A SUM OF RECIPROCALS OF LEAST COMMON MULTIPLES

BY

D. BORWEIN

The purpose of this note is to prove the following theorem conjectured by P. Erdős.

**Theorem.** Let \( a_0, a_1, \ldots, a_k \) be integers satisfying \( 1 < a_0 < a_1 < \cdots < a_k \), and let \([a_{i-1}, a_i]\) denote the least common multiple of \( a_{i-1} \) and \( a_i \). Then

\[
\frac{1}{[a_0, a_1]} + \frac{1}{[a_1, a_2]} + \cdots + \frac{1}{[a_{k-1}, a_k]} \leq 1 - \frac{1}{2^k},
\]

with equality occurring if and only if \( a_i = 2^i \) for \( 1 \leq i \leq k \).

**Proof.** For \( i = 1, 2, \ldots, k \), let \( c_i = [a_{i-1}, a_i] \), and let

\[
s_i = \frac{1}{c_1} + \frac{1}{c_2} + \cdots + \frac{1}{c_i}.
\]

Then \( c_i = u_ia_{i-1} = v_ia_i \) where \( u_i > v_i \geq 1 \). Hence

\[
\frac{1}{c_i} \leq \frac{1}{a_i},
\]

and, since \( c_i^{-1} \leq (u_i - v_i)c_i^{-1} \),

\[
\frac{1}{c_i} \leq \frac{1}{a_{i-1}} - \frac{1}{a_i}.
\]

It follows from (3) that

\[
s_i \leq \frac{1}{a_0} - \frac{1}{a_i}.
\]

To establish (1) we consider three cases which exhaust all possible conditions on the integers \( a_0, a_1, \ldots, a_k \).

**Case 1.** \( a_k \leq 2^k \). Then, by (4),

\[
s_k \leq 1 - \frac{1}{a_k} \leq 1 - \frac{1}{2^k}.
\]
CASE 2. \(a_i > 2^i\) for \(1 \leq i \leq k\). Then, by (2),
\[
s_k \leq \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} < \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} = 1 - \frac{1}{2^k}.\]

CASE 3. \(a_i \leq 2^i\) for some positive integer \(j < k\), and \(a_i > 2^i\) for \(j + 1 \leq i \leq k\). Then, by (2) and (4),
\[
s_k = s_j + \frac{1}{c_{j+1}} + \cdots + \frac{1}{c_k} < \frac{1}{2^j} + \frac{1}{2^{j+1}} + \cdots + \frac{1}{2^k} = 1 - \frac{1}{2^k}.\]

Thus (1) holds in all three cases. Further, it is immediate that equality occurs in (1) when \(a_i = 2^i\) for \(1 \leq i \leq k\).

Suppose next that
\[
s_k = 1 - \frac{1}{2^k}.\]

Then, by (4), we have \(1 - 2^{-k} \leq 1 - a_k^{-1}\) so that \(a_k \geq 2^k\); and we cannot have \(a_k > 2^k\) for Case 2 and Case 3 show that this would lead to \(s_k < 1 - 2^{-k}\). Hence
\[
a_k = 2^k.\]

If \(k = 1\) there is nothing further to prove. For \(k > 1\), we have, by (1) with \(k - 1\) in place of \(k\), and (2), that
\[
1 - \frac{1}{2^{k-1}} \geq s_{k-1} = s_k - \frac{1}{c_k} \geq 1 - \frac{1}{2^{k-1}} - \frac{1}{2^k} = 1 - \frac{1}{2^{k-1}}.\]

Hence
\[
s_{k-1} = 1 - \frac{1}{2^{k-1}},\]

and repetition yields the desired conclusion that
\[
a_i = 2^i\quad \text{for } 1 \leq i \leq k.\]