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ON WEIGHTED INDUCTIVE LIMITS OF SPACES OF FRÉCHET-VALUED CONTINUOUS FUNCTIONS

ANTONIO GALBIS

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Abstract

In this article we continue the study of weighted inductive limits of spaces of Fréchet-valued continuous functions, concentrating on the problem of projective descriptions and the barrelledness of the corresponding "projective hull." Our study is related to the work of Vogt on the study of pairs (E, F) of Fréchet spaces such that every continuous linear mapping from E into F is bounded and on the study of the functor $\operatorname{Ext}^1(E, F)$ for pairs (E, F) of Fréchet spaces.

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In this article we continue the study of weighted inductive limits of spaces of Fréchet-valued continuous functions (see [4], [5], [6], [7], [8]), concentrating on the problem of projective descriptions and the barrelledness of the corresponding "projective hull". We complete and extend certain results of [4]. Our study is related to the work of Vogt on the study of pairs (E, F) of Fréchet spaces such that $L(E, F) = LB(E, F)$ and on the study of the functor $\operatorname{Ext}^1(E, F)$ for pairs (E, F) of Fréchet spaces (see [14], [15]).

To treat countable inductive limits of weighted spaces of continuous or holomorphic functions, Bierstedt, Meise and Summers [2], [3] developed the method of a "projective description". One of the original aims was to give a canonical and useful description of the inductive limit topology and its continuous seminorms by associating a certain "projective hull." A particular

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case of the problem of projective description stated in [2, 0.5] is the following. Given a decreasing sequence $\mathscr{V} = (v_n)$ of strictly positive continuous weights on a locally compact topological space X and a locally convex space E find conditions to ensure that the weighted inductive limit $\mathscr{V}_0C(X, E)$ is a topological subspace of (or coincides algebraically with) its projective hull $C\overline{V}_0(X, E)$.

The problem on the algebraic identity $\mathscr{V}_0C(X, E) = C\overline{V}_0(X, E)$ for a Fréchet space E has been completely solved by Bonet and the author in [7], in connection with the identity $L(E, F) = LB(E, F)$ for pairs of Fréchet spaces E and F (see [14]). Bonet [6] proved that a condition introduced by Vogt in [14] on a Fréchet space E and a sequence of weights $\mathscr{V} = (v_n)$ on $\mathbb{N}$ is necessary and sufficient for $k_0(\mathscr{V}, E)$ to be a topological subspace of $K_0(\overline{V}, E)$, and the same condition works in the setting of co-echelon spaces of order p, $1 \leq p < \infty$, as was proved in [8].

In Section 1 we shall use an adequate extension of this condition to function spaces to characterize when $\mathscr{V}_0C(X, E)$ is a topological subspace of $C\overline{V}_0(X, E)$, E being a Fréchet space (Theorem 1.1). Our new proof is more elementary and differs from the ones in [6] and [8] in the sense that it does not use duality. Our general characterization is complemented by some evaluations of the conditions (see Corollary 1.3 and Observation 1.4).

We observe that if $\mathscr{V}_0C(X, E)$ is a topological subspace of $CV_0(X, E)$ then $CV_0(X, E)$ is barrelled, but the converse do not hold (see, for example, [4, 2.10]). In Section 2 we study the barrelledness of $C\overline{V}_0(X, E)$ for a Fréchet space E and certain sequences $\mathscr{V} = (v_n)$ of weights on X. We apply the methods of [12] and [16]. The results we deduce complement and slightly improve those obtained by Bierstedt and Bonet in [4] for the discrete case.

Our notation for locally convex spaces is standard and can be seen in [10], [11] and [13]. The notation for weighted inductive limits can be seen in [2], and the vector valued cases in [5]. For a recent report on the projective description of weighted inductive limits of vector-valued continuous functions see [4].

1. Projective descriptions of $\mathscr{V}_0C(X, E)$

1.1. THEOREM. *Let X be a locally compact topological space, $\mathscr{V} = (v_n)$ a decreasing sequence of strictly positive continuous weights on X. If E is a Fréchet space with a fundamental system of seminorms $(\|\cdot\|_k)$, the following conditions are equivalent:*

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(1) *$\mathscr{V}_0C(X, E)$ is a topological subspace of $C\overline{V}_0(X, E)$;*

(2) *for every strictly increasing sequence $(k(l))$ in $\mathbb{N}$ there is $k \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ there are $l_0 = l_0(n)$, $C_n > 0$ with*

$$\frac{\|u\|_k^*}{v_n(x)} \le C_n \max\left(\frac{\|u\|_{k(l)}^*}{v_l(x)} : l = 1, \dots, l_0(n)\right)$$

for all $x \in X$ and $u \in E'$.

Remarks. (i) $\|\cdot\|_k^*$ denotes the dual seminorms of E'.

(ii) More information about condition (2) can be seen in [14].

Proof. (1) implies (2). We suppose $(\mathscr{V}, E)$ does not satisfy condition (2) for a given sequence $(k(l))$. We can assume $k(1) = 1$. Proceeding by recurrence we can easily find sequences $(n(i)) \subset \mathbb{N}$, $(x_i) \subset X$, $(u_i) \subset E'$, such that

(1) for every i there is a compact neighbourhood U_i of x_i, with (U_i) pairwise disjoint and

$$\text{(2)} \qquad \frac{\|u_i\|_i^*}{v_{n(i)}(x_i)} > i \max\left(\frac{\|u_i\|_{k(l)}^*}{v_l(x_i)} : l = 1, \dots, i\right).$$

We put $w_n(i) := v_n(x_i)$ $(n, i \in \mathbb{N})$. Then $\omega = (w_n)$ is a decreasing sequence of weights on $\mathbb{N}$ and we denote by $\overline{W}$ its maximal Nachbin family [2]. We can apply [6, Theorem] to deduce that $k_0(\omega, E)$ is not a topological subspace of $K_0(\overline{W}, E)$. Let B a 0-neighbourhood in $k_0(\omega, E)$ which is not a 0-neighbourhood for the topology induced by $K_0(\overline{W}, E)$ and let $l(n)$ be such that $\{z \in c_0(w_n, E) : \sup w_n(i)\|z(i)\|_{l(n)} \le 1\}$ is contained in B, $n \in \mathbb{N}$.

We take $A_n := \{f \in C(v_n)_0(X, E) : \sup v_n(x)\|f(x)\|_{l(n)} \le 1\}$, $n \in \mathbb{N}$. Then $A := acx(\bigcup(A_n : n \in N))$ is a 0-neighbourhood in $\mathscr{V}_0C(X, E)$ which is not a 0-neighbourhood for the topology induced by $C\overline{V}_0(X, E)$. In fact, given $\overline{v} = \inf \alpha_n v_n \in \overline{V}$ and $k \in \mathbb{N}$ let $\overline{w} \in \overline{W}$ be given by $\overline{w} := \inf \alpha_n w_n$. There is $z \in k_0(\omega, E)$ such that $\sup w(i)\|z(i)\|_k < 1$ but $z \notin B$. We can find a strictly increasing sequence $(n(i))$ in $\mathbb{N}$ with $\alpha_{n(i)} v_{n(i)}(x_i)\|z(i)\|_k < 1$, $i \in \mathbb{N}$. Let Ω_i be a compact neighbourhood of x_i, $\Omega_i \subset U_i$, such that $v_j(x)/v_j(y) \le 2$ for every $x, y \in \Omega_i$ and every $j = 1, \dots, n(i)$. Let φ_i be a continuous function with compact support contained in Ω_i, $0 \le \varphi_i \le 1$, $\varphi_i(x_i) = 1$ $(i \in \mathbb{N})$. Then $f := \sum(\varphi_i \otimes z(i) : i \in \mathbb{N})$ is an element of $\mathscr{V}_0C(X, E)$ satisfying $\sup \overline{v}(x)\|f(x)\|_k \le 2$, but $f \notin A$.

(2) implies (1). Let W be an *absx* 0-neighbourhood in $\mathscr{V}_0C(X, E)$. There is an increasing sequence $(k(n))$ of natural numbers such that $\{f \in C(v_n)_0(X, E) : \sup v_n(x)\|f(x)\|_{k(n)} \le 1\}$ is contained in W, $n \in \mathbb{N}$. Applying condition (2) and a polarity argument we conclude that there is $k > k(1)$

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such that for all $n \in \mathbb{N}$ there are $l_0 = l_0(n)$, $C_n > 0$ with

$$\frac{U_k}{v_n(x)C_n} \subset \sum_{l=1}^{l_0(n)} 2^{-l} \frac{U_{k(l)}}{v_l(x)} \quad \text{for every } x \in X. \tag{1}$$

We take $\overline{v} := \inf 2^n C_n v_n \in \overline{V}$ and fix $f \in C(X, E)$ with compact support and satisfying $\sup \overline{v}(x)\|f(x)\|_k \leq 1$. We shall prove that $f \in 5W$. To do this we observe that we can find $m \in \mathbb{N}$ such that for every $x \in \operatorname{supp} f$ there is $1 \leq n \leq m$ with $2^n C_n v_n \|f(x)\|_k < 1$, and we take an open cover $(\Omega_j : 1 \leq j \leq p)$ of $\operatorname{supp} f$ with the following properties.

(2) $\operatorname{supp} f$ is not contained in $\bigcup(\Omega_i : i \neq j)$, $j = 1, \ldots, p$.

(3) $C_1 v_1(x)\|f(x) - f(y)\|_k < 1$ for every $x, y \in \Omega_j$. Consequently
- (a) $\overline{v}(x)\|f(x) - f(y)\|_k < 1$ and
- (b) $v_1(x)\|f(x) - f(y)\|_{k(1)} < 1$.

(4) $v_l(x)/v_l(y) \leq 2$ for every $x, y \in \Omega_j$, $l = 1, \ldots, m$.

We fix $x_j \in \Omega_j$, $x_j \notin \bigcup(\Omega_i : i \neq j)$ and take $(f_j : 1 \leq j \leq p)$ a partition of unity subordinated to $(\Omega_j : 1 \leq j \leq p)$, $e_j := f(x_j)$. Now we put

$$g := \sum_{j=1}^{p} f_j \otimes e_j.$$

Then $g(x_j) = e_j$, $\sup \overline{v}(x)\|g(x)\|_k < 2$ (from (3)(a)) and $f - g \in W$ (from (3)(b)). For every $1 \leq j \leq p$ there is $1 \leq n(j) \leq m$ such that $2^{n(j)} C_{n(j)} v_{n(j)} \|e_j\|_k < 2$. We set $J_s := \{1 \leq j \leq p : n(j) = s\}$ and

$$g_s := \sum_{j \in J_s} f_j \otimes e_j, \qquad (s = 1, \ldots, m).$$

Given $j \in J_s$, we apply (1) to write

$$e_j = w^{-s+1} \sum_{l=1}^{l_0(s)} e_{j,l}, \qquad e_{j,l} \in 2^{-l} \frac{U_{k(l)}}{v_l(x_j)}.$$

We define

$$g_{s,l} := 2 \sum_{j \in J_s} f_j \otimes e_{j,l}, \qquad (1 \leq s \leq m,\ 1 \leq l \leq l_0(s)).$$

Clearly

$$v_l(x)\|g_{s,l}(x)\|_{k(l)} \leq 2 \sum_{j \in J_s} f_j(x) 2^{-l} \frac{v_l(x)}{v_l(x_j)} \leq 2^{-l+2}, \qquad x \in X.$$

Consequently every g_s belongs to $4W$ and we deduce $g \in 4W$, which implies $f \in 5W$.

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Since the subspace of all continuous functions with compact support is dense in $\mathcal{V}_0C(X, E)$ we conclude that $\mathcal{V}_0C(X, E)$ is a topological subspace of $C\overline{V}_0(X, E)$.

1.2. REMARK. Let X be a locally compact topological space, $\mathcal{V} = (v_n)$ a decreasing sequence of strictly positive and continuous weights on X and (U_n) a sequence of open relatively compact pairwise disjoint subsets of X satisfying $v_j(x)/v_j(y) \leq 2$ for every $x, y \in U_n$, $j = 1, \dots, n$ $(n \in \mathbb{N})$. For each $n \in \mathbb{N}$ let φ_n be a positive continuous function with support contained in U_n and taking the value 1 at some point $x_n \in U_n$. We define $\omega_n(i) := v_n(x_i)$ $(i, n \in \mathbb{N})$. Then $\omega = (\omega_n)$ is a decreasing sequence of weights on $\mathbb{N}$. It is implicit in the proof above ((1) implies (2)) that the map $C(\mathbb{N}, E) \to C(X, E)$, given by $(z(i)) \to \sum(\varphi_i \otimes z(i) \colon i \in N)$, induces a topological imbedding from $k_0(\omega, E)$ into $\mathcal{V}_0C(X, E)$ (E being an arbitrary locally convex space).

Proceeding as [14, 4.2] and [14, 5.2] we obtain the following

1.3. COROLLARY. *Let X be locally compact, $0 < v < 1$ a continuous weight on X and $\mathcal{V} = (v_n)$ defined by $v_n := v^n$ (respectively $v_n := v^{1-1/n}$). For a Fréchet space E the following holds.*

(1) *If E has the property (LB^∞) (respectively property $(\overline{\overline{\Omega}})$) (see [14]) then $\mathcal{V}_0C(X, E)$ is a topological subspace of $C\overline{V}_0(X, E)$.*

(2) *If there is a sequence $(x_j) \subset X$ such that $(v(x_j))$ is decreasing and converges to zero and $\sup(\log v(x_j)/\log v(x_{j+1}) \colon j \in \mathbb{N}) < \infty$ then the converse of (1) holds.*

The following result of universal type is a consequence of [17, 0.3] and [5, 2.7] and can be deduced as in [8, 1.6]. In fact, that (2) implies (1) was already extended in [7, 3.5].

1.4. OBSERVATION. Let E be a Fréchet space. The following conditions are equivalent:

(1) $\mathcal{V}_0C(X, E)$ is a topological subspace of $CV_0(X, E)$ for every locally compact topological space X and every decreasing sequence $\mathcal{V} = (v_n)$ of strictly positive and continuous weights in X;

(2) E'' is a quojection (cf. [1]).

2. Barrelledness of $C\overline{V}_0(X, E)$

In this section we study the barrelledness of the projective hull $C\overline{V}_0(X, E)$ of the inductive limit $\mathcal{V}_0C(X, E)$. We observe that $C\overline{V}_0(X, E)$ is barrelled

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if $\mathscr{V}_0C(X, E)$ is a topological subspace of $C\overline{V}_0(X, E)$ but the converse does not hold, as an adaptation of a classical example shows [4, 2.10].

The following result improves [4, 2.10 (2)] and [12, Theorem 2.2, (1) implies (3)].

2.1. **Lemma.** *Let X be a locally compact topological space, $\mathscr{V} = (v_n)$ a decreasing sequence of strictly positive and continuous weights on X, $A = (a_k(j))$ a Köthe matrix $(k \in \mathbb{N},\ j \in I)$ and E a Banach space. If $(\mathscr{V}, \lambda(A))$ satisfies the condition*

(S) $\forall \mu\ \exists n_0, k\ \forall K, m\ \exists n, S\ \forall x, j$:

$$\frac{1}{a_k(j)v_m(x)} \leq S \max\left(\frac{1}{a_k(j)v_n(x)}, \frac{1}{a_\mu(j)v_{n_0}(x)}\right)$$

and if F is a quotient of $\lambda(A, E)$, then $CV_0(X, E)$ is barrelled.

Remark. Let $B = (b_n)$ be given by $b_n(x) := v_n(x)^{-1}$ $(n \in \mathbb{N},\ x \in X)$. Condition (S) is equivalent to $(\lambda(B), \lambda(A)) \in (S_2^*)$ (see [16, page 359]).

Proof. (a) We first assume $F = \lambda(A, E)$ and we adapt the proof of [12, 2.2]. Let $\|\cdot\|$ denote the norm on E and

$$\|u\|_k := \sum_{i=1}^{\infty} a_k(i)\|u(i)\|, \qquad u = (u(i)) \in \lambda(A, E),\ k \in \mathbb{N}.$$

We set $B_{k,m} := \{f \in C(v_m)_0(X, F) \colon \sup v_m(x)\|f(x)\|_k < 1\}$ $(k, m \in \mathbb{N})$. Let T be a bornivorous barrel in $C\overline{V}_0(X, E)$. We claim that there is $k \in \mathbb{N}$ such that T absorbes $B_{k,m}$ for every $m \in \mathbb{N}$. If this is not satisfied, we can find an increasing sequence $(m(k))$ in $\mathbb{N}$ such that T does not absorb $B_{k,m(k)}$ and we can suppose, proceeding as in [12, 1.5], that

$$\frac{1}{a_k(j)v_{m(k)}(x)} \leq \max\left(\frac{1}{a_{k+1}(j)v_{m(k+1)}(x)}, 2^{-k}\frac{1}{a_{k-1}(j)v_{m(k-1)(x)}}\right).$$

We observe that, for given x, j,

$$\frac{1}{a_k(j)v_{m(k)}(x)} \leq \frac{1}{a_{k+1}(j)v_{m(k+1)}(x)} \tag{1}$$

implies

$$\frac{1}{a_l(j)v_{m(l)}(x)} \leq \frac{1}{a_{l+1}(j)v_{m(l+1)}(x)} \quad \text{for all } l \geq k$$

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and

(2) $$\frac{1}{a_k(j)v_{m(k)}(x)} \leq 2^{-k} \frac{1}{a_{k-1}(j)v_{m(k-1)}(x)}$$

implies

$$\frac{1}{a_l(j)v_{m(l)}(x)} \leq 2^{-l} \frac{1}{a_{l-1}(j)v_{m(l-1)}(x)} \quad \text{for every } 2 \leq l \leq k.$$

1. First we prove

$$B_{k,m(k)} \subset \overline{\bigcap(2B_{l,m(l)}\colon k \leq l \leq p) + \bigcap(2B_{l,m(l)}\colon 1 \leq l \leq k)}$$

for every $k \in \mathbb{N}$ and every $p > k$ (the closure taken in $C(v_{m(k)})_0(X, F)$). To do this we fix $f \in B_{k,m(k)}$ with compact support, $l > k$, $l \in \mathbb{N}$, and $\varepsilon > 0$. We take an open cover $(\Omega_j\colon 1 \leq j \leq m)$ of $\operatorname{supp} f$ such that, for each $1 \leq j \leq m$, $\operatorname{supp} f$ is not contained in $\bigcup(\Omega_i\colon i \neq j)$ and

$$v_{m(k)}(x)\|f(x) - f(y)\|_l < \varepsilon v_{m(k)}(x)/v_{m(k)}(y) \leq 2 \quad \text{for every } x, y \in \Omega_j.$$

Let $x_j \in \Omega_j \bigcap \operatorname{supp} f$ be a point which does not belong to $\bigcup(\Omega_i\colon i \neq j)$ and take f_j $(1 \leq j \leq m)$ a continuous function with compact support contained in Ω_j, $0 \leq f_j \leq 1$, satisfying $\sum(f_j(x)\colon 1 \leq j \leq m) = 1$ on $\operatorname{supp} f$ and $\sum(f_j(x)\colon 1 \leq j \leq m) \leq 1$ on X. We write

$$f(x_j) = \sum_{i=1}^{\infty} u_{ji} e_i \in \lambda(A, E)$$

(e_i being the canonical unit vectors of $\lambda(A)$, $u_{ji} \in E$) and we suppose $f \in B_{k,m(k)}$. We observe that

$$v_{m(k)}(x_j) \sum_{i=1}^{\infty} a_k(i)\|u_{ji}\| \leq 1, \qquad 1 \leq j \leq m.$$

We put

$$\begin{aligned} u_{ji}^+ &:= u_{ji} && \text{if } \tfrac{1}{a_k(i)v_{m(k)}(x_j)} \leq \tfrac{1}{a_{k+1}(i)v_{m(k+1)}(x_j)}, \\ u_{ji}^+ &:= 0 && \text{otherwise}, \\ u_{ji}^- &= u_{ji} - u_{ji}^+, \end{aligned}$$

and we define

$$f^+ := \sum_{j=1}^{m} f_j \otimes \left(\sum_{i=1}^{\infty} u_{ji}^+ e_i\right), \qquad f^- := \sum_{j=1}^{m} f_j \otimes \left(\sum_{i=1}^{\infty} u_{ji}^- e_i\right).$$

For $k \leq l \leq p$, (1) implies

$$v_{m(l)}(x)\|f^+(x)\|_l \leq 2\sum_{j=1}^{m} f_j(x) \left(\sum_{i=1}^{\infty} \|u_{ji}^+\| a_k(i) v_{m(k)}(x_j)\right) \leq 2$$

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and for $1 \le l \le k$, (2) implies

$$v_{m(l)}(x)\|f^{-}(x)\|_l \le 2\sum_{j=1}^{m} f_j(x)\left(\sum_{i=1}^{\infty}\|u_{ji}^{-}\|a_k(i)v_{m(k)}(x_j)\right) \le 2.$$

Then, $g := f^{+} + f^{-} \in \bigcap(2B_{l,m(l)}: k \le l \le p) + \bigcap(2B_{l,m(l)}: 1 \le l \le k)$ and

$$v_{m(k)}(x)\|f(x) - g(x)\|_l = v_{m(k)}(x)\left\|\sum_{j=1}^{m} f_j(x)(f(x) - f(x_j))\right\|_l \le \varepsilon$$

for every $x \in X$.

2. Now we can proceed as in [19, 5.3] to derive a contradiction and the claim is proved. Hence, we can find $k \in \mathbb{N}$, $(\alpha_m : m \in \mathbb{N})$, $\alpha_m > 0$, such that $\{f \in C(v_m)_0(X, F) : \sup \alpha_m v_m(x)\|f(x)\|_k < 1\}$ is contained in T. Now we take $\overline{v} := \inf 2^m \alpha_m v_m$. By a partition of unity argument one obtains $\{f \in C\overline{V}_0(X, F) : \sup \overline{v}(x)\|f(x)\|_k < 1\} \subset T$ (see [2, Lemma 1.1]).

(b) Now let us assume there is a quotient map $Q: \lambda(A, E) \to F$. Let $\tilde{Q}$ denote the induced map $\tilde{Q}: C\overline{V}_0(X, \lambda(A, E)) \to C\overline{V}_0(X, F)$, $\tilde{Q}(f) = Q \circ f$. Let T be a barrel in $C\overline{V}_0(X, F)$. Then $\tilde{Q}^{-1}(T)$ is a 0-neighbourhood in $C\overline{V}_0(X, \lambda(A, E))$ because of (a) and, since $C\overline{V}_0(X)$ is an ε-space ([5, 2.4]), there is a 0-neighbourhood W in $C\overline{V}_0(X) \otimes_\varepsilon F$ which is contained in $\tilde{Q}(\tilde{Q}^{-1}(T) \cap (C\overline{V}_0(X) \otimes_\varepsilon \lambda(A, E))) \subset T$.

At this point we observe that in a similar way to [4, 10] (also compare with [12, 2.3]) it is possible to prove the following

2.2. Proposition. *Let X be a locally compact topological space, $\mathscr{V} = (v_n)$ a decreasing sequence of strictly positive and continuous weights on X and F a Fréchet space. If $C\overline{V}_0(X, F)$ is barrelled then $(\mathscr{V}, F)$ satisfies the condition*

$$(\mathrm{S}_2^*) \qquad \begin{gathered} \forall \mu\ \exists n_0, k\ \forall K, m\ \exists n, S\ \forall x \in X\ \forall u \in F' \\ \frac{\|u\|_k^*}{v_m(x)} \le S \max\left(\frac{\|u\|_K^*}{v_n(x)}, \frac{\|u\|_\mu^*}{v_{n_0}(x)}\right). \end{gathered}$$

2.3. Observation. Given Köthe matrices $A = (a_i(n))$, $B = (b_j(n))$, $(n \in \mathbb{N},\ i \in I,\ j \in J)$ such that $\lambda(B)$ is a Schwartz space and given a Banach space E, the following conditions are equivalent:

(1) $L_b(\lambda(B), \lambda(A, E))$ is barrelled;

(2) $(\lambda(B), \lambda(A)) \in (\mathrm{S}_2^*)$.

Observe that our next result avoids the nuclearity assumptions in [16, 4.9].

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2.4. COROLLARY. *Let* X *be locally compact,* $0 < v < 1$ *a continuous weight on* X *and* $\mathscr{V} = (v_n)$ *given by* $v_n := v^n$ *(respectively* $v_n := v^{1-1/n}$*). For a Fréchet space* E *the following hold.*

(1) *If* E *has the property* (Ω) *(respectively the property* $(\overline{\overline{\Omega}})$ *) (see* [14]*) then* $C\overline{V}_0(X, E)$ *is barrelled.*

(2) *If there is a sequence* $(x_j) \subset X$ *such that* $(v(x_j))$ *is decreasing and converges to zero and* $\sup(\log v(x_j)/\log v(x_{j+1}) \colon j \in \mathbb{N}) < \infty$ *then the converse of* (1) *holds.*

PROOF. (1) If E satisfies the property (Ω) and $v_n := v^n$ $(n \in \mathbb{N})$ then there is an index set I such that E is a quotient of $l_1(I)\hat{\otimes}s$ [18, 3.1] and we can apply Lemma 2.1. If E satisfies the property $(\overline{\overline{\Omega}})$ and $v_n := v^{1-1/n}$ $(n \in \mathbb{N})$ then $\mathscr{V}_0C(X, E)$ is a dense topological subspace of $C\overline{V}_0(X, E)$ (Corollary 1.3). To prove (2) one proceeds as in [15, 4.1] and [15, 4.2].

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Facultad de Matemáticas
Universitat de València
Burjasot (Valencia)
Spain

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