A group is *covered* by a collection of subgroups if it is the union of the collection. The intersection of an irredundant cover of $n$ subgroups is known to have index bounded by a function of $n$, though in general the precise bound is not known. Here we confirm a claim of Tompkinson that the correct bound is 16 when $n$ is 5. The proof depends on determining all the 'minimal' groups with an irredundant cover of five maximal subgroups.

1. Introduction

A *covering* or *cover* of a group $G$ is a collection of subgroups of $G$ whose union is $G$. We use the term $n$-cover for a cover with $n$ members. The cover is *irredundant* if no proper sub-collection is also a cover. Neumann [5] obtained a uniform bound for the index of the intersection of an irredundant $n$-cover; see Tompkinson [7] for an improved bound. We shall write $f(n)$ for the largest index $|G : D|$ over all groups $G$ with an irredundant $n$-cover with intersection $D$. An immediate consequence is that such a group $G$ has a permutation representation of degree at most $f(n)$, with kernel $\text{core}_G(D)$. In particular $G/\text{core}_G(D)$ is a finite group with an irredundant $n$-cover whose intersection is core-free.

The groups with an irredundant core-free intersection covering are known precisely when $n = 3$ (Scorza [6]) and when $n = 4$ (Greco [4, p.58]): see Propositions 2.3 and 2.4 below. Partial results are known for $n = 5$: Greco [3] lists all groups with an irredundant 5-cover in which all pairwise intersections are the same; and Tompkinson [7] claims that $f(5) = 16$.

The aim of the present article is to fill in some of the missing detail when $n = 5$. We are concerned with irredundant, core-free intersection 5-covers in which all five subgroups of the cover are maximal. A cover in which all subgroups are maximal we shall call *maximal*.

**Theorem 1.1.** Let $G$ be a group with a maximal irredundant cover of five subgroups with core-free intersection $D$. Then either

(a) $D = 1$ and $G$ is elementary Abelian of order 16; or

(b) $D = 1$ and $G \cong \text{Alt}_4$; or

(c) $|D| = 3$, $|G| = 48$ and $G$ embeds in $\text{Alt}_4 \times \text{Alt}_4$.
THEOREM 1.2. \( f(5) = 16 \).

2. PRELIMINARY RESULTS

The following results will be needed below. Where no proof is given it is either very easy or a reference is given.

LEMMA 2.1. Let \( \{A_i; 1 \leq i \leq m\} \) be a (maximal) irredundant covering of a group \( G \) with intersection \( D \). If \( N \) is a normal subgroup of \( G \) contained in \( D \) then \( \{A_i/N; 1 \leq i \leq m\} \) is a (maximal) irredundant cover of \( G/N \).

LEMMA 2.2. (See [1, Lemma 2.2]) Let \( A = \{A_i; 1 \leq i \leq m\} \) be an irredundant covering of a group \( G \) whose intersection is \( D \).

(a) If \( p \) is a prime, \( x \) a \( p \)-element of \( G \) and \( |\{i: x \in A_i\}| = n \) then either \( x \in D \) or \( p^m - n \).

(b) \( \bigcap_{j \neq i} A_j = D \) \( (1 \leq i \leq m) \).

(c) If \( \bigcap_{i \in S} A_i = D \) whenever \( |S| = n \) then \( \left| \bigcap_{i \in T} A_i : D \right| \leq m - n + 1 \) whenever \( |T| = n - 1 \).

(d) If \( A \) is maximal and \( U \) is an Abelian minimal normal subgroup of \( G \) then, if \( |\{i: U \subseteq A_i\}| = n \), either \( U \subseteq D \), or \( |U| \leq m - n \).

PROPOSITION 2.3. (Scorza [6]) Let \( \{A_i; 1 \leq i \leq 3\} \) be an irredundant cover with core-free intersection \( D \) of a group \( G \). Then \( D = 1 \) and \( G \cong C_2 \times C_2 \).

PROPOSITION 2.4. (Greco [4]) Let \( \{A_i; 1 \leq i \leq 4\} \) be an irredundant cover with core-free intersection \( D \) of a group \( G \). If the cover is maximal then either

(a) \( D = 1 \) and \( G \cong Sym_3 \) or \( G \cong C_3 \times C_3 \); or

(b) \( |D| = 2, \ |G| = 18 \) and \( G \) embeds into \( Sym_3 \times Sym_3 \).

If the cover is not maximal then either

(c) \( D = 1 \) and \( G \cong D_8 \), or \( G \cong C_4 \times C_2 \), or \( G \cong C_2 \times C_2 \times C_2 \); or

(d) \( |D| = 2 \) and \( G \cong D_8 \times C_2 \).

LEMMA 2.5. Let \( G \) be a group with a maximal irredundant \( 5 \)-cover with core-free intersection \( D \).

(a) \( G \) is a 2-group if and only if \( D = 1 \) and \( G \) is elementary of order 16.

(b) \( G \) is not a 3-group.

PROOF: Let \( G = M_1 \cup M_2 \cup M_3 \cup M_4 \cup M_5 \) be a maximal irredundant cover for a \( p \)-group \( G \), with core-free intersection \( D \). Now \( \Phi(G) \subseteq D \) so \( D \leq G \), therefore \( D = 1 \), and \( G \) is elementary Abelian. By Lemma 2.2(b), (c), \( |M_i \cap M_j \cap M_k| \leq 2 \) whenever \( i, j, k \) are distinct. When \( p = 2 \), therefore, \( |G| \leq 16 \). Also \( |G| \geq 8 \) since otherwise \( G \)
does not have five maximal subgroups. However \( |G| = 8 \) is impossible. For, if \( |G| = 8 \)
and \( |M_1 \cap M_2 \cap M_3| = 2 \) then \( G = M_1 \cup M_2 \cup M_3 \), contradicting the irredundance
of the cover; and if \( M_1 \cap M_2 \cap M_3 = 1 \) then \( |M_1 \cup M_2 \cup M_3| = 7 \), so \( G \) is covered by four
of the \( M_i \), again a contradiction. Conversely if \( (a, b, c, d) \) is elementary of order 16,
then \( (a, b, c), (a, b, d), (a, c, d), (b, c, d), (ab, bc, cd) \) provide a maximal irredundant
core-free intersection cover.

When \( p = 3 \) we conclude that \( M_i \cap M_j \cap M_k = 1 \) for all distinct \( i, j, k \). \( |G| > 9 \)
since an elementary Abelian group of order 9 has only four maximal subgroups; in particular,
no pairwise intersection is trivial. Hence \( |M_i \cap M_j| = 3 \ (i \neq j) \). By the
inclusion-exclusion principle \( |G| = 5.9 - 10.3 + 10.1 - 5.1 + 1 = 21 \), which is not a power
of 3, a contradiction.

**Lemma 2.6.** Let \( F \) be finite field with \( q \) elements. Suppose that

\[
F^2 = S_1 \cup S_2 \cup \ldots \cup S_m
\]

where \( S_i \) is a translate of a one dimensional subspace \( U_i \) \((1 \leq i \leq m)\). Then \( m \geq q \)
and

(a) if \( m = q \), \( U_1 = U_i \ (1 \leq i \leq q) \);

(b) if \( m = q + 1 \) and the union (2.1) is irredundant, then the subspaces \( U_i \)
are distinct and, for some \( r \in F \), \( S_i = U_i + r \ (1 \leq i \leq q + 1) \);

and

(c) if \( m = q + 2 \) and the union (2.1) is irredundant then the subspaces \( U_i \)
\((1 \leq i \leq q + 2) \) do not cover \( F^2 \).

**Proof:** Firstly note that \( mq \geq q^2 \) so \( m \geq q \). Now observe that \( F^2 \) can be
thought of as an affine plane in which the lines are the translates of one-dimensional vector subspaces. The result then has an easy, and presumably well known, geometrical
proof. We give a sketch.

(a) In this case the space is covered by the \( q \) lines \( S_i \), each containing exactly \( q \)
points. Hence these lines are parallel and one of them passes through the origin.

(b) We are to prove that \( q + 1 \) lines have a common point if their union is irredundant and equal to \( F^2 \). There are at most \( q \) mutually parallel lines, so \( S_1 \) and \( S_2 \) say,
meet at a point \( P \). Let \( A = S_1 \cup S_2 \). Every line \( S_i \ (3 \leq i \leq q + 1) \) meets \( A \) in at least
one point. Since \( |F^2 \setminus A| = q^2 - (2q - 1) = (q - 1)^2 \) no line \( S_i \ (3 \leq i \leq q + 1) \) meets
\( A \) in more than one point. If \( q = 2 \) then \( S_3 \) is incident with \( P \) since neither \( S_1 \) nor
\( S_2 \) is redundant. Hence we may suppose that \( q > 2 \). Now no two \( S_i \ (3 \leq i \leq q + 1) \)
meet outside \( A \). Suppose \( P \in S_i \) but \( P \notin S_j \) for some \( i, j \) satisfying \( 3 \leq i, j \leq q + 1 \).
Then \( S_j \) is parallel to just one of \( S_1, S_2 \), say to \( S_1 \), and also parallel to just one of \( S_2, S_i \)
therefore to \( S_i \), a contradiction since \( S_1 \) and \( S_i \) are not parallel. That is, if three of
the lines $S_i$ ($1 \leq i \leq q + 1$) pass through $P$ then all do, and we are done. Suppose that none of $S_i$ ($3 \leq i \leq q + 1$) is incident with $P$. Then all $S_j$ ($3 \leq j \leq d$) are parallel to $S_1$ and all $S_k$ $(d + 1 \leq k \leq q + 1)$ are parallel to $S_2$, for some $d$ satisfying $3 \leq d < q + 1$, or else the union (2.1) is redundant. It follows that $|S_i \cap S_j| = 1$ if $i \in \{1, 3, \ldots, d\}$ and $j \in \{2, d + 1, \ldots, q + 1\}$, and is zero otherwise; in particular all three-fold intersections are empty. Hence, counting points, 

$$q^2 = q(q + 1) - (d - 1)(q - d + 2)$$

whence $q = (d - 1)(q - d + 2)$. However, both right-side factors are greater than 1, and hence have a prime common factor which therefore divides both $q$ and $q + 1$, a contradiction.

(c) In this case $q > 2$. Two of the lines, say $S_1$ and $S_2$, are parallel. It is enough to show that there is another pair of parallels. If there is not, all the lines $S_i$ ($3 \leq i \leq q + 2$) are incident in pairs, and each is incident with each of $S_1$ and $S_2$. Since the complement of $S_1 \cup S_2$ has cardinality $q^2 - 2q = q(q - 2)$, it follows that $S_i \cap S_j \subseteq S_1 \cup S_2$ ($3 \leq i < j \leq q + 2$). If all these intersections are the same, say lying in $S_1$, then counting shows that $S_2$ is redundant. Hence $S_1 \cap S_h \neq S_1 \cap S_k$ for some $h$, $k \in \{3, \ldots, q + 2\}$. Then $S_h \cap S_k$ is incident with $S_2$, and there is some $S_t$ ($3 \leq t \leq q + 2$) for which $S_2 \cap S_t \neq S_2 \cap S_h$. But then one of $S_h \cap S_t$ or $S_k \cap S_t$ is not incident with $S_1$, a contradiction.

**Lemma 2.7.** Let $G$ be a group with the following structure: $O_3(G)$ is elementary Abelian of index 2 in $G$, and $G$ has trivial centre. There does not exist a maximal irredundant 5-cover of $G$.

**Proof:** Let us suppose that the result is false, and that $G$ is a minimal counterexample. Note that $|G| > 6$ since Sym$_3$ is not a counterexample.

Let

$$G = M_1 \cup M_2 \cup M_3 \cup M_4 \cup M_5$$

be a maximal irredundant cover of $G$ with core-free intersection $D$. Then $|M_i| > 2$ ($1 \leq i \leq 5$). Therefore either

(a) for some $i$, $M_i = V := O_3(G)$ and $|G : M_j| = 3$ ($j \neq i$); or

(b) $|G : M_j| = 3$ for all $j$.

Now $D \cap V = 1$ by Lemma 2.1 since $D \cap V \leq G$. Let $a$ be an involution of $G$. Since (a) is a Sylow 2-subgroup of $G$ every 2-element of $G$ is conjugate to $a$. Define

$$S_i := \{x \in V : a^x \in M_i\}, \quad 1 \leq i \leq 5.$$ 

Either $S_i = \emptyset$ or $S_i$ is a coset of $X_i := V \cap M_i$ in $V$, and there is at most one of the first type. For all $x \in V$, there is an $i$ for which $a^x \in M_i$ so

$$V = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5.$$
From Lemma 2.2(c) the intersection of every triple of the subgroups $X_i \ (1 \leq i \leq 5)$ is trivial. In the case (a) suppose that $M_5 = V$, so that the pairwise intersections $X_i \cap X_j \ (1 \leq i < j \leq 4)$ are all trivial. In particular $|V| = 9$. Also $S_5 = \emptyset$ and

$$V = S_1 \cup S_2 \cup S_3 \cup S_4.$$ 

In this union all the $S_i$ are essential since if, say, $S_1$ were omissible, then $M_1$ would be omissible in (2.2). However Lemma 2.6 now shows that the subgroups $X_i \ (1 \leq i \leq 4)$ are distinct. They therefore cover $V$ making $M_5$ redundant, a contradiction. This shows that case (a) does not arise.

In case (b) we have $V = X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5$. From Lemma 2.5 this union is redundant; and from Proposition 2.3 just one term, say $X_i$, is omissible. Since $1 = \bigcap_{j \neq i} X_j$, by Lemma 2.2(b), it follows from Proposition 2.4 that $|V| = 9$. Now we apply Lemma 2.6. Firstly, by (c) of that result, the union (2.3) is redundant, and at most two terms on the right are omissible. If omitting $S_5$ say, leaves an irredundant union then, by Lemma 2.6(b), $V = X_1 \cup X_2 \cup X_3 \cup X_4$ and $M_5$ is omissible from (2.2), contradiction. If omitting $S_4$ and $S_5$ from (2.3) leaves $V = S_1 \cup S_2 \cup S_3$ then Lemma 2.6(a) yields $X_1 = X_2 = X_3 \subseteq D \cap V = 1$, another contradiction.

Finally we note the following well known fact which is used repeatedly, and without explicit reference, throughout what follows: if $M$ is a maximal subgroup, and $U$ an Abelian minimal normal subgroup, of a group then either $U \subseteq M$ or $U \cap M = 1$.

3. PROOF OF THEOREM 1.1

We have already determined the 2-groups which have maximal irredundant core-free intersection 5-covers. The next lemma addresses non-2-groups

**Lemma 3.1.** Suppose that the intersection of a maximal irredundant cover of five subgroups of a group $G$ is core-free. If $G$ is not a 2-group then every minimal normal subgroup of $G$ has order 4.

**Proof:** By Lemma 2.2(a) $G$ is a $\{2, 3\}$-group. Since $G$ is soluble, by Burnside's Theorem, every minimal normal subgroup $U$ of $G$ is Abelian. Moreover, by Lemma 2.2(d), $|U| \leq 4$.

If $|U| = 2$ then, again by Lemma 2.2(d), $U$ is contained in at most three of the subgroups $A_i$, say $U \not\subseteq A_4 \cup A_5$. Since $U$ is central, and since $G = A_4 U = A_5 U$, every 3-element of $G$ is in $A_4 \cap A_4$. However if $1 \neq u \in U$ and if $y$ is a 3-element, then $uy \not\subseteq A_4 \cup A_5$. Hence $uy \in A_1 \cup A_2 \cup A_3$ and therefore $y \in A_1 \cup A_2 \cup A_3$. It follows that a Sylow 3-subgroup $S$ of $G$ is in $A_1 \cup A_2 \cup A_3$ and therefore, by Proposition 2.3, in one of $A_i \ (1 \leq i \leq 3)$, say in $A_3$. Therefore $S \subseteq A_3 \cap A_4 \cap A_5$ and so, by Lemma
2.2(c), \( S \subseteq D \). Since, therefore, every 3-element of \( G \) is in \( D \) so is the subgroup \( T \) which they generate. Of course \( T \subseteq G \) so \( T = 1 \). But this contradicts the fact that \( G \) is not a 2-group. Therefore \( G \) has no normal subgroups of order 2.

If \( |U| = 3 \) then \( U \) is contained in at most two of the subgroups \( A_i \), say \( U \nsubseteq A_3 \cup A_4 \cup A_5 \). It follows that \( G = UA_i \) (3 \( \leq i \leq 5 \)). An argument similar to that of the last paragraph shows that every 2-element of \( C := C_G(U) \) is in \( D \). Since the subgroup they generate is normal it is 1, and we see that \( C \) is a 3-group. Also, \( \Phi(C) \subseteq \Phi(G) \subseteq D \), so \( \Phi(C) = 1 \). That is, \( C \) is elementary Abelian. By Lemma 2.5(b) \( C \neq G \). That is, no minimal normal subgroup of \( G \) is central. However \( |G : C| = 2 \), and so \( G \) satisfies the hypotheses of Lemma 2.7, contradiction.

PROOF OF THEOREM 1.1: Let \( G \) be a group with a maximal irredundant cover \( \bigcup_{i=1}^{5} A_i \) with core-free intersection \( D \). By Lemma 2.5 we may suppose that \( G \) is not a 2-group. Suppose that \( U \) is a minimal normal subgroup of \( G \). It follows from Lemma 3.1 that \( |U| = 4 \). Also, by Lemma 2.2(d), \( U \) is in at most one of the subgroups \( A_i \), say \( U \nsubseteq A_2 \cup A_3 \cup A_4 \cup A_5 \). A familiar argument gives that \( C := C_G(U) \) is an elementary 2-group. Moreover \( G/C \) embeds into \( \text{Aut}(U) \cong \text{Sym}_3 \), and \( O_3(G/C) \neq 1 \). As \( G/C \)-module, \( C \) has no non-trivial fixed points for the action of \( O_3(G/C) \), using Lemma 3.1. It follows that \( C \) is the first or second nilpotent residual of \( G \). Therefore \( C \) is complemented in \( G \), using the result in [2, (5.18) p.383], say \( G = CH \) where \( H \cong C_3 \) or \( H \cong \text{Sym}_3 \). As \( H \)-module \( C \) is completely reducible, and every minimal normal subgroup of \( G \) is of order 4.

If \( C = U \) then \( G \cong \text{Alt}_4 \) or \( G \cong \text{Sym}_4 \). The first case is (b) of the theorem. The second does not arise because \( \text{Sym}_4 \) has no maximal irredundant cover of five subgroups. For, \( D \) is core-free, does not contain the monolith of \( \text{Sym}_4 \) so, by Lemma 2.2(d), four of the five subgroups of the cover are copies of \( \text{Sym}_3 \) whilst the fifth, therefore, contains all the elements of \( \text{Sym}_4 \) of order 4. However this is a contradiction because these elements generate \( \text{Sym}_4 \).

If \( C \neq U \) then \( C_{A_i}(U) \neq 1 \) (2 \( \leq i \leq 5 \), and \( C = U \times C_{A_i}(U) \). Since \( D \) is core-free, it follows from Lemma 3.1 and Lemma 2.2(d) that \( 1 = C_{A_i}(U) \cap C_{A_j}(U) \) (2 \( \leq i < j \leq 5 \)). Then, for \( i \neq j \),

\[
|C_{A_j}(U)| |U| = |C| \geq |C_{A_i}(U)C_{A_j}(U)| = |C_{A_i}(U)||C_{A_j}(U)|
\]

so that \( |U| \geq |C_{A_i}(U)| \geq |U| \). It follows that each \( C_{A_i}(U) \) is minimal normal in \( G \). That is, \( C \) is the direct product of two minimal normal subgroups of \( G \). If \( H \) were isomorphic to \( \text{Sym}_3 \) then \( C \), as \( H \)-module, would contain just three proper non-zero submodules instead of the (at least) five it does contain. Hence \( |H| = 3 \).

Now we examine the nature of this cover for \( G \). Choose \( a \in G \) of order 3. Then \( \langle a \rangle \) is a Sylow 3-subgroup of \( G \), and every 3-element of \( G \) is conjugate either to \( a \) or
Covering groups with subgroups

to $a^2$. Define $S_i := \{w \in C: a^w \in A_i\}$ ($1 \leq i \leq 5$) and $N_i := A_i \cap C$ ($1 \leq i \leq 5$). $S_i$ is a coset on $N_i$ in $C$: it is not empty since otherwise $A_i$ would contain no 3-element, would therefore be equal to $C$, and some $N_j$ would be in two of the $A_k$ whence, by Lemma 2.2(d), in $D$, which is core-free. We have

(3.1) $C = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$

since every $a^w$ is in some $A_i$. We may regard $C$ as a space of dimension 2 over the field $F$ of 4 elements, where $(a)$ is the multiplicative group of $F$, and apply Lemma 2.6(b). If the union (3.1) is irredundant then $S_i = N_i c$ ($1 \leq i \leq 5$) for some $c \in C$. Hence $a^c \in A_i$ ($1 \leq i \leq 5$), so $|D| = 3$ and $G$ has the structure required by (c) of the theorem. If, however, (3.1) is redundant then, by Lemma 2.6, at most one term, say $S_5$, is omissible and $N_i = N_i$ ($1 \leq i \leq 4$). This gives $N_1 = \bigcap_{i=1}^{4} N_i \subseteq \bigcap_{i=1}^{4} A_i = D_1$, a contradiction to the core-freeness of $D$.

4. Proof of Theorem 1.2

If the result is false, let $G$ be a group with an irredundant cover $C$ of five subgroups, with core-free intersection $D$, for which $|G : D| > 16$. In the light of Theorem 1.1, $C$ is not maximal. Suppose $C$ chosen from among such 5-covers of $G$ with as many maximal subgroups as possible. Let $C^*$ be a cover of $G$ got from $C$ by replacing one of its non-maximal subgroups by a maximal subgroup containing it. Write $D^*$ for the intersection of $C^*: D^* \supseteq D$. $C^*$ is redundant; for, if not, $D^* = D$ by Lemma 2.2(b), and so is core-free, while $C^*$ has more maximal subgroups than does $C$. It follows that we may write $C = \bigcup_{i=1}^{5} A_i$ where $A_1$ is not maximal, and if $A^*_1$ is a maximal subgroup containing it, then $C^* = \{A^*_1, A_2, A_3, A_4, A_5\}$ is redundant as a cover for $G$.

If $G$ is an irredundant union of four of the subgroups in $C^*$, then we may suppose that

(4.1) $G = A^*_1 \cup A_2 \cup A_3 \cup A_4$

since $A^*_1$ is certainly essential. If $D_1 := A^*_1 \cap A_2 \cap A_3 \cap A_4$ then it follows from Proposition 2.4 that $|G : D_1| \leq 9$ with equality only if $A^*_1 \cap A_i = D_1$ ($2 \leq i \leq 4$). If we have equality therefore, it follows that

(4.2) $A^*_1 = A_1 \cup D_1 \cup (A_5 \cap A^*_1)$,

an irredundant union. However from (4.1) we deduce that $|A^*_1 : D_1| = 3$, and from (4.2) and Proposition 2.3 that $|A^*_1 : D_1| = 2$, a contradiction. Hence $|G : D_1| \leq 8$. Then,
since $D_1 = A_2 \cap A_3 \cap A_4$, we have $|D_1 : D| \leq 2$ by Lemma 2.2(c), so $|G : D| \leq 16$, a contradiction.

Lastly, if $G$ is an irredundant union of three of the subgroups in $C^*$, we may suppose that

(4.3) \[ G = A_1^* \cup A_2 \cup A_3 \]

since $A_1^*$ is surely included. Let us write $N := A_2 \cap A_3 \cap A_4 = A_1^* \cap A_3$. Now

(4.4) \[ A_1^* = A_1 \cup N \cup (A_1^* \cap A_4) \cup (A_1^* \cap A_5). \]

If the union (4.4) is irredundant then $|A_1^* : D| = |A_1^* : A_1 \cap N \cap A_4 \cap A_5| \leq 9$. However, by (4.3), $|A_1^* : A_2 \cap A_3| = 2$, so $|A_1^* : D| \neq 9$. Hence $|G : D| = |G : A_1^*||A_1^* : D| \leq 16$, a contradiction. On the other hand if the union (4.4) is redundant then three of the subgroups on the right side are essential, and the possible intersections $I$ satisfy $|I : D| \leq 2$, using Lemma 2.2(c). Hence $|G : D| = |G : A_1^*||A_1^* : I||I : D| \leq 2.4.2 = 16$. This contradiction completes the proof of Theorem 1.2. \[ \square \]

REFERENCES