INEQUALITIES FOR THE SCHATTEN p-NORM II

by FUAD KITTANEH

(Received 15 August, 1985)

This paper is a continuation of [**3**] in which some inequalities for the Schatten p-norm were considered. The purpose of the present paper is to improve some inequalities in [**3**] as well as to give more inequalities in the same spirit.

Let H be a separable, infinite dimensional complex Hilbert space, and let $B(H)$ denote the algebra of all bounded linear operators acting on H. Let $K(H)$ denote the closed two-sided ideal of compact operators on H. For any compact operator A, let $|A| = (A^*A)^{1/2}$ and $s_1(A), s_2(A), \ldots$ be the eigenvalues of $|A|$ in decreasing order and repeated according to multiplicity. A compact operator A is said to be in the Schatten p-class $C_p (1 \leq p < \infty)$, if $\sum s_i(A)^p < \infty$. The Schatten p-norm of A is defined by $\|A\|_p = (\sum s_i(A)^p)^{1/p}$. This norm makes C_p into a Banach space. Hence C_1 is the trace class and C_2 is the Hilbert–Schmidt class. It is reasonable to let C_∞ denote the ideal of compact operators $K(H)$, and $\|.\|_\infty$ stand for the usual operator norm.

If $A \in C_1$ and $\{e_i\}$ is an orthonormal basis of H, then the trace of A, denoted by $\operatorname{tr} A = \sum (Ae_i, e_i)$ is independent of the choice of $\{e_i\}$. If $A \in C_p$ and $B \in C_q$, then $AB \in C_1$, $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, and $|\operatorname{tr}(AB)| \leq \|A\|_p \|B\|_q$ whenever $1/p + 1/q = 1$. If $\{e_i\}$ is any orthonormal set in H, then for $A \in C_p$, $\|A\|_p^p \geq \sum |(Ae_i, e_i)|^p$. The reader is referred to [**4**] or [**5**] for further properties of the Schatten p-classes.

In [**6**], G. Weiss proved that if N is a normal operator in $B(H)$ and if $X \in C_2$ is such that $NX - XN \in C_1$, then $\operatorname{tr}(NX - XN) = 0$. This result admits the following extension.

THEOREM 1. *If N is normal in $B(H)$, $X \in C_2$, and $A \in B(H)$ is such that $AX - XN \in C_1$, then $|\operatorname{tr}(AX - XN)| \leq \|X\|_2 \|A - N\|_2$.*

Proof. If $A - N$ is not in C_2, then the result is trivial. We therefore assume that $A - N \in C_2$. Thus $(A - N)X \in C_1$ and so $AX - XN = NX - XN + (A - N)X$ implies that $NX - XN \in C_1$. Now Weiss's result implies that $\operatorname{tr}(NX - XN) = 0$. Therefore, $\operatorname{tr}(AX - XN) = \operatorname{tr}((N - A)X)$ from which it follows that $|\operatorname{tr}(AX - XN)| \leq \|X\|_2 \|A - N\|_2$.

If H is finite dimensional, then every commutator, that is an operator of the form $AX - XA$, has zero trace. In fact, it is well-known [**1**, p. 128] that an operator on a finite dimensional Hilbert space is a commutator if and only if it has trace 0. Thus if A, B, and X are operators on H with $\dim H = n$, then $AX - XB = AX - XA + X(A - B)$. Since $\operatorname{tr}(AX - XA) = 0$, it follows that $|\operatorname{tr}(AX - XB)| \leq \|X\|_q \|A - B\|_p$ whenever $1/p + 1/q = 1$. In particular if X is the identity operator, then $|\operatorname{tr}(A - B)| \leq n^{1/q} \|A - B\|_p$ which is known. This inequality may be useful in approximation problems of operators on a finite dimensional Hilbert space. For, if C is an operator with $\operatorname{tr} C = 0$, then for any operator A we have $|\operatorname{tr} A| = |\operatorname{tr}(A - C)| \leq n^{1/q} \|A - C\|_p$ and so $\|A - C\|_p \geq \dfrac{|\operatorname{tr} A|}{n^{1/q}}$. But if we choose

Glasgow Math. J. **29** (1987) 99–104.

$C = A - \dfrac{\operatorname{tr} A}{n}$, and hence $\operatorname{tr} C = 0$, then

$$\|A - C\|_p = \left\|\frac{\operatorname{tr} A}{n}\right\|_p = \frac{|\operatorname{tr} A|}{n} n^{1/p} = \frac{|\operatorname{tr} A|}{n^{1/q}}.$$

Thus we have proved the following result.

THEOREM 2. *If* $\dim H = n$ *and* $A \in B(H)$, *then* $\min\{\|A - C\|_p : \operatorname{tr} C = 0\} = \dfrac{|\operatorname{tr} A|}{n^{1/q}}$ *whenever* $1/p + 1/q = 1$.

Theorem 2 can be formulated in terms of commutators to yield the following inequality.

COROLLARY 1. *If* $\dim H = n$, *then for any operators* A, B, *and* X *acting on* H, *we have* $\|A + BX - XB\|_p \geq \dfrac{|\operatorname{tr} A|}{n^{1/q}}$ *whenever* $1/p + 1/q = 1$.

It has been shown in **[3]** that if A is an operator in $B(H)$ such that $\operatorname{Im} A \geq a \geq 0$, then for any $X \in B(H)$, $\|AX - XA^*\|_p \geq a\|X\|_p$ $(1 \leq p \leq \infty)$. Replacing A by iA, this inequality becomes $\|AX + XA^*\|_p \geq a\|X\|_p$ whenever $\operatorname{Re} A \geq a \geq 0$. The remarkable Clarkson–McCarthy inequalities **[5]** can be used to improve Theorem 3 in **[3]** for $1 < p < \infty$ as follows.

THEOREM 3. *If* A *is an operator in* $B(H)$ *such that* $\operatorname{Im} A \geq a \geq 0$, *then for any* $X \in B(H)$

$$\|AX - XA^*\|_p \geq (4^{1/p}a)\|X\|_p \qquad (2 \leq p \leq \infty)$$

and

$$\|AX - XA^*\|_p \geq (4^{1/q}a)\|X\|_p \qquad (1 \leq p \leq 2),$$

where $1/p + 1/q = 1$. *In particular,* $\|AX - XA^*\|_2 \geq (2a)\|X\|_2$.

Proof. If $T = \operatorname{Re} T + i \operatorname{Im} T$ is the cartesian decomposition of an operator T in $B(H)$, then it is not hard to conclude from the Clarkson–McCarthy inequalities that

$$\|\operatorname{Re} T\|_p^p + \|\operatorname{Im} T\|_p^p \leq \|T\|_p^p \leq \frac{2^p}{4}(\|\operatorname{Re} T\|_p^p + \|\operatorname{Im} T\|_p^p) \qquad (2 \leq p < \infty)$$

and

$$\frac{2^p}{4}(\|\operatorname{Re} T\|_p^p + \|\operatorname{Im} T\|_p^p) \leq \|T\|_p^p \leq \|\operatorname{Re} T\|_p^p + \|\operatorname{Im} T\|_p^p \qquad (1 < p \leq 2).$$

Let $X = Y + iZ$ be the cartesian decomposition of X, then $AY - YA^* = i \operatorname{Im}(AX - XA^*)$ and $AZ - ZA^* = -i \operatorname{Re}(AX - XA^*)$. But, as in the proof of Theorem 3 in **[3]**, we have $\|AY - YA^*\|_p \geq (2a)\|Y\|_p$ and $\|AZ - ZA^*\|_p \geq (2a)\|Z\|_p$ for all $1 \leq p \leq \infty$. Now if

$2 \leq p < \infty$, then

$$\begin{aligned}\|AX - XA^*\|_p^p &\geq \|AY - YA^*\|_p^p + \|AZ - ZA^*\|_p^p \\ &\geq (2a)^p(\|Y\|_p^p + \|Z\|_p^p) \\ &\geq (2a)^p\left(\frac{4}{2^p}\right)\|X\|_p^p.\end{aligned}$$

Hence $\|AX - XA^*\|_p \geq (4^{1/p}a)\,\|X\|_p$.

If $1 < p \leq 2$, then

$$\begin{aligned}\|AX - XA^*\|_p^p &\geq \frac{2^p}{4}(\|AY - YA^*\|_p^p + \|AZ - ZA^*\|_p^p) \\ &\geq \left(\frac{2^p}{4}\right)(2a)^p(\|Y\|_p^p + \|Z\|_p^p) \\ &\geq \left(\frac{2^p}{4}\right)(2a)^p\,\|X\|_p^p.\end{aligned}$$

Hence

$$\|AX - XA^*\|_p \geq \frac{4a}{4^{1/p}}\,\|X\|_p$$

and so

$$\|AX - XA^*\|_p \geq (4^{1/q}a)\,\|X\|_p,$$

where $1/p + 1/q = 1$.

Employing Berberian's trick also enables us to generalize Theorem 3. First we need a lemma.

LEMMA. *Let X be an operator in $B(H)$. If $Y = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$ is an operator defined on $H \oplus H$, then $\|Y\|_p = \|X\|_p$ for $1 \leq p \leq \infty$.*

Proof. It is clear that $X \in C_p$ on H if and only if $Y \in C_p$ on $H \oplus H$. The desired conclusion now follows from the observation that $Y^*Y = \begin{bmatrix} 0 & 0 \\ 0 & X^*X \end{bmatrix}$ or equivalently $|Y| = \begin{bmatrix} 0 & 0 \\ 0 & |X| \end{bmatrix}$.

THEOREM 4. *If A and B are operators in $B(H)$ such that $\operatorname{Im} A \geq a \geq 0$ and $\operatorname{Im} B \geq b \geq 0$, then for any $X \in B(H)$,*

$$\|AX - XB^*\|_p \geq (4^{1/p}\min(a, b))\,\|X\|_p \qquad (2 \leq p \leq \infty)$$

and

$$\|AX - XB^*\|_p \geq (4^{1/q}\min(a, b))\,\|X\|_p \qquad (1 \leq p \leq 2),$$

where $1/p + 1/q = 1$.

Proof. On $H \oplus H$, let $T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and let $Y = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$. Then $\operatorname{Im} T \geq \min(a, b) \geq 0$

and $TY - YT^* = \begin{bmatrix} 0 & AX - XB^* \\ 0 & 0 \end{bmatrix}$. The result now follows by applying the lemma and Theorem 3 to the operators T and Y.

It should be noticed that if B^* is replaced by B in Theorem 4, then the result is no longer true.

Whether Theorem 3, and hence its generalization, Theorem 4, can be improved further so that $\|AX - XA^*\|_p \geq (2a)\|X\|_p$ for all $1 \leq p \leq \infty$ is not known to the author. This result is obtained when either $p = 2$ or X is taken to be self-adjoint or more generally seminormal (X or X^* is hyponormal). As an application of the last assertion we now obtain Proposition 3.2 in **[2]** and extend it so that it includes the case $p = \infty$ as well.

THEOREM 5. *If A and B are self-adjoint operators in $B(H)$ and $A + B \geq a \geq 0$, then $\|A^2 - B^2\|_p \geq a\|A - B\|_p$ for $1 \leq p \leq \infty$.*

Proof. Let $S = A - B$ and $T = i(A + B)$. Then $\operatorname{Im} T \geq a \geq 0$, S is self-adjoint, and $TS - ST^* = 2i(A^2 - B^2)$. The result now follows by appealing to the inequality $\|TS - ST^*\|_p \geq (2a)\|S\|_p$.

The case $p = \infty$ is of particular importance. It asserts that the square root function is continuous on the interior of the positive cone of $B(H)$. A similar remark has been made about C_1 in **[2]**. It also follows from Theorem 5 that if A and B are self-adjoint operators in $B(H)$ such that $A^2 = B^2$ and $A + B \geq a > 0$, then $A = B$. Of course these equality signs may be taken modulo C_p. For general operators A and B in $B(H)$, it may be that $A^2 = B^2$ and $A^3 = B^3$, yet $A \neq B$. For example, take A and B to be distinct nilpotent operators of index two. The following theorem is a positive result in this direction.

THEOREM 6. *Let A and B be operators in $B(H)$ such that $A^2 = B^2$ and $A^3 = B^3$. If* $\ker A \subset \ker A^*$ *and* $\ker B \subset \ker B^*$ (ker A *denotes the kernel of* A), *then* $A = B$.

Proof. Let $C = A - B$. Now $A^2C = A^3 - A^2B = A^3 - B^3 = 0$. Thus from the assumption that $\ker A \subset \ker A^*$, it follows that $A^*AC = 0$. Hence $(AC)^*(AC) = 0$ and so $AC = 0$. Thus $A^*C = 0$. Similarly $B^2C = 0$ and $\ker B \subset \ker B^*$ imply that $BC = 0$ and $B^*C = 0$. Therefore $C^*C = (A^* - B^*)C = A^*C - B^*C = 0$. Whence $C = 0$ and so $A = B$ as required.

REMARKS. 1. Algebraic manipulations and induction show that the powers 2 and 3 in Theorem 6 can be replaced by any two relatively prime powers n and m.

2. The following two conditions are important special cases of the kernel assumption given in Theorem 6: (a) A is one-to-one and B is one-to-one; (b) A and B are hyponormal operators.

Next we establish the following inequality, the proof of which has a flavor similar to that of Theorems 2 and 3 in **[3]**.

THEOREM 7. *If A and B are operators in $B(H)$ such that $A + B \geq a \geq 0$, then for any seminormal operator X in $B(H)$, $\|XAX^* + X^*BX\|_p \geq a\|X\|_{2p}^2$ for $1 \leq p \leq \infty$.*

https://doi.org/10.1017/S0017089500006716 Published online by Cambridge University Press

Proof. We consider two cases.

Case 1. $p=\infty$. Without loss of generality we may assume that X is hyponormal. Hence there exists a sequence $\{f_n\}$ of unit vectors in H such that $(X-t)f_n \to 0$ as $n \to \infty$ where $|t| = \|X\|$.

Since $X-t$ is also hyponormal, it follows that $(X-t)^* f_n \to 0$ as $n \to \infty$. Now

$$\begin{aligned}\|XAX^*+X^*BX\| &\geq |((XAX^*+X^*BX)f_n, f_n)| \\ &= |(AX^*f_n, (X-t)^*f_n) + t(A(X-t)^*f_n, f_n) + |t|^2(Af_n, f_n) \\ &\quad + (BXf_n, (X-t)f_n) + \bar{t}(B(X-t)f_n, f_n) + |t|^2(Bf_n, f_n)|\end{aligned}$$

$\geq a|t|^2$ minus a term which goes to zero as $n \to \infty$. Hence $\|XAX^*+X^*BX\| \geq a\|X\|^2$ as required.

Case 2. $1 \leq p < \infty$. There is nothing to prove if XAX^*+X^*BX is not in C_p. We therefore assume that $XAX^*+X^*BX \in C_p$, and hence it is in particular compact. If $\pi: B(H) \to B(H)/C_\infty$ is the quotient map of $B(H)$ onto the Calkin algebra $B(H)/C_\infty$, then we have $\pi(X)\pi(A)\pi(X)^* + \pi(X)^*\pi(B)\pi(X) = 0$. Applying case 1 now implies that $\pi(X) = 0$, whence X is compact. But is is known [**1**, p. 110] that a compact hyponormal operator is diagonalizable, and hence $Xe_n = t_n e_n$ where $\{e_n\}$ is an orthonormal basis of H. Thus

$$\begin{aligned}\|XAX^*+X^*BX\|_p^p &\geq \sum |((XAX^*+X^*BX)e_n, e_n)|^p \\ &= \sum |(AX^*e_n, X^*e_n) + (BXe_n, Xe_n)|^p \\ &= \sum ||t_n|^2((A+B)e_n, e_n)|^p \\ &\geq a^p \sum |t_n|^{2p} \\ &= a^p \|X\|_{2p}^{2p}.\end{aligned}$$

Therefore $\|XAX^*+X^*BX\|_p \geq a\|X\|_{2p}^2$.

We point out here that Theorem 7 is not true if the semi-normality assumption on X is removed. For example, consider

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

which act on a two-dimensional Hilbert space. Also if X is hyponormal and $A+B \geq a \geq 0$, then it need not be true that $XAX^*+X^*BX \geq aXX^*$. For example, let $X=U$, the unilateral shift operator, $A=UU^*$, and $B=1-UU^*$. Then $XAX^*+X^*BX = U^2U^{*2}$ and $aXX^* = UU^*$. The assertion now follows since U is a nonunitary isometry.

Finally, we state the following theorem. The proof is omitted since it is similar to that of Theorem 7.

https://doi.org/10.1017/S0017089500006716 Published online by Cambridge University Press

THEOREM 8. *If A and B are operators in $B(H)$ such that $A+B\geq a\geq 0$, then for any seminormal operator X in $B(H)$, $\|AX+XB\|_p\geq a\,\|X\|_p$ for $1\leq p\leq\infty$.*

If A and B are self-adjoint operators in $B(H)$ with $A+B\geq a\geq 0$, and $X=A-B$, then $AX+XB=A^2-B^2$. Hence Theorem 5 is obtained as a special case of Theorem 8.

ADDED IN PROOF. It has been shown recently by the author in Inequalities for the Schatten p-norm III, *Comm. Math. Phys.* **104** (1986), 307–310 that if A is an operator in $B(H)$ such that $\operatorname{Im} A\geq a\geq 0$, then for any X in $B(H)$, we have

$$\|AX-XA^*\|_p\geq 2a\,\|X\|_p\ (1\leq p\leq\infty),$$

which is the desired improvement of Theorem 3.

REFERENCES

1. P. R. Halmos, *A Hilbert space problem book* (Springer-Verlag, 1982).

2. J. L. van Hemmen and T. Ando, An inequality for trace ideals, *Comm. Math. Phys.* **76** (1980), 143–148.

3. F. Kittaneh, Inequalities for the Schatten p-norm, *Glasgow Math. J.* **26** (1985), 141–143.

4. J. R. Ringrose, *Compact non-self-adjoint operators* (Van Nostrand Reinhold Co., 1971).

5. B. Simon, *Trace ideals and their applications* (Cambridge University Press, 1979).

6. G. Weiss, The Fuglede commutativity theorem modulo the Hilbert–Schmidt class and generating functions for matrix operators. I, *Trans. Amer. Math. Soc.* **246** (1978), 193–209.

DEPARTMENT OF MATHEMATICS
UNITED ARAB EMIRATES UNIVERSITY
P.O. BOX 15551
AL-AIN, U.A.E.