THE NON-ABELIAN TENSOR PRODUCT OF GROUPS AND RELATED CONSTRUCTIONS

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Introduction. The tensor product of two arbitrary groups acting on each other was introduced by R. Brown and J.-L. Loday in [5, 6]. It arose from consideration of the pushout of crossed squares in connection with applications of a van Kampen theorem for crossed squares. Special cases of the product had previously been studied by A. S.-T. Lue [10] and R. K. Dennis [7]. The tensor product of crossed complexes was introduced by R. Brown and the second author [3] in connection with the fundamental crossed complex \( \pi(X) \) of a filtered space \( X \), which also satisfies a van Kampen theorem. This tensor product provides an algebraic description of the crossed complex \( \pi(X \otimes Y) \) and gives a symmetric monoidal closed structure to the category of crossed complexes (over groupoids). Both constructions involve non-abelian bilinearity conditions which are versions of standard identities between group commutators. Since any group can be viewed as a crossed complex of rank 1, a close relationship might be expected between the two products. One purpose of this paper is to display the direct connections that exist between them and to clarify their differences.

Given a group \( A \), we denote by \( A \) the crossed complex which has \( A \) in dimension 1 but is otherwise trivial. For two arbitrary groups \( A \) and \( B \), without actions, the tensor product \( A \otimes B \), as defined in [3], can be easily described; it is effectively the crossed module \( A \otimes B \to A \ast B \), where \( A \otimes B \) is the Cartesian subgroup of \( A \ast B \) (the kernel of the canonical homomorphism \( A \ast B \to A \times B \)).

The tensor product \( G \otimes H \) of two groups acting on one another is more subtle. It is a quotient of \( G \square H \) and is a crossed module over a group \( G \ltimes H \) introduced by J. H. C. Whitehead [11] which we here call the Peiffer product of \( G \) and \( H \) (because of its connection with Peiffer identities). The tensor product \( G \otimes H \) does not have the functorial properties enjoyed by the tensor product of crossed complexes but it essentially includes \( A \otimes B \) as a special case: if \( A \) and \( B \) are groups without actions then \( A \otimes B \) is the crossed module \( \tilde{A} \otimes \tilde{B} \to \tilde{A} \ltimes \tilde{B} \) where \( \tilde{A} \) and \( \tilde{B} \) are obtained by freely generating from \( A \) and \( B \) two groups acting compatibly on each other.

On the other hand, for groups \( G, H \) acting on each other, the crossed module \( G \square H \to G \ltimes H \) cannot in general be written in the form \( A \otimes B \) since the latter is always infinite, whereas \( G \otimes H \) and \( G \ltimes H \) are finite whenever \( G \) and \( H \) are finite [8]. We compute some examples which show that reasonable conjectures on how to obtain \( G \otimes H \) from \( G \otimes H \) are false.

1. Two tensor products of groups. A group \( G \) may be regarded as a crossed complex \( G \) of rank 1; thus \( G \) has \( G \) in dimension 1 and is otherwise trivial. Given two groups \( G \) and \( H \), we may therefore form the tensor product of the crossed complexes \( G \)

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and $H$, as defined in [3]. This product is a crossed complex of rank 2, that is a crossed module, and it has an explicit description which we now recall (cf. [3, Proposition 6.1]).

For arbitrary groups $G, H$ we denote by $G \square H$ the Cartesian subgroup of the free product $G \ast H$ (that is, the kernel of the canonical homomorphism $G \ast H \to G \times H$).

1.1 Proposition. Given arbitrary groups $G$ and $H$, the tensor product of the crossed complexes $G$ and $H$ is the crossed module $i: G \square H \to G \ast H$ where $i$ is the inclusion map.

Now suppose that $G$ and $H$ act on each other on the right. Then the free product $G \ast H$ acts on both $G$ and $H$, each group acting on itself by conjugation. We assume that the actions are compatible, that is

$$g_1 g_2 = (g_1 g_2 h)(g_2 h),$$

$$h_1 h_2 = (g_1 h_2)(g_1 h_2 h).$$

We note that these relations have the form of standard commutator identities when $g \otimes h$ is replaced by $[g, h] = g^{-1}h^{-1}gh$ and the actions by conjugation.

1.2 Lemma. (i) $G \ast H$ acts on $G \otimes H$ with

$$(g \otimes h)^w = g^w \otimes h^w, \quad (w \in G \ast H).$$

(ii) $G \otimes H$ is the $G \ast H$-group generated by symbols $g \otimes h$ with defining relations

$$g_1 g \otimes h = (g_1 \otimes h^g)(g \otimes h),$$

$$g \otimes h_1 h = (g \otimes h^g h^h),$$

$$g \otimes h^w = g^w \otimes h^w.$$  

Proof. It is straightforward to check that the subgroup of the free group on symbols $g \otimes h$ generated by the relators corresponding to the relations (1) and (2) admits the action of $G \ast H$ given by $(g \otimes h)^w = g^w \otimes h^w$. Note that compatibility is crucial here. This proves (i) and then (ii) is immediate.

1.3 Lemma. In $G \otimes H$ the following relations hold and are equivalent to (1) and (2):

$$gg_0 \otimes h = (g \otimes h^g)(g \otimes h),$$

$$g \otimes hh_0 = (g \otimes h^g)(g^h \otimes h_0).$$

Proof. Setting $g_1 = gg_0 g^{-1}$ and $h_1 = hh_0 h^{-1}$ in (1) and (2) yields (6) and (7).

We now have the two tensor products: however, the notation $G \otimes H$ will always refer to the non-abelian tensor product of Brown and Loday. In Proposition 1.1 we
obtained a crossed module as a tensor product of crossed complexes. We now show that \( G \otimes H \) is naturally a crossed module and to do this we need to introduce another product of pairs of groups and actions.

Given \( G \) and \( H \) acting compatibly on each other, we define their *Peiffer product* \( G \Join H \) as the quotient of \( G \ast H \) by the normal closure \( K \) of all elements of the form

\[
h^{-1}g^{-1}hg^h \text{ or } g^{-1}h^{-1}gh^g
\]

where \( g \in G \) and \( h \in H \) (see [11, p. 428]). Compatibility ensures that these elements act trivially on \( G \) and \( H \) so that \( G \Join H \) acts on \( G \) and \( H \). Moreover, the canonical maps

\[
\begin{align*}
G \ast H & \rightarrow G \Join H \\
H & \rightarrow G \Join H
\end{align*}
\]

make \( G \) and \( H \) into crossed \( G \Join H \)-modules and the original actions are induced by the action of \( G \Join H \).

1.4 Proposition. The tensor product \( G \otimes H \) is a crossed \( G \Join H \)-module and there is a morphism \((\phi, \psi)\) of crossed modules

\[
\begin{align*}
G \Join H & \rightarrow G \otimes H \\
G \ast H & \rightarrow G \Join H
\end{align*}
\]

in which \( \psi \) is the quotient map and \( \phi([g, h]) = g \otimes h \). Here \( \delta(g \otimes h) = \psi([g, h]) \) and the action of \( G \Join H \) on \( G \otimes H \) is induced by that of \( G \ast H \). Furthermore, \( L = \ker \phi \) is the normal closure in \( G \Join H \) of all elements

\[
w^{-1}[g, h]w[g^w, h^w]^{-1}
\]

where \( w \in G \ast H \), and is a normal subgroup of \( G \ast H \).

Proof. The group \( G \Join H \) is freely generated by all elements \([g, h], g \in G, h \in H, g \neq 1, h \neq 1 \) (see [9]). There is therefore a morphism of groups \( \phi : G \Join H \rightarrow G \otimes H \) with \( \phi([g, h]) = g \otimes h \) for all \( g \in G, h \in H \). Now \( G \ast H \) acts on \( G \Join H \) by conjugation and on \( G \otimes H \) by the action described in Lemma 1.2. The map \( \phi \) is compatible with these actions because, for \( g, g_1 \in G \) and \( h, h_1 \in H \),

\[
\phi(g^{-1}[g_1, h]g) = \phi([g_1g, h][g, h]^{-1}) = (g_1g \otimes h)(g \otimes h)^{-1} = (g_1 \otimes h)^g
\]

and similarly \( \phi(h^{-1}[g, h_1]h) = (g \otimes h_1)^h \). It follows that \( L = \ker \phi \) is a normal subgroup of \( G \ast H \). Furthermore, for \( w \in G \ast H \), the element \( u = w^{-1}[g, h]w[g^w, h^w]^{-1} \) is in \( L \) since
\( \phi(u) = \phi([g, h])^{w}\phi([g^w, h^w])^{-1} = (g \otimes h)^{w}(g^w \otimes h^w)^{-1} = 1. \) However, by definition of \( G \otimes H, L \) is generated, as a normal subgroup of \( G \square H, \) by all elements
\[
[g_1g, h][g, h]^{-1}[g^{-1}g_1g, h^s]^{-1} = g^{-1}[g_1, h]g[g^{-1}g_1g, h^s]^{-1}
\]
and
\[
[g, h]^{-1}[g, h_1h][g^h, h^{-1}h_1h]^{-1} = h^{-1}[g, h_1]h[g^h, h^{-1}h_1h]^{-1}.
\]
Since these elements are of the form \( w^{-1}[g, h]w[g^w, h^w]^{-1}, \) the last part of the proposition follows.

Now, modulo \( K = \ker \psi, \) we have \( g^w = w^{-1}gw \) and \( h^w = w^{-1}hw \) so \( L \subseteq K \) and \( \psi(L) = 1. \) Consequently, there is a unique morphism \( \delta : G \otimes H \to G \rtimes H \) with \( \delta \phi = \psi, \) that is \( \delta(g \otimes h) = \psi([g, h]). \) It remains only to show that \( K \) acts trivially on \( G \otimes H, \) for then \( \psi \) induces an action of \( G \rtimes H \) on \( G \otimes H \) and it is immediate that \( \delta : G \otimes H \to G \rtimes H \) is a crossed module and \((\phi, \psi)\) is a morphism of crossed modules. However, \( K \) is normally generated by all \( w = h^{-1}g^{-1}hg\) and \( w' = g^{-1}h^{-1}gh^s \) so it is enough to show that \( w \) and \( w' \) act trivially on \( G \otimes H. \) This follows from the fact, already noted, that they act trivially on \( G \) and \( H. \)

Let \( A \) and \( B \) be groups, with no actions assumed. We shall show how to construct the tensor product of the crossed complexes \( A \) and \( B \) as a crossed module \( G \otimes H \to G \rtimes H \) by judicious choices of \( G \) and \( H. \)

Define \( \tilde{A} \) to be the universal \( B \)-group on \( A, \) that is, \( \tilde{A} \) is the group generated by symbols \( a^b \) \((a \in A, b \in B)\) with defining relations \((a_1a_2)^b = a_1^ba_2^b. \) Thus \( \tilde{A} \) is the free product of copies \( A^b \) of \( A, \) one for each \( b \in B, \) and \( B \) permutes these copies according to \((a^b)^{b'} = a^{bb'} \). We identify \( A \) with the subgroup \( A_1 \) of \( \tilde{A} \) and so we write \( a = a^1 \). Similarly we define \( \tilde{B} \) to be the universal \( A \)-group on \( B. \)

The action of \( B \) on \( \tilde{A} \) can be extended to an action of \( \tilde{B} \) on \( \tilde{A} \) by the rule \((a_1^b)^{b'} = a_1^{b+b'a} \) where \( a \) is identified with \( a_1^a \) and acts by conjugation in \( \tilde{A}. \) Thus, in normal form,
\[
(a_1^{b})^{b'} = a^{-1}a^ba_1^{b'b}(a^b)^{-1}a.
\]
Similarly we can define an action of \( \tilde{A} \) on \( \tilde{B} \) and it is easy to see that the two actions are compatible.

We may now form \( \tilde{A} \otimes \tilde{B} \): it is a crossed module over \( \tilde{A} \rtimes \tilde{B}. \)

1.5 Proposition. The crossed module \( \tilde{A} \otimes \tilde{B} \to \tilde{A} \rtimes \tilde{B} \) is isomorphic to the tensor product of \( A \) and \( B \) regarded as crossed complexes of rank 1, that is, to the crossed module \( A \square B \to A \ast B. \)

Proof. Consider the composite morphism of crossed modules where \( i, j, k, l \) are inclusion maps and \((\phi, \psi)\) is the morphism given by Proposition 1.4 applied to \( \tilde{A} \) and \( \tilde{B}. \)
The kernel $K$ of $\psi$ is the normal subgroup of $\tilde{A} \ast \tilde{B}$ generated by all $x^{-1}y^{-1}xy$ and $y^{-1}x^{-1}yx$ for $x \in \tilde{A}$, $y \in \tilde{B}$. Modulo this kernel we have $a^b = b^{-1}ab$ and $b^a = a^{-1}ba$ for $a \in A$, $b \in B$. So $\tilde{A} \ast \tilde{B}$ is generated by $A \ast B$ and $K$. Since $\psi$ is surjective, it follows that $\psi l$ is surjective.

On the other hand, because of the freeness of $\tilde{A}$ and $\tilde{B}$, there is a morphism $\theta : \tilde{A} \ast \tilde{B} \to A \ast B$ with

$$\theta(a^b) = b^{-1}ab, \quad \theta(b^a) = a^{-1}ba \quad \text{for} \quad a \in A, \quad b \in B. \tag{9}$$

Clearly $\theta l$ is the identity on $A \ast B$. Also, if $x \in \tilde{A}$, one may deduce from (9) firstly that $\theta(x^b) = b^{-1}\theta(x)b$ and then that $\theta(x^{ba}) = \theta(b^{-1})^{-1}\theta(x)\theta(b^a)$, where $a \in A$ and $b \in B$. Hence, for all $x \in \tilde{A}$ and $y \in \tilde{B}$ we have $\theta(x^y) = \theta(y)^{-1}\theta(x)\theta(y)$ and similarly $\theta(y^x) = \theta(x)^{-1}\theta(y)\theta(x)$. Thus $\theta(K) = 1$. If now, $u \in A \ast B$ and $\psi l(u) = 1$, then $l(u) \in K$ and therefore $u = \theta l(u) = 1$. This proves that $\psi l : A \ast B \to \tilde{A} \ast \tilde{B}$ is an injection and therefore an isomorphism.

Now consider $\phi k : A \square B \to \tilde{A} \otimes \tilde{B}$. It is clearly an injection, because $\delta \phi k = \theta j$ is an injection. Thus the theorem will be proved if we can show that $\phi k$ is a surjection. This we will do by showing that $\tilde{A} \otimes \tilde{B}$ is generated as crossed $\tilde{A} \ast \tilde{B}$-module by the elements $\phi k([a, b]) = a \otimes b$. Since the action of $\tilde{A} \ast \tilde{B}$ on $\tilde{A} \otimes \tilde{B}$ is induced from that of $\tilde{A} \ast \tilde{B}$, this is equivalent to showing that $\tilde{A} \otimes \tilde{B}$ is generated as $\tilde{A} \ast \tilde{B}$-group by all $a \otimes b$ with $a \in A$, $b \in B$. Now $\tilde{A} \otimes \tilde{B}$ is certainly generated as a group by all $x \otimes y$, $x \in \tilde{A}$, $y \in \tilde{B}$ and the relations (3), (4) can be used to express any such $x \otimes y$ as a product of elements of the form $(a_i \otimes b_j)^w$, where $a, a_1 \in A$, $b, b_1 \in B$ and $w \in \tilde{A} \ast \tilde{B}$. Finally,

$$a_i^b \otimes b_i^a = (a_i \otimes b_i)^{-1}(a_1 \otimes b b_1)$$

by (7)

$$= (a_i \otimes b)^{-1}(a_i \otimes b_i a_1 \otimes b_1)$$

by (4)

$$= (a_i \otimes b)^{-1}(a_1 \otimes b_1 a_i \otimes b_i)$$

by (6) and (3)

and this completes the proof. \[\Box\]

2. The Peiffer product. We shall return in Section 3 to the crossed module morphism of Proposition 1.4. Before doing so we consider the Peiffer product $G \rtimes H$ in more detail. As mentioned in the introduction, this construction was introduced by Whitehead in [11]. There he posed his famous question on the asphericity of subcomplexes of aspherical 2-complexes and reformulated it as part of the wider problem of finding conditions under which the groups $G$ and $H$ are embedded in $G \rtimes H$.

Let $G$ and $H$ be groups acting compatibly on each other and let $K$ be the kernel of the natural map $\psi : G \ast H \to G \rtimes H$. Then modulo $K$, $hg = gh^h$, so that every element of $G \rtimes H$ can be written as $\psi(g)\psi(h)$ for suitable $g, h$. We write $\langle g, h \rangle$ for $\psi(g)\psi(h)$. By considering the implied presentation of $G \rtimes H$ as $(G \ast H)/K$ it is easy to see that the relations

$$\langle g, h \rangle \langle g_1, h_1 \rangle = \langle gg_1, h g_1 h_1 \rangle = \langle g_1, h_1 \rangle$$

are defining relations for $G \rtimes H$ on the generators $\langle g, h \rangle$ and so $G \rtimes H$ is a homomorphic image of both the semidirect products $G \ltimes H$ and $G \rtimes H$. This explains our choice of
notation. The group $G \rtimes H$ is obtained from $G \ltimes H$ (or from $G \ltimes H$) by imposing the relations
\[(g^{-1}gh, 1) = (1, h^{-g}h).\] (10)

These facts were proved by R. Brown in [1].

Given two crossed $P$-modules $\lambda : G \to P$ and $\mu : H \to P$, we can form the Peiffer product $G \rtimes H$ using the actions of $G$ and $H$ on each other induced via $P$. Such actions are always compatible. R. Brown also proved in [1] that in this case $G \rtimes H$ is itself a crossed $P$-module with boundary map $\partial : G \rtimes H \to P$ given by $(g, h) \mapsto \lambda(g)\mu(h)$ and is the coproduct of $G$ and $H$ in the category of crossed $P$-modules. The expression of $G \rtimes H$ as a quotient of $G \ltimes H$ greatly facilitates the study of the kernel of $\partial : G \rtimes H \to P$.

On the other hand, if $G$ and $H$ act compatibly on one another, then each is a crossed $G \rtimes H$-module with boundary map induced by the respective inclusion into $G \ast H$ and the given actions then coincide with those obtained via $G \rtimes H$. It follows that the coproduct of $G$ and $H$ as crossed $G \rtimes H$-modules is just the identity map $G \rtimes H \to G \rtimes H$.

We now consider some special cases in which the Peiffer product $G \rtimes H$ of groups $G$ and $H$ acting compatibly on one another may be described explicitly in terms of $G$ and $H$.

We write $D^H_G(G)$ for the displacement subgroup of $G$ relative to the action of $H$, that is, the subgroup of $G$ generated by all elements $g^{-1}gh$ where $g \in G$ and $h \in H$. Then $D^H_G(G)$ is normal in $G$ and $G/D^H_G(G)$ is the largest quotient of $G$ on which $H$ acts trivially: we denote this quotient by $G/H$.

2.1 PROPOSITION. Let $\lambda : G \to P$ and $\mu : H \to P$ be crossed $P$-modules such that $\lambda(G) \subseteq \mu(H)$ and suppose that $\mu : H \to H$ is split by a homomorphism $\sigma : H \to H$. Then the Peiffer product $G \rtimes H$ formed with respect to the actions of $G$ and $H$ on each other via $P$ is isomorphic as a group to $G/H \times H$.

Proof. Form the semidirect product $G \ltimes H$ and define a map $\xi : G \rtimes H \to G \times H$ by $(g, h) \mapsto (g, \sigma\lambda(g)h)$. Then $\xi$ is an isomorphism, for it is clearly bijective and
\[
\xi((g_1, h_1)(g, h)) = \xi(g_1g, h_1h) \\
= (g_1g, \sigma\lambda(g_1)h_1h) \\
= (g_1g, \sigma\lambda(g)\sigma\lambda(g)h_1h).
\]

Now $h_1^y = h_1^{\lambda(g)} = h_1^{\mu(y)}$ for some $y \in H$ and $\sigma\lambda(g) = y \pmod{\ker \mu}$. Hence $h_1^y = h_1^{\mu(y)} = h_1^{-1}y = \sigma\lambda(g)^{-1}h_1\sigma\lambda(g)$ since $\ker \mu$ is central in $H$. So
\[
\xi((g_1, h_1)(g, h)) = (g_1g, \sigma\lambda(g_1)h_1, \sigma\lambda(g)h) = \xi(g_1, h_1)\xi(g, h).
\]

Further, $\xi$ maps the relation (10) to
\[(g^{-1}gh, 1) = (1, h^{-g}h).
\]

Now
\[
\sigma\lambda(g^{-1}gh) = \sigma((\lambda(g), \mu(h))) \\
= [\sigma\lambda(g), h] \text{ (since $\ker \mu$ is central)} \\
= \sigma\lambda(g^{-1})h^{-1}\sigma\lambda(g)h \\
= h^{-g}h.
\]
So the kernel of the map $G \times H \to G \rtimes H$ is mapped to the normal closure in $G \times H$ of the elements $(g^{-1}g^h, 1)$ and so $G \rtimes H \cong G_H \ltimes H$.

Note that under the hypotheses of Proposition 2.1, $G_H$ is abelian and since $H$ is a crossed module with a splitting of its boundary map, we have a split central extension

$$0 \to \ker \mu \to H \to \mu(H) \to 1$$

and $H \cong \ker \mu \times \mu(H)$ as groups. Further, if we can find a $P$-equivariant splitting $\sigma$ then this isomorphism and that of Proposition 2.1 are isomorphisms of crossed $P$-modules.

2.2 Corollary. Under the hypotheses of Proposition 2.1 the canonical map $H \to G \rtimes H$ is an embedding, but the canonical map $G \to G \rtimes H$ is an embedding if and only if $\ker \lambda \cap D_H(G) = 1$.

Proof. Identifying $G \rtimes H$ with $G_H \times H$, the canonical maps in question are $g \mapsto (gD_H(G), \sigma \lambda(g))$ and $h \mapsto (1, h)$. So $H$ certainly embeds in $G \rtimes H$ and the statement for $G$ follows since $\sigma$ is injective.

2.3 Corollary. If $\lambda: G \to P$ is a crossed $P$-module then $G \rtimes P$ and $G_P \times P$ are isomorphic as crossed $P$-modules.

In particular, if $M$ is a normal subgroup of $P$ we can form the Peiffer product of $M$ and $P$ with respect to the conjugation actions and $M \rtimes P \cong M/[M, P] \times P$. So, putting $M = P$, we find $P \rtimes P \cong P_{ab} \times P$.

We now return to the general case of groups $G$ and $H$ given as crossed $P$-modules acting on each other via $\lambda: G \to P$ and $\mu: H \to P$. The kernel of $\delta: G \rtimes H \to P$ has been investigated by R. Brown in [1]. Let $G \times_P H$ be the pullback: this is again a crossed $P$-module under the diagonal action of $P$ with boundary map given by $\delta(g, h) = \lambda(g) = \mu(h)$. It is easy to verify that in fact $G \times_P H$ is the product of $G$ and $H$ in the category of crossed $P$-modules. Define the function $\zeta: G \times H \to G \times_P H$ by $\zeta(g, h) = (g^{-1}g^h, h^{-g})$ and let $J$ be the subgroup of $G \times_P H$ generated by the image of $\zeta$. Then $J$ is normal in $G \times_P H$ and contains the commutator subgroup. Let us write $M = \lambda(G)$ and $N = \mu(H)$: then there are exact sequences of groups, [1, Propositions 2.5 and 2.8],

$$0 \to (G \times_P H)/J \to G \rtimes H \to P,$$  \hspace{1cm} (11)

and

$$0 \to (\ker \lambda \oplus \ker \mu) \cap J \to \ker \lambda \oplus \ker \mu \to (G \times_P H)/J \to (M \cap N)/[M, N] \to 0,$$ \hspace{1cm} (12)

where the map $j$ in (11) is induced by the map $G \times_P H \to G \rtimes H$ given by $(g, h) \mapsto (g, h^{-1})$. If $\lambda$ and $\mu$ are injective then (11) and (12) show that $\ker \delta = (M \cap N)/[M, N]$. In particular, for any normal subgroups $M$ and $N$ of a group $P$, there is a short exact sequence

$$0 \to \frac{M \cap N}{[M, N]} \to G \rtimes N \to MN \to 1$$
showing how $M \rtimes N$ depends on the normal structure of $M$ and $N$ relative to each other. Note further that both $M$ and $N$ embed in $M \rtimes N$.

**3. Induced crossed modules and the tensor products.** We recall from [2] the definition of an induced crossed module. Suppose that $d: A \rightarrow P$ is a crossed $P$-module and that $f: P \rightarrow S$ is a homomorphism of groups. Then there is a crossed $S$-module $C = f_* A$ and a morphism of crossed modules

$$
\begin{array}{ccc}
A & \longrightarrow & f_* A \\
\downarrow & & \downarrow \\
P & \longrightarrow & S
\end{array}
$$

which is universal for morphisms from $A$ to crossed $S$-modules which induce $f: P \rightarrow S$. The crossed $S$-module $f_* A$ is said to be *induced* by $f$ and $f_*$ is a functor from crossed $P$-modules to crossed $S$-modules. A presentation of $f_* A$ is given in [2]. For our present purposes we have need only of the simpler description that applies when $f$ is surjective.

**3.1 Proposition [2].** If $f: P \rightarrow S$ is a surjective homomorphism and $A$ is a crossed $P$-module then $f_* A = A_{\ker f}$. ■

R. Brown has asked the following questions. Does the crossed module morphism of Proposition 1.4 present $G \otimes H$ as the crossed $G \rtimes H$-module induced from $G \rtimes H$ by the natural map $\psi: G \rtimes H \rightarrow G \rtimes H$? Or is $G \otimes H$ the crossed $S$-module induced from $G \smash H$ by some other morphism $G \rtimes H \rightarrow S$?

From Proposition 3.1 the induced crossed module $\psi_*(G \rtimes H)$ is obtained from $G \rtimes H$ by killing the action of $K = \ker \psi$. Since this action is by conjugation we have

$$
\psi_*(G \rtimes H) = (G \rtimes H)/[G \rtimes H, K].
$$

Let $\kappa: G \rtimes H \rightarrow \psi_*(G \rtimes H)$ be the natural map. By the universal property of induced crossed modules there is a morphism $\tau: \psi_*(G \rtimes H) \rightarrow G \otimes H$ of crossed $G \rtimes H$-modules such that $\kappa \tau = \phi$. The question at issue now is whether or not $\tau$ is an isomorphism.

We consider the simplest case, in which $G$ and $H$ act on one another trivially. In this case $G \rtimes H$ is just the direct product $G \times H$ and $K = G \rtimes H$. Thus $\psi_*(G \rtimes H)$ is $(G \rtimes H)^{ab}$ which is free abelian on the basis $\{[g, h] \mid g \neq 1, h \neq 1\}$ of mixed commutators in $G \rtimes H$, which we now wish to regard merely as a set of ordered pairs.

Since we are assuming that $G$ and $H$ act trivially on one another, from (6) we obtain the relation

$$
gg_0 \otimes h = (g_0 \otimes h)(g \otimes h).
$$

Now $G \otimes H$ is abelian (it is a homomorphic image of $(G \rtimes H)^{ab}$) and so

$$
gg_0 \otimes h = (g \otimes h)(g_0 \otimes h) = g_0 g \otimes h.
$$

Similarly

$$
g \otimes hh_0 = (g \otimes h)(g \otimes h_0) = g \otimes hh_0.
$$
It follows that $G \otimes H \cong G^{ab} \otimes_z H^{ab}$ (see [6, Proposition 2.4]) and that $G \ast H$ and $G \times H$ act trivially on $G \otimes H$.

It is now clear that the map $\tau$ from $\psi_*(G \square H) = (G \square H)^{ab}$ to $G \otimes H = G^{ab} \otimes_z H^{ab}$ is not an isomorphism unless one of $G$, $H$ is trivial; for if $g \in G$, $h \in H$, $g \neq 1$, $h \neq 1$, then $[g, h][g^{-1}, h] \neq 1$ in $(G \square H)^{ab}$ but $(g \otimes h)(g^{-1} \otimes h) = 1$ in $G \otimes H$. In fact it follows from Lemma 1.2 and Proposition 3.1 that $G \otimes H$ is obtained from $G \square H$ by killing the action of $G \ast H$, that is:

3.2 Proposition. If $G$ and $H$ act trivially on each other, $G \otimes H$ is the crossed module over the trivial group induced from $G \square H \to G \ast H$ by the map $G \ast H \to 1 \cdot 1$.

Thus there remains the possibility that $G \otimes H$ is in all cases the crossed module induced from $G \square H \to G \ast H$ by some quotient map $\chi: G \ast H \to S$. We show that this is not the case by means of the example of two infinite cyclic groups acting non-trivially on each other.

3.3 Proposition. Let $X$ and $Y$ be infinite cyclic groups generated by $x$ and $y$ respectively, acting on each other by

$$x^y = x^{-1}, \quad y^x = y^{-1}.$$ 

These actions are compatible and the Peiffer product $X \bowtie Y$ is the quaternion group $Q = \{ \pm 1, \pm i, \pm j, \pm k \}$ with canonical map $\psi: X \ast Y \to Q$ given by $\psi(x) = i$ and $\psi(y) = j$. The induced crossed $Q$-module $\psi_*(X \square Y)$ is $\mathbb{Z}^4$ and $X \otimes Y$ is $\mathbb{Z}^2$ with bases, $Q$-actions and boundary maps given by the formulae (13), . . . , (17) and (20), . . . , (24) below.

Proof. The compatibility of the given actions is easily checked. Further, it is clear that if we are given presentations of groups $G$ and $H$ that act compatibly on one another, we obtain a presentation of $G \bowtie H$ by adjoining to the natural presentation of $G \ast H$ the relations (8) between generators. Hence

$$X \bowtie Y = \langle x, y \mid y^{-1}xy = x^{-1}, x^{-1}yx = y^{-1} \rangle.$$

The map $x \mapsto i$, $y \mapsto j$ defines a surjection $X \bowtie Y \to Q$. However, in $X \bowtie Y$, $[x, y] = x^{-2}$ and $[y, x] = y^{-2}$ and so $x^2 = y^{-2} = [y, x]$. Hence

$$yx = xy[y, x] = xy^{-1} = x^3y.$$

Therefore $x = y^{-1}x^3y = x^{-3}$ and so $x^4 = 1$ and since $x^2 = y^{-2}$, $y^4 = 1$. Every element of $X \bowtie Y$ can now be written as $x^ry^s$ where $r, s = 0, 1, 2, 3$ and $x^2y^2 = 1$. This implies $|X \bowtie Y| \leq 8$ and so $X \bowtie Y \cong Q$.

We now compute $\psi_*(X \square Y)$ where $\psi: X \ast Y \to Q$ is given by $x \mapsto i$, $y \mapsto j$. Let $K = \ker \psi$, $M = \psi_*(X \square Y)$, $\phi: X \square Y \to M$ and $N = \ker(\delta: M \to Q)$. For each generator $[x^m, y^n]$ of $X \square Y$ we have

$$\psi([x^m, y^n]) = [i^m, j^n] = (-1)^{mn}.$$
Thus $\delta(M) = \{\pm 1\}$ and $N$ is central in $M$ and of index 2: hence $M$ is abelian and $\delta(M)$ acts trivially on $M$. So $i^2$, $j^2$ and $k^2$ act trivially on $M$. We write $\{x^m, y^n\}$ for $\phi([x^m, y^n])$.

Now

$$\{x^m, y^n\} = (x^m, y^n)^2 = \phi([x^m, y^n]^2) = \phi([x^{m+2}, y^n][x^2, y^n]^{-1}) = \{x^{m+2}, y^n\} \{x^2, y^n\}^{-1}.$$

So $\{x^{m+2}, y^n\} = \{x^m, y^n\} \{x^2, y^n\}$ and similarly $\{x^m, y^{n+2}\} = \{x^m, y^n\} \{x^m, y^2\}$. Thus $M$ is generated as a group by the four elements

$$\{x, y\}, \{x^2, y\}, \{x, y^2\}, \{x^2, y^2\}.$$

The actions of $i$ and $j$ on these generators are easily computed and we find

$$\begin{align*}
\{x, y\}^i &= \{x^2, y\}\{x, y\}^{-1}, & \{x, y\}^j &= \{x, y^2\}\{x, y\}^{-1}, \\
\{x^2, y\}^i &= \{x^2, y\}, & \{x^2, y\}^j &= \{x^2, y^2\}\{x^2, y\}^{-1}, \\
\{x, y^2\}^i &= \{x^2, y^2\}\{x, y^2\}^{-1}, & \{x, y^2\}^j &= \{x, y^2\}, \\
\{x^2, y^2\}^i &= \{x^2, y^2\}, & \{x^2, y^2\}^j &= \{x^2, y^2\}.
\end{align*}$$

So $M$, as a $Q$-module, is a homomorphic image of $\mathbb{Z}^4$ with basis $b_1, b_2, b_3, b_4$ mapping respectively to $\{x, y\}, \{x^2, y\}, \{x, y^2\}, \{x^2, y^2\}$ and with the action of $i$ and $j$ given by

$$\begin{align*}
b_1^i &= b_2 - b_1, & b_1^j &= b_3 - b_1, \\
b_2^i &= b_2, & b_2^j &= b_4 - b_2, \\
b_3^i &= b_4 - b_3, & b_3^j &= b_3, \\
b_4^i &= b_4, & b_4^j &= b_4.
\end{align*}$$

It is easy to verify that the map $d: \mathbb{Z}^4 \to Q$ given by

$$b_1 \mapsto -1, \quad b_r \mapsto 1 \quad (r \geq 2),$$

is a crossed module and that the map $\theta: X \square Y \to \mathbb{Z}^4$ given by

$$\begin{align*}
\theta([x^{2r}, y^{2s}]) &= rsb_4, \\
\theta([x^{2r+1}, y^{2s}]) &= sb_3 + rsb_4, \\
\theta([x^{2r}, y^{2s+1}]) &= rb_2 + rsb_4, \\
\theta([x^{2r+1}, y^{2s+1}]) &= b_1 + rb_2 + sb_3 + rsb_4,
\end{align*}$$

is a crossed module.
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gives a morphism of crossed modules

\[ \begin{array}{ccc}
X \square Y & \rightarrow & Z^4 \\
\downarrow & & \downarrow \\
F(x, y) & \rightarrow & Q.
\end{array} \]

By the universal property of \( \delta : M \rightarrow Q \) there is a morphism of crossed modules

\[ \begin{array}{ccc}
M & \rightarrow & Z^4 \\
\downarrow & & \downarrow \\
Q & \rightarrow & Q.
\end{array} \]

and it follows that \( M \) is isomorphic as a crossed \( Q \)-module to \( Z^4 \).

Now \( X \otimes Y \) is abelian (since \( M \) is) and in \( X \otimes Y \),

\[ x \otimes y^2 = (x \otimes y)(x^{-1} \otimes y), \]
\[ x \otimes y = x^{-1}x^2 \otimes y = (x^{-1} \otimes y)(x^2 \otimes y), \]

and so

\[ (x \otimes y^2)(x^2 \otimes y) = (x \otimes y)^2. \] (18)

Further, \( x \otimes y^2 = x^{-1}x^2 \otimes y^2 = (x^{-1} \otimes y^2)(x^2 \otimes y^2) \) so that

\[ x^{-1} \otimes y^2 = (x \otimes y^2)(x^2 \otimes y^2)^{-1}. \]

But also

\[ x^{-1} \otimes y^2 = (x \otimes y^2)^y = (x \otimes y)^{-1}(x \otimes y^3) \]
\[ = (x \otimes y)^{-1}(x \otimes y^2)(x \otimes y) \]
\[ = x \otimes y^2. \]

Therefore

\[ x^2 \otimes y^2 = 1. \] (19)

(18) and (19) show that \( x \otimes y \) and \( x^2 \otimes y \) generate \( X \otimes Y \) as an abelian group and the action of \( Q \) is given by

\[ (x \otimes y)^i = x \otimes y^{-1} = (x^2 \otimes y)(x \otimes y)^{-1}, \] (20)
\[ (x \otimes y)^i = x^{-1} \otimes y = (x^2 \otimes y)^{-1}(x \otimes y), \] (21)
\[ (x^2 \otimes y)^i = x^2 \otimes y^{-1} = x^2 \otimes y, \] (22)
\[ (x^2 \otimes y)^i = x^{-2} \otimes y = (x^2 \otimes y)^{-1}. \] (23)

It is now straightforward to show that \( Z^2 \), with basis

\[ x \otimes y, \quad x^2 \otimes y \] (24)

and the action just given, is a crossed \( Q \)-module and that the defining relations for \( X \otimes Y \) are satisfied. ■
As a consequence of this computation, we see that in the diagram

$$X \boxdot Y \longrightarrow \mathbb{Z}^4 \longrightarrow \mathbb{Z}^2 = X \otimes Y$$

$$X \star Y \longrightarrow Q$$

the only elements of $Q$ which act trivially on $X \otimes Y$ are $\pm 1$, and these already act trivially on $\mathbb{Z}^4$. Therefore, all elements of $X \star Y$ which act trivially on $X \otimes Y$ already act trivially on $\mathbb{Z}^4$ and it follows that $X \otimes Y$ cannot be obtained from $X \boxdot Y$ by killing the action of a normal subgroup of $X \star Y$. Thus $X \otimes Y$ is not the crossed $S$-module induced from $X \boxdot Y \rightarrow X \star Y$ by any surjection $X \star Y \rightarrow S$. It seems to be difficult (but less interesting) to determine when $X \otimes Y$ is induced by a non-surjective map $X \star Y \rightarrow S$.

We conclude with a discussion of another special case of the tensor product. Given any group $G$ we may form its tensor square $G \otimes G$ using the conjugation action of $G$ on itself. Then $G \otimes G$ is a $G \star G$-group and the images $g_1$ and $g_2$ of $g \in G$ in the factors of $G \star G$ each act via

$$(x \otimes y)^g = g^{-1}xg \otimes g^{-1}yg \quad (i = 1, 2).$$

It follows that the kernel of the folding map $G \star G \rightarrow G$ which identifies the two copies of $G$ in $G \star G$ acts trivially on $G \otimes G$ and that $G \otimes G$ is a crossed $G$-module with $G$-action given by $(x \otimes y)^g = g^{-1}xg \otimes g^{-1}yg$ and boundary map $\delta : G \otimes G \rightarrow G$ by $x \otimes y \mapsto [x, y]$. We refer to [4] and [6] for further results on and applications of the tensor square.

The question of the relationship between $G \boxdot G$ and $G \otimes G$ first arose in conversations between H. J. Baues and R. Brown. We shall show that $G \otimes G$ is not induced from the inclusion map $G \boxdot G \rightarrow G \star G$ by the folding map $G \star G \rightarrow G$.

Let $(G, G)$ denote the induced crossed module just described. Then $(G, G)$ is obtained from $G \boxdot G$ by killing the action of the kernel of the folding map, that is by making the two images of $g \in G$ in $G \star G$ act in the same way. It follows that $(G, G)$ is the group generated by all pairs $(x, y)$ where $x, y \in G$, subject to defining relations

$$(1, x) = 1 = (x, 1),$$

$$(x, y)(xy, z) = (x, zy)(y, z).$$

The $G$-action is given by

$$(x, y)^g = (xg, y)(g, y)^{-1} = (x, g)^{-1}(x, yg)$$

and the boundary map is $d : (x, y) \mapsto [x, y]$. So if $G$ is abelian, $d$ is the zero map and $(G, G)$ is abelian. It is easy to see that if $G$ is cyclic of order 2 then $(G, G)$ is infinite cyclic, whereas $G \otimes G \cong G$.

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