square$

Since $b_1(Y) = 0$ and $\dim(Y) = 2$, the cohomology algebra $H^*(0 \times Y, K)$ is a subalgebra of $SH^2(X, K)^{\leq 4}$. Restriction of ϕ therefore extends the K-linear isomorphism ψ from (28) to an isomorphism

$$\psi : H^*(Y, K) \xrightarrow{\sim} H^*(Y^\sigma, K) \tag{30}$$

of graded K-algebras which we denote with the same letter. Since ρ in the proof of Claim 2 is the identity, it follows from (26), (27) and (29) that ψ satisfies (P2).

In order to prove (P1), we note that

$$\ker(\cup[D_5] \colon F_1 \oplus h \cdot K \longrightarrow H^4(X, K)) = (i^*h - h) \cdot K,$$

where $i^*h \in F_1 = H^2(0 \times Y, K)$. A similar statement holds on X^σ. Since ϕ maps F_1 to F^σ_1, $[D_5] \cdot K$ to $[D^\sigma_5] \cdot K$ and $h \cdot K$ to $h^\sigma \cdot K$, it follows that ϕ maps $i^*h \cdot K$ to $i^{\sigma*}h^\sigma \cdot K$. This finishes the proof of Proposition 19.

4.2. Proof of Proposition 20. As in the proof of Proposition 19, we use (12) in order to identify classes of degree $\leq n$ on T with classes on X. Further, $SH^{1,1}(-, K)$ denotes the subalgebra of $H^*(-, K)$ that is generated by $H^{1,1}(-, K)$; its quotient by elements of degree $\geq r + 1$ is denoted by $SH^{1,1}(-, K)^{\leq r}$.

Let us now suppose that there is a K-linear isomorphism

$$\phi' : H^{1,1}(T, K) \xrightarrow{\sim} H^{1,1}(T^\sigma, K), \tag{31}$$

which induces a weak isomorphism between the respective intersection forms. Then we have the following analogue of Claim 1 in the proof of Proposition 19:

CLAIM 3. *The isomorphism from* (11) *induces a unique isomorphism*

$$\phi \colon SH^{1,1}(X,K)^{\leq 4} \xrightarrow{\sim} SH^{1,1}(X^\sigma, K)^{\leq 4}$$

of graded K-algebras.

Proof. As in the proof of Claim 1, this claim reduces to showing the following: suppose that we have K-rational $(1,1)$-classes $\alpha_1, \ldots, \alpha_r$ and $\beta_1, \ldots, \beta_r$ on T such that

$$\sum_i \phi'(\alpha_i) \cup \phi'(\beta_i) \cup \eta \cup [T^\sigma] = 0 \quad \text{in } H^{2N+8}(X^\sigma, K), \tag{32}$$

for all $\eta \in SH^{1,1}(X^\sigma, K)^{2n-4}$. Then, $\sum_i \phi'(\alpha_i) \cup \phi'(\beta_i)$ vanishes.

In order to prove the latter, let ω be the hyperplane class on X^σ with $[T^\sigma] = \omega^{N+4-n}$. We obtain with respect to this Kähler class a decomposition into primitive pieces:

$$\sum_i \phi'(\alpha_i) \cup \phi'(\beta_i) = \delta_0 \cdot \omega^2 + \delta_1 \cup \omega + \delta_2,$$

where $\delta_j \in H^{j,j}(X, \mathbb{C})_{pr}$. Since ω is an integral class, it follows that δ_j lies in $H^{j,j}(X, K)_{pr}$. The above identity then shows $\delta_2 \in SH^{1,1}(X, K)$.

At this point, we use the assumption in Proposition 20 which ensures that K is stable under complex conjugation. Indeed, this assumption allows us to choose for $j = 0, 1, 2$ the following K-rational classes:

$$\eta_j := \overline{\delta_j} \cdot \omega^{n-2-j} \in SH^{1,1}(X^\sigma, K)^{2n-4}.$$

For $j = 0, 1, 2$, we put $\eta = \eta_j$ in (32). Then, the Hodge–Riemann bilinear relations yield $\delta_j = 0$ for $j = 0, 1, 2$. This finishes the proof of Claim 3. $\qquad\square$

Exploiting the isomorphism of K-algebras ϕ from Claim 3, the proof of Proposition 20 is now obtained by changing the notation in the corresponding part of the proof of Proposition 19. This finishes the proof of Proposition 20.

5. Some simply connected surfaces with special automorphisms

In this section we construct for any integer $g \geq 1$ a simply connected surface Y_g of geometric genus g and with special automorphisms. In the proof of Theorem 6 in Section 6, we will then apply the construction from Section 4 to these surfaces. In Section 7, we will use the examples from Section 6 in order to prove Theorem 7. It is only for the proof of the latter theorem that it will become important that $b_2(Y_g)$ tends to infinity if g does.

5.1. Hyperelliptic curves with special automorphisms. For $g \geq 1$, let C_g denote the hyperelliptic curve with affine equation $y^2 = x^{2g+1} - 1$; see [16]. The complement of this affine piece in C_g is a single point which we denote by ∞. For a primitive $(2g + 1)$th root of unity ζ_{2g+1}, the maps

$$(x, y) \mapsto (\zeta_{2g+1} \cdot x, y) \quad \text{and} \quad (x, y) \mapsto (x, -y)$$

induce automorphisms of C_g which we denote by η_g and ι respectively. Then, ι has the $(2g + 2)$ fixed points

$$(1, 0), (\zeta_{2g+1}, 0), \ldots, (\zeta_{2g+1}^{2g}, 0) \quad \text{and} \quad \infty.$$

The automorphism η_g fixes ∞ and performs a cyclic permutation on the remaining fixed points. The corresponding permutation matrix has eigenvalues $1, \zeta_{2g+1}, \ldots, \zeta_{2g+1}^{2g}$.

The holomorphic 1-forms

$$\frac{x^{i-1}}{y} \cdot dx,$$

where $i = 1, \ldots, g$, form a basis of $H^{1,0}(C_g)$. Therefore, η_g^* has eigenvalues $\zeta_{2g+1}, \ldots, \zeta_{2g+1}^g$ on $H^{1,0}(C_g)$. Moreover, ι acts on $H^1(C_g, \mathbb{Z})$ by multiplication with -1.

5.2. The elliptic curve E_i. Let E_i be the elliptic curve $\mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$; see Section 2.3. Multiplication by i and -1 induces automorphisms η_i and ι of E_i respectively. The involution ι has four fixed points. The action of η_i fixes two of those fixed points and interchanges the remaining two. On $H^{1,0}(E_i)$, the automorphisms ι and η_i act by multiplication with -1 and i respectively.

5.3. Products modulo the diagonal involution. For $g \geq 1$, we consider the product $C_g \times E_i$, where C_g and E_i are defined above. On this product, the involution ι acts via the diagonal. This action has $8g+8$ fixed points. Let $\widetilde{C_g \times E_i}$ be the blow-up of these fixed points. Then,

$$Y_g := \widetilde{C_g \times E_i}/\iota \tag{33}$$

is a smooth surface. For instance, $Y_1 = K3(C_1 \times E_i)$ is a Kummer K3 surface; see Section 2.4.

LEMMA 24. *The surface Y_g is simply connected.*

Proof. It suffices to prove that the normal surface

$$Y_g' := (C_g \times E_i)/\iota$$

is simply connected. Projection to the second coordinate induces a map

$$\pi : Y'_g \longrightarrow \mathbb{P}^1.$$

Let $U \subseteq \mathbb{P}^1$ be the complement of the four branch points of $E_i \to \mathbb{P}^1$. Then, restriction of π to $V := \pi^{-1}(U)$ yields a fiber bundle $\pi|_V : V \to U$ with fiber C_g. Since U is homotopic to a wedge of three circles, the long exact homotopy sequence yields a short exact sequence

$$0 \longrightarrow \pi_1(C_g) \longrightarrow \pi_1(V) \longrightarrow \pi_1(U) \longrightarrow 0.$$

Since π has a section, this sequence splits. Since V is the complement of a divisor in Y'_g, the natural map $\pi_1(V) \to \pi_1(Y'_g)$ is surjective; see [9, Proposition 2.10]. Therefore, the above split exact sequence shows that $\pi_1(Y'_g)$ is generated by the fundamental group of a general fiber together with the image of the fundamental group of a section of π. The latter is clearly trivial. Furthermore, the inclusion of a general fiber $C_g \hookrightarrow Y'_g$ is homotopic to the inclusion of a special fiber $C_g/\iota \cong \mathbb{P}^1$, which is simply connected. It follows that the image of $\pi_1(C_g) \to \pi_1(Y'_g)$ is trivial. This proves the lemma. $\qquad\square$

DEFINITION 25. Let Y_g be as in (33). Then we define the automorphisms f and f' of Y_g to be induced by $\eta_g \times \mathrm{id}$ and $\mathrm{id} \times \eta_i$ respectively.

LEMMA 26. *The surface Y_g with automorphisms f and f' as above satisfies* (A1)– (A3).

Proof. In order to describe the second cohomology of Y_g, we denote the exceptional $\mathbb{P}^1$-curves of Y_g by $D_1, \ldots, D_{8g+8}$. Then, for any field K,

$$H^2(Y_g, K) = H^2(C_g \times E_i, K) \oplus \left(\bigoplus_{i=1}^{8g+8} [D_i] \cdot K \right). \tag{34}$$

It follows from the discussion in Section 5.1 (respectively 5.2) that the action of f (respectively f') on $H^2(Y_g, \mathbb{C})$ has eigenvalues $1, \zeta_{2g+1}, \ldots, \zeta_{2g+1}^{2g}$ (respectively $\pm 1, \pm i$). Moreover, the same statement holds for their actions on $H^{1,1}(Y_g, \mathbb{C})$. This proves (A2) and (A3).

By (34), nontrivial rational (1, 1)-classes on C_g and E_i induce classes α and β in $H^{1,1}(Y_g, \mathbb{Q})$ which satisfy (A1). This finishes the proof of the lemma. $\qquad\square$

6. Multilinear intersection forms on $H^2(-, \mathbb{R})$ and $H^{1,1}(-\mathbb{C})$

Here we prove Theorem 6. This will be achieved by Lemma 27 and Theorem 28 below, where more precise statements are proven.

Let $n \geq 4$ and $g \geq 1$. Moreover, let Y_g be the simply connected surface with automorphisms f and f' from Definition 25. We pick an ample divisor on Y_g which is fixed by f and f'. A sufficiently large multiple of this divisor gives an embedding

$$i \colon Y_g \hookrightarrow \mathbb{P}^N$$

with $n \leq N+4$ such that the actions of f and f' fix the pullback of the hyperplane class.

Next, let

$$X_g := Bl_{Z_1 \cup \cdots \cup Z_5}(Y_g \times Y_g \times \mathbb{P}^N)$$

be the blow-up of $Y_g \times Y_g \times \mathbb{P}^N$ along $Z_1 \cup \cdots \cup Z_5$, where Z_i is defined in (10). Since $n \leq N+4$, X_g contains a smooth n-dimensional complete intersection subvariety

$$T_{g,n} \subseteq X_g. \tag{35}$$

Since Y_g, f and f' are defined over $\mathbb{Q}[\zeta_{8g+4}] = \mathbb{Q}[\zeta_{2g+1}, i]$, so is X_g and we may assume that the same holds true for $T_{g,n}$.

LEMMA 27. *Let $n \geq 2$; then the variety $T_{g,n}$ from (35), as well as each of its conjugates, is simply connected.*

Proof. Since Y_g is simply connected by Lemma 24, so is X_g. By the Lefschetz hyperplane theorem, $T_{g,n}$ is then simply connected for $n \geq 2$.

Since the curves C_g and E_i in the definition of Y_g are defined over $\mathbb{Z}$, it follows that Y_g is isomorphic to any conjugate Y_g^σ. Thus, Y_g^σ is simply connected and the above reasoning shows that the same holds true for $T_{g,n}^\sigma$, as long as $n \geq 2$. This proves the lemma. □

The next theorem, which implies Theorem 6 from the introduction, shows that certain automorphisms $\sigma \in \mathrm{Aut}(\mathbb{C})$ which act nontrivially on $\mathbb{Q}[\zeta_{8g+4}]$ change the analytic topology as well as the complex Hodge structure of $T_{g,n}$.

THEOREM 28. *Let $g \geq 1$ and $n \geq 4$ be integers and let $\sigma \in \mathrm{Aut}(\mathbb{C})$ with $\sigma(i) = i$ and $\sigma(\zeta_{2g+1}) \neq \zeta_{2g+1}$ or vice versa. Then, the $\mathbb{R}$-multilinear intersection forms on $H^2(T_{g,n}, \mathbb{R})$ and $H^2(T_{g,n}^\sigma, \mathbb{R})$, as well as the $\mathbb{C}$-multilinear intersection forms on $H^{1,1}(T_{g,n}, \mathbb{C})$ and $H^{1,1}(T_{g,n}^\sigma, \mathbb{C})$, are not weakly isomorphic.*

Proof. For ease of notation, we assume $\sigma(i) = i$ and $\sigma(\zeta_{2g+1}) = \zeta_{2g+1}^{-1}$. The general case is proven similarly.

Since the curves C_g and E_i from Sections 5.1 and 5.2 are defined over $\mathbb{Z}$, it follows that the isomorphism type of Y_g is invariant under any automorphism

of $\mathbb{C}$. Hence, we may identify Y_g with Y_g^σ. Under this identification, $f'^\sigma = f'$ since i is fixed by σ. Moreover, $f^\sigma = f^{-1}$, since it is induced by the automorphism

$$\eta_g^{-1} \times \mathrm{id} \in \mathrm{Aut}(C_g \times E_i).$$

Suppose that the $\mathbb{R}$-multilinear intersection forms on $H^2(T_{g,n}, \mathbb{R})$ and $H^2(T_{g,n}^\sigma, \mathbb{R})$ are weakly isomorphic. By Lemma 26, Proposition 19 applies and we obtain an $\mathbb{R}$-algebra automorphism of $H^*(Y_g, \mathbb{R})$ with properties (P1) and (P2). By (P1),

$$\psi(i^*h) = b \cdot i^*h$$

for some $b \in \mathbb{R}^\times$. Since the square of i^*h generates $H^4(Y_g, \mathbb{R})$, it follows that in degree four, the automorphism ψ is given by multiplication with a positive real number.

We extend ψ now $\mathbb{C}$-linearly and obtain an automorphism

$$\psi : H^*(Y_g, \mathbb{C}) \xrightarrow{\sim} H^*(Y_g, \mathbb{C}),$$

which we denote by the same letter and which satisfies

$$\psi \circ f = f^{-1} \circ \psi \quad \text{and} \quad \psi \circ f' = f' \circ \psi. \tag{36}$$

Let us now pick nontrivial classes $\omega \in H^{1,0}(C_g)$ and $\omega' \in H^{1,0}(E_i)$ with $\eta_g^* \omega = \zeta_{2g+1} \cdot \omega$ and $\eta_i^* \omega' = i \cdot \omega'$. Then, $\omega \cup \overline{\omega'}$ lies in $H^{1,1}(Y_g)$ and we consider $\psi(\omega \cup \overline{\omega'})$ in $H^2(Y_g, \mathbb{C})$. By (36), f^{-1} and f' act on this class by multiplication with ζ_{2g+1} and $-i$ respectively. We claim that the only classes in $H^2(Y_g, \mathbb{C})$ with these with this properties are multiples of $\overline{\omega} \cup \omega'$ and so

$$\psi(\omega \cup \overline{\omega'}) = \lambda \cdot \overline{\omega} \cup \omega' \tag{37}$$

for some nonzero $\lambda \in \mathbb{C}$. Indeed, since η_i interchanges two of the fixed points of ι on E_i and fixes the remaining two, f'^* has eigenvalues ± 1 on the subspace of exceptional divisors in (34). Therefore, $\psi(\omega \cup \overline{\omega'})$ needs to be contained in $H^2(C_g \times E_i, \mathbb{C})$. On this subspace, f^{-1*} and f'^* are given by $(\eta_g^{-1} \times \mathrm{id})^*$ and $(\mathrm{id} \times \eta_i)^*$ respectively. Our claim follows by the explicit description of η_g and η_i in Sections 5.1 and 5.2.

Together with its complex conjugate, Equation (37) shows

$$\psi(\omega \cup \overline{\omega'} \cup \overline{\omega} \cup \omega') = -|\lambda|^2 \cdot \omega \cup \overline{\omega'} \cup \overline{\omega} \cup \omega'.$$

Since the above degree-four class generates $H^4(Y_g, \mathbb{C})$, we deduce that ψ is given in degree four by multiplication with $-|\lambda|^2$. As we have seen earlier, this number should be positive, which is a contradiction. This finishes the proof of the first assertion in Theorem 28.

For the proof of the second assertion, assume that the $\mathbb{C}$-multilinear intersection forms on $H^{1,1}(T_{g,n}, \mathbb{C})$ and $H^{1,1}(T_{g,n}^\sigma, \mathbb{C})$ are weakly isomorphic.

By Lemma 26 and Proposition 19, this yields an automorphism ψ of $H^{1,1}(Y_g, \mathbb{C})$ which satisfies (36). Then, f^{-1} and f' act on $\psi(\omega \cup \overline{\omega'})$ by multiplication with ζ_{2g+1} and $-i$ respectively. This is a contradiction, since $H^{1,1}(Y_g, \mathbb{C})$ does not contain such a class. This finishes the proof of the theorem. $\qquad\square$

Recall from (35) that $T_{g,n}$ is defined over the cyclotomic number field $\mathbb{Q}[\zeta_{8g+4}]$. This number field contains the totally real subfield

$$K_g := \mathbb{Q}[\zeta_{8g+4} + \zeta_{8g+4}^{-1}].$$

For instance, $K_1 = \mathbb{Q}[\sqrt{3}]$. From Theorem 28, we deduce the following:

COROLLARY 29. *Let $K_g \subseteq K \subseteq \mathbb{C}$ be fields, and let $\sigma \in \mathrm{Aut}(\mathbb{C})$ with $\sigma(i) = i$ and $\sigma(\zeta_{2g+1}) \neq \zeta_{2g+1}$ or vice versa. Then the intersection forms on the equidimensional vector spaces $H^{1,1}(T_{g,n}, K)$ and $H^{1,1}(T_{g,n}^{\sigma}, K)$ are not weakly isomorphic.*

Proof. By Theorem 28 it suffices to prove that the $(1, 1)$-classes on $T_{g,n}$ are spanned by K_g-rational ones. Modulo divisor classes, $H^{1,1}(T_{g,n})$ is given by $H^{1,1}(Y_g) \oplus H^{1,1}(Y_g)$. Furthermore, modulo divisors, $H^{1,1}(Y_g)$ is given by the ι-invariant classes on $E_i \times C_g$. The complex Hodge structure of E_i and C_g is generated by $\mathbb{Q}[i]$-rational and $\mathbb{Q}[\zeta_{2g+1}]$-rational classes respectively; see [16] for the latter. We may now arrange that the induced generators of $H^{1,1}(Y_g)$ are invariant under complex conjugation and thus lie in the subspace of K_g-rational classes. This concludes the proof of the corollary. $\qquad\square$

REMARK 30. Our types of arguments are consistent with Conjecture 2 in the sense that they cannot detect conjugate varieties with nonisomorphic algebras of $\mathbb{Q}$-rational (p, p)-classes. This is because the essential ingredient in the proof of Theorem 28 is a variety Y with an automorphism whose action on $H^{p,p}(Y, K)$ has a set of eigenvalues which is not $\mathrm{Aut}(\mathbb{C})$-invariant. (In our arguments, this role is played by the surface Y_g with the automorphism $f \circ f'$.) For $K = \mathbb{Q}$, the characteristic polynomial of such an action has rational coefficients and so the above situation cannot happen.

REMARK 31. Using Freedman's classification of simply connected topological 4-manifolds, one can prove that simply connected conjugated smooth complex projective surfaces are always homeomorphic. On the other hand, Theorem 6 shows that in any dimension at least four, there are simply connected conjugate smooth complex projective varieties which are not homeomorphic. The case of dimension three remains open.

7. Nonhomeomorphic conjugate varieties in each birational equivalence class

In this section we prove Theorem 7. For this purpose, let Z be a given smooth complex projective variety of dimension ≥ 10. Next, let $T_{g,4}$ be the four-dimensional smooth complex projective variety, defined in (35). By (13) and (34), the second Betti number of $T_{g,4}$ equals $24g + 26$. We may therefore choose an integer $g \geq 1$ with

$$b_2(T_{g,4}) > b_4(Z) + 4. \tag{38}$$

From some projective space, Z is cut out by finitely many homogeneous polynomials. We denote the field extension of $\mathbb{Q}$ which is generated by the coefficients of these polynomials by L. Since L is finitely generated, and after possibly replacing g by a suitable larger integer, we may pick an automorphism σ of $\mathbb{C}$ which fixes L and i but not ζ_{2g+1}.

Since $T_{g,4}$ has dimension four, it can be embedded into $\mathbb{P}^9$. The assumption $\dim(Z) \geq 10$ therefore ensures that we may fix an embedding of $T_{g,4}$ into the exceptional divisor of the blow-up $\hat{Z}$ of Z at a point $p \in Z$. We then define the following element in the birational equivalence class of Z:

$$W := Bl_{T_{g,4}}(\hat{Z}). \tag{39}$$

Since conjugation commutes with blow-ups, the σ-conjugate of W is given by

$$W^\sigma = Bl_{T_{g,4}^\sigma}(\hat{Z}^\sigma), \tag{40}$$

where $\hat{Z}^\sigma$ is the blow-up of Z^σ at a point $p^\sigma \in Z^\sigma$ and $T_{g,4}^\sigma$ is embedded in the exceptional divisor of this blow-up. Since σ fixes L, we have $Z^\sigma \cong Z$. Therefore, W and W^σ are both birational to Z. Hence, Theorem 7 follows from the following result:

THEOREM 32. *Let W and σ be as above. Then the graded even-degree real cohomology algebras of W and W^σ are nonisomorphic.*

Proof. For a contradiction, let us assume that there is an isomorphism

$$\gamma : H^{2*}(W, \mathbb{R}) \longrightarrow H^{2*}(W^\sigma, \mathbb{R})$$

of graded $\mathbb{R}$-algebras. Using pullbacks, we regard $H^{2*}(Z, \mathbb{R}) \subseteq H^{2*}(\hat{Z}, \mathbb{R})$ and $H^{2*}(Z^\sigma, \mathbb{R}) \subseteq H^{2*}(\hat{Z}^\sigma, \mathbb{R})$ as subalgebras of $H^{2*}(W, \mathbb{R})$ and $H^{2*}(W^\sigma, \mathbb{R})$ respectively. By Lemma 9,

$$H^2(W, \mathbb{R}) = H^2(Z, \mathbb{R}) \oplus [H] \cdot \mathbb{R} \oplus [D] \cdot \mathbb{R}, \tag{41}$$

$$H^2(W^\sigma, \mathbb{R}) = H^2(Z^\sigma, \mathbb{R}) \oplus [H^\sigma] \cdot \mathbb{R} \oplus [D^\sigma] \cdot \mathbb{R}, \tag{42}$$

where $H \subset \hat{Z}$ and $H^\sigma \subset \hat{Z}^\sigma$ are the exceptional divisors above the blown-up points, and

$$j : D \hookrightarrow W \quad \text{and} \quad j^\sigma : D^\sigma \hookrightarrow W^\sigma$$

are the exceptional divisors of the blow-ups along $T_{g,4}$ and $T_{g,4}^\sigma$ respectively.

Any cohomology class of positive degree on Z is Poincaré dual to a homology class which does not meet the center of the blow-up $\hat{Z} \to Z$. This shows that for any $\eta \in H^k(Z, \mathbb{R})$, with $k \geq 1$, and for any $\alpha \in H^*(D, \mathbb{R})$,

$$\eta \cup [H] = 0 \quad \text{and} \quad \eta \cup j_*(\alpha) = 0.$$

A similar statement holds on W^σ and we will use these properties tacitly.

The restriction of $-[H]$ to $H \subset \hat{Z}$ is given by $c_1(\mathcal{O}_H(1))$; its restriction to $T_{g,4}$ is therefore ample. By Lemma 9, we have

$$b_4(W) = b_4(Z) + b_2(T_{g,4}) + 2.$$

It then follows from (38) that the second primitive Betti number of $T_{g,4}$ is bigger than $b_4(W)/2$. Since $T_{g,4}$ is four-dimensional, and since $-[H]$ restricts to an ample class on $T_{g,4}$, it follows that $H^2(Z, \mathbb{R}) \oplus [H] \cdot \mathbb{R}$ inside $H^2(W, \mathbb{R})$ is given by those classes whose multiplication on $H^4(W, \mathbb{R})$ has kernel of dimension bigger than $b_4(W)/2$. A similar statement holds for $H^2(Z^\sigma, \mathbb{R}) \oplus [H^\sigma] \cdot \mathbb{R}$ inside $H^2(W^\sigma, \mathbb{R})$ and so γ needs to take $H^2(Z, \mathbb{R}) \oplus [H] \cdot \mathbb{R}$ to $H^2(Z^\sigma, \mathbb{R}) \oplus [H^\sigma] \cdot \mathbb{R}$. Since γ is an isomorphism, it follows that

$$\gamma([D]) = \alpha^\sigma + a \cdot [H^\sigma] + b \cdot [D^\sigma] \tag{43}$$

holds for some $\alpha^\sigma \in H^2(Z^\sigma, \mathbb{R})$ and $b \neq 0$.

The cup product with $[D]$ on $H^2(W, \mathbb{R})$ has a two-dimensional image, spanned by $[D] \cup [H]$ and $[D]^2$. For any $\beta^\sigma \in H^2(Z^\sigma, \mathbb{R})$, the following classes are therefore linearly dependent:

$$\gamma([D]) \cup \beta^\sigma, \quad \gamma([D]) \cup [H^\sigma] \quad \text{and} \quad \gamma([D]) \cup [D^\sigma].$$

Since $b \neq 0$, this is only possible if $\alpha^\sigma \cup \beta^\sigma = 0$ for all β^σ. Hence, $\alpha^\sigma = 0$.

Since $\alpha^\sigma = 0$, it follows from $[D] \cup [H] \neq 0$ that $\gamma([H]) \in H^2(Z^\sigma, \mathbb{R}) \oplus [H^\sigma] \cdot \mathbb{R}$ cannot be contained in $H^2(Z^\sigma, \mathbb{R})$ and hence

$$\gamma([H]) = \tilde{\alpha}^\sigma + c \cdot [H^\sigma]$$

for some $\tilde{\alpha}^\sigma \in H^2(Z^\sigma, \mathbb{R})$ and $c \neq 0$. As the cup product with $[H]$ on $H^2(W, \mathbb{R})$ has a two-dimensional image, the above argument which showed that $\alpha^\sigma = 0$ also implies that $\tilde{\alpha}^\sigma = 0$. Thus, γ takes $[H] \cdot \mathbb{R}$ to $[H^\sigma] \cdot \mathbb{R}$. It follows that γ takes $H^2(Z, \mathbb{R})$ to $H^2(Z^\sigma, \mathbb{R})$, since these are the kernels of the cup product with $[H]$ and $[H^\sigma]$ respectively.

Since $T_{g,4}$ is four-dimensional, we have $[H]^5 \cup [D] = 0$. Then application of γ yields

$$c^5 \cdot [H^\sigma]^5 \cup (a \cdot [H^\sigma] + b \cdot [D^\sigma]) = 0.$$

Since $[H^\sigma]^5 \cup [D^\sigma]$ vanishes, whereas $[H^\sigma]^6$ is nontrivial, it follows from $c \neq 0$ that a vanishes. Thus, γ maps $[D] \cdot \mathbb{R}$ to $[D^\sigma] \cdot \mathbb{R}$ and we conclude that γ respects the decompositions (41) and (42).

The latter implies that γ induces an $\mathbb{R}$-linear isomorphism between the ideals $([D]) \subseteq H^{2*}(W, \mathbb{R})$ and $([D^\sigma]) \subseteq H^{2*}(W^\sigma, \mathbb{R})$. In order to state the key property of this isomorphism, we identify cohomology classes on $T_{g,4}$ and $T_{g,4}^\sigma$ with their pullbacks to the exceptional divisors D and D^σ respectively.

LEMMA 33. *For every* $\alpha \in H^{2k}(T_{g,4}, \mathbb{R})$, *there exists a unique* $\alpha^\sigma \in H^{2k}(T_{g,4}^\sigma, \mathbb{R})$ *such that*

$$\gamma([D] \cup j_*(\alpha)) = [D^\sigma] \cup j_*^\sigma(\alpha^\sigma).$$

Proof. For $0 \leq k \leq 2$, let us fix some $\alpha \in H^{2k}(T_{g,4}, \mathbb{R})$ and note that

$$H^{2k+2}(W^\sigma, \mathbb{R}) = H^{2k+2}(Z^\sigma, \mathbb{R}) \oplus [H^\sigma]^{k+1} \cdot \mathbb{R} \oplus j_*^\sigma(H^{2k}(D^\sigma, \mathbb{R})).$$

Since γ maps $[D]$ to a multiple of $[D^\sigma]$, and since products of $[D^\sigma]$ with positive-degree classes on Z^σ always vanish, the above identity shows

$$\gamma([D] \cup j_*(\alpha)) = [D^\sigma] \cup j_*^\sigma(\alpha^\sigma) + e \cdot [D^\sigma] \cup [H^\sigma]^{k+1}$$

for some $\alpha^\sigma \in H^{2k}(D^\sigma, \mathbb{R})$ and $e \in \mathbb{R}$. The restrictions of $-[H]$ to $T_{g,4}$ and $-[H^\sigma]$ to $T_{g,4}^\sigma$ are ample classes $\omega \in H^2(T_{g,4}, \mathbb{R})$ and $\omega^\sigma \in H^2(T_{g,4}^\sigma, \mathbb{R})$ respectively.

Now suppose that α in the above formula is primitive with respect to ω. Then the cup product of the above class with $\gamma([H])^{5-2k}$ vanishes. Since $\gamma([H])$ is a multiple of $[H^\sigma]$,

$$[D^\sigma] \cup j_*^\sigma(\alpha^\sigma \cup (\omega^\sigma)^{5-2k}) + e \cdot (-1)^{k+1} j_*^\sigma((\omega^\sigma)^{6-k}) = 0.$$

This implies firstly that $e = 0$ and secondly that $\alpha^\sigma \cup (\omega^\sigma)^{5-2k}$ vanishes as a class on D^σ. By the hard Lefschetz theorem, the latter already implies that α^σ, which *a priori* is only a class on D^σ, is in fact a primitive class on $T_{g,4}^\sigma$.

For arbitrary $\alpha \in H^k(T_{g,4}, \mathbb{R})$, the existence of α^σ now follows – since γ takes $[H] \cdot \mathbb{R}$ to $[H^\sigma] \cdot \mathbb{R}$ – from the Lefschetz decompositions with respect to ω and ω^σ; the uniqueness is immediate from Lemma 9. This concludes Lemma 33. $\square$

By Lemma 33, we are now able to define an $\mathbb{R}$-linear map

$$\phi \colon H^{2*}(T_{g,4}, \mathbb{R}) \longrightarrow H^{2*}(T_{g,4}^\sigma, \mathbb{R}),$$

by requiring

$$\gamma([D] \cup j_*(\alpha)) = b \cdot \gamma([D]) \cup j_*^\sigma(\phi(\alpha))$$

for all $\alpha \in H^*(T_{g,4}, \mathbb{R})$, where b is, as above, the nontrivial constant with $\gamma([D]) = b \cdot [D^\sigma]$. Applying the same argument to γ^{-1}, we obtain an $\mathbb{R}$-linear inverse of ϕ.

By Theorem 28, ϕ cannot be an isomorphism of algebras and so we will obtain a contradiction as soon as we have seen that ϕ respects the product structures. For this purpose, let α and β denote even-degree cohomology classes on $T_{g,4}$. Then, by Lemmas 9 and 10, it suffices to prove

$$b \cdot \gamma([D])^3 \cup j_*^\sigma(\phi(\alpha \cup \beta)) = b \cdot \gamma([D])^3 \cup j_*^\sigma(\phi(\alpha) \cup \phi(\beta)).$$

Using (3), the latter is seen as follows:

$$\begin{aligned}
b \cdot \gamma([D])^3 \cup j_*^\sigma(\phi(\alpha \cup \beta)) &= \gamma([D])^2 \cup \gamma([D] \cup j_*(\alpha \cup \beta)) \\
&= \gamma([D]^2 \cup j_*(1) \cup j_*(\alpha \cup \beta)) \\
&= \gamma([D] \cup j_*(\alpha) \cup [D] \cup j_*(\beta)) \\
&= b^2 \cdot \gamma([D])^2 \cup j_*^\sigma(\phi(\alpha)) \cup j_*^\sigma(\phi(\beta)) \\
&= b^2 \cdot \gamma([D])^2 \cup j_*^\sigma(1) \cup j_*^\sigma(\phi(\alpha) \cup \phi(\beta)) \\
&= b \cdot \gamma([D])^3 \cup j_*^\sigma(\phi(\alpha) \cup \phi(\beta)).
\end{aligned}$$

This concludes the proof of Theorem 32. $\qquad\square$

8. Examples with nonisotrivial deformations

In this section we prove that the examples in Theorem 7 may be chosen to have nonisotrivial deformations. Here, a family $(X_s)_{s \in S}$ of varieties over a connected base S is called nonisotrivial if there are two points $s_0, s_1 \in S$ with $X_{s_0} \not\cong X_{s_1}$. The idea of the proof is to vary the blown-up point $p \in Z$ in the construction of Section 7. In order to state our result, we write $X \sim Y$ if two varieties X and Y are birationally equivalent.

THEOREM 34. *Let Z be a smooth complex projective variety of dimension $\geq$ 10. Then there is a nonisotrivial family $(W_p)_{p \in U}$ of smooth complex projective varieties W_p over some smooth affine variety U, and an automorphism $\sigma \in$ Aut$(\mathbb{C})$ such that, for all $p \in U$,*

$$W_p \sim Z \sim W_p^\sigma \quad \text{and} \quad H^{2*}(W_p, \mathbb{R}) \not\cong H^{2*}(W_p^\sigma, \mathbb{R}).$$

Proof. As in Section 7, we may pick some $\sigma \in$ Aut$(\mathbb{C})$ and some $g \geq 1$ such that

$$Z \cong Z^\sigma, \quad \sigma(i) = i, \quad \sigma(\zeta_{2g+1}) \neq \zeta_{2g+1} \quad \text{and} \quad b_2(T_{g,4}) > b_2(Z) + 4.$$

Next, let $U \subseteq Z$ be a Zariski open and dense subset with trivial tangent bundle. Let $\Delta \subseteq U \times Z$ be the graph of the inclusion $U \hookrightarrow Z$ and consider the blow-up

$Bl_\Delta(U \times Z)$. The normal bundle of Δ in $U \times Z$ is trivial, since U has trivial tangent bundle. Hence, the exceptional divisor of $Bl_\Delta(U \times Z)$ is isomorphic to $\Delta \times \mathbb{P}^{n-1}$. Since $n \geq 10$, we may fix an embedding of $\Delta \times T_{g,4}$ into this exceptional divisor and consider the blow-up

$$Bl_{\Delta \times T_{g,4}}(Bl_\Delta(U \times Z)).$$

Projection to the first coordinate then gives a family

$$(W_p)_{p \in U}$$

of smooth complex projective varieties, birational to Z. Then, for all $p \in U$, the conjugate varieties W_p and W_p^σ are as in (39) and (40) respectively. Thus, $W_p \sim Z$ and $W_p^\sigma \sim Z^\sigma$. By Theorem 32 and since $Z \cong Z^\sigma$, we obtain, for all $p \in U$,

$$W_p \sim Z \sim W_p^\sigma \quad \text{and} \quad H^{2*}(W_p, \mathbb{R}) \not\cong H^{2*}(W_p^\sigma, \mathbb{R}).$$

To conclude Theorem 34, it therefore remains to prove:

CLAIM 4. *After replacing Z by another representative of its birational equivalence class, and for a suitable choice of U, the family $(W_p)_{p \in U}$ is nonisotrivial.*

Let us prove this claim. By the arguments of Theorem 32, one sees that any isomorphism $g \colon W_p \to W_q$ induces an isomorphism g^* on cohomology which respects the decomposition (41). This implies that g respects the exceptional divisors and thus induces an isomorphism of Z which takes p to q.

The above argument, applied to $p = q$, shows that W_p admits no automorphism which takes points from the exceptional divisors to $Z - \{p\}$. In particular, W_p contains a Zariski open subset with trivial tangent bundle and with two points that cannot be interchanged by an automorphism of W_p. Since W_p is birational to Z, we may therefore, after possibly replacing Z by another representative of its birational equivalence class, assume that U already contains points p and q which cannot be interchanged by any automorphism of Z. Then, as we have seen, W_p and W_q are not isomorphic. This finishes the proof of Claim 4 and so concludes Theorem 34. □

REMARK 35. In contrast to Theorem 34, most of the previously known examples of nonhomeomorphic pairs of conjugate varieties tend to be rather rigid and do not in general occur in nonisotrivial families. This was already observed by Reed in [11]. However, it is often possible to obtain nonisotrivial families as products of previously known examples with nonrigid varieties; for example one could take products of Serre's examples [13] with a smooth hypersurface of degree at least four in $\mathbb{P}^3$, since the latter are simply connected and come in nonisotrivial families.

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