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THE DISTRIBUTIONAL k-HESSIAN IN FRACTIONAL SOBOLEV SPACES

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Abstract

We introduce the notion of a distributional k-Hessian ($k = 2, \ldots, n$) associated with fractional Sobolev functions on Ω, a smooth bounded open subset in $\mathbb{R}^n$. We show that the distributional k-Hessian is weakly continuous on the fractional Sobolev space $W^{2-2/k,k}(\Omega)$ and that the weak continuity result is optimal, that is, the distributional k-Hessian is well defined in $W^{s,p}(\Omega)$ if and only if $W^{s,p}(\Omega) \subseteq W^{2-2/k,k}(\Omega)$.

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1. Introduction and main results

For $k = 1, \ldots, n$ and $u \in C^2(\Omega)$, where Ω is a smooth bounded open subset in $\mathbb{R}^n$, the k-Hessian operator $H_k[u]$ is defined by

$$H_k[u] = \sigma_k(\lambda) = \sum_{i_1<\cdots<i_k} \lambda_{i_1} \cdots \lambda_{i_k},$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of the Hessian matrix D^2u. Alternatively,

$$H_k[u] = [D^2u]_k,$$

where $[A]_k$ denotes the sum of the $k \times k$ principal minors of an $n \times n$ matrix A, which may also be called the k-trace of A. It is well known that the k-Hessian is the Laplace operator when $k = 1$ and the Monge–Ampère operator when $k = n$.

This paper is devoted to the study of the k-Hessian of a nonsmooth map u from Ω into $\mathbb{R}$ with $2 \leqslant k \leqslant n$. From the seminal work of Trudinger and Wang [12, 13], it has been known that the k-Hessian makes sense as a Radon measure and enjoys the weak continuity property for k-admissible functions. In [5], Fu introduced the space of Monge–Ampère functions for which all minors of the Hessian matrices are well defined as signed Radon measures and weakly continuous in a certain natural sense. Other generalised notions of the k-Hessian measures are considered in [3, 4].

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Our purpose in this paper is to extend the definition of $H_k[u]$ to certain fractional Sobolev spaces so that the k-Hessian $H_k[u]$ is a distribution on Ω. Our first result is the following theorem.

THEOREM 1.1. *For $2 \leqslant k \leqslant n$, the k-Hessian operator $u \mapsto H_k[u] : C^2(\overline{\Omega}) \to \mathcal{D}'(\Omega)$ can be extended uniquely as a continuous mapping $u \mapsto H_k[u] : W^{2-2/k,k}(\Omega) \to \mathcal{D}'(\Omega)$. In particular, the distributional k-Hessian $H_k[u]$ with $k \geqslant 3$ can be expressed as*

$$\langle H_k[u], \varphi\rangle = \sum_{|\alpha|=k} \sum_{i\in\alpha+(n+1)} \sum_{j\in\alpha+(n+2)} \int_{\Omega\times(0,1)^2} \operatorname{adj}\,((D^2U)^{\alpha+(n+1)}_{\alpha+(n+2)})^i_j \partial_{ij}\Phi\, d\widetilde{x} \tag{1.1}$$

for any extensions $U \in W^{2,k}(\Omega\times(0,1)^2)$ and $\Phi \in C^2_c(\Omega\times[0,1)^2)$ of $u \in W^{2-2/k,k}(\Omega)$ and $\varphi \in C^2_c(\Omega)$, respectively. Moreover, for all $u_1, u_2 \in W^{2-2/k,k}(\Omega)$,

$$|\langle H_k[u_1] - H_k[u_2], \varphi\rangle| \leqslant C\|u_1 - u_2\|_{W^{2-2/k,k}}(\|u_1\|^{k-1}_{W^{2-2/k,k}} + \|u_2\|^{k-1}_{W^{2-2/k,k}})\|D^2\varphi\|_{L^\infty}. \tag{1.2}$$

We refer to Section 2 for the notation. In the case when $k = 2$, Theorem 1.1 follows directly from the result in [7]. In the remaining cases, $2 < k \leqslant n$, our results are inspired by work of Brezis–Nguyen and Baer–Jerison. Brezis–Nguyen [2] show that the Jacobian determinant operator $u \mapsto \det(Du) : C^1(\overline{\Omega}, \mathbb{R}^n) \to \mathcal{D}'(\Omega)$ admits a unique continuous extension from the fractional Sobolev space $W^{1-1/n,n}(\Omega, \mathbb{R}^n)$ to the space of distributions $\mathcal{D}'(\Omega)$; then Baer–Jerison [1] show that the Hessian determinant operator $u \mapsto \det(D^2u) : C^2_c(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ admits a unique continuous extension from $W^{2-2/n,n}(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$.

While the weak continuity result (1.2) can be proved using Theorem 1.1 in [1] combined with the 'Fubini-type' characterisation of the space $W^{2-2/k,k}(\mathbb{R}^n)$ (see the discussion in Remark 3.2), we choose to prove it by using an extension identity, Lemma 3.1 in Section 3, which is inspired by the work of Brezis–Nguyen in [2] and the concept of minors. The advantage of using Lemma 3.1 is that it not only provides a shorter proof of the statements established in [1] (such as Lemma 2.1), but also gives a fundamental representation for the distributional k-Hessian.

In analogy with [1, 2], Theorem 1.1 immediately gives several consequences. In particular, the k-Hessian as a distribution is continuous in spaces $W^{1,p}(\Omega) \cap W^{2,q}(\Omega)$ with $1 < p, q < \infty$, $2/p + (k-2)/q = 1$ and $k \geqslant 3$. Furthermore, the following optimality result ensures that the definition and weak continuity results for the k-Hessian are optimal in the framework of fractional Sobolev spaces $W^{s,p}$. More precisely, the distributional k-Hessian is well defined in $W^{s,p}(\Omega)$ if and only if $W^{s,p}(\Omega) \subseteq W^{2-2/k,k}(\Omega)$.

THEOREM 1.2. *Suppose that $2 \leqslant k \leqslant n$, $1 < p < \infty$ and $0 < s < \infty$ are such that $W^{s,p}(\Omega) \nsubseteq W^{2-2/k,k}(\Omega)$. Then there exist a sequence $\{u_m\} \subset C^2(\overline{\Omega})$ and a function $\varphi \in C^2_c(\Omega)$ such that*

$$\lim_{m\to\infty} \|u_m\|_{s,p} = 0$$

and

$$\lim_{m\to\infty} \int_\Omega H_k[u_m]\varphi\, dx = \infty.$$

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According to the embedding properties of fractional Sobolev spaces, it is enough to prove Theorem 1.2 in three cases (see Section 4). While the optimality results in Cases 2 and 3 follow essentially immediately from the counterexample sequences identified in [1], it is hard to prove the results in Case 1 in the same manner since the choice of ρ in (4.3) depends not only on s, p, n but also on k. It is necessary to establish an explicit formula (Lemma 4.1) in Section 4 for the purpose of proving Theorem 1.2.

2. Preliminaries

We recall that, for $0 < s < \infty$ and $1 \leqslant p < \infty$, the fractional Sobolev space $W^{s,p}(\Omega)$ is defined as follows: for $s < 1$,

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \,\middle|\, \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{1/p} < \infty \right\}$$

with the norm

$$\|u\|_{W^{s,p}} := \|u\|_{L^p} + \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{1/p};$$

for $s > 1$ and not an integer,

$$W^{s,p}(\Omega) := \left\{ u \in W^{[s],p}(\Omega) \mid D^{[s]}u \in W^{s-[s],p}(\Omega) \right\}$$

with the norm

$$\|u\|_{W^{s,p}} := \|u\|_{W^{[s],p}} + \left(\int_\Omega \int_\Omega \frac{|D^{[s]}u(x) - D^{[s]}u(y)|^p}{|x - y|^{n+(s-[s])p}} \, dx \, dy \right)^{1/p}.$$

For integer $n \geqslant 2$, we use the standard conventions for ordered multi-indices

$$I(k, n) := \{ \alpha = (\alpha_1, \ldots, \alpha_k) \mid \alpha_i \text{ integers}, 1 \leqslant \alpha_1 < \cdots < \alpha_k \leqslant n \}.$$

Set $I(0, n) = \{0\}$ and $|\alpha| = k$ if $\alpha \in I(k, n)$. For $\alpha \in I(k, n)$, $k = 0, 1, \ldots, n$, we let $\overline{\alpha}$ denote the element of $I(n - k, n)$ which complements α in $\{1, 2, \ldots, n\}$ in the natural increasing order (so that, for instance, $\overline{0} = (1, 2, \ldots, n)$). For $\alpha \in I(k, n)$ and $i \in \alpha$, we use the notation $\alpha - i$ to refer to the multi-index of length $k - 1$ obtained by removing i from α. Similarly, for $\alpha \in I(k, n)$ and $j \notin \alpha$, we use $\alpha + j$ to denote the multi-index of length $k + 1$ obtained by reordering naturally the multi-index $(\alpha_1, \ldots, \alpha_k, j)$.

Let $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ be $n \times n$ matrices. Given two ordered multi-indices with $\alpha, \beta \in I(k, n)$, we use the notation A^β_α for the associated minor, consisting of the $k \times k$-submatrix of A with rows $(\alpha_1, \ldots, \alpha_k)$ and columns $(\beta_1, \ldots, \beta_k)$. Its determinant will be denoted by

$$M^\beta_\alpha(A) := \det A^\beta_\alpha.$$

The adjoint of A^β_α is defined by the formula

$$(\operatorname{adj} A^\beta_\alpha)^i_j := \sigma(i, \beta - i)\sigma(j, \alpha - j) \det A^{\beta-i}_{\alpha-j} \quad \text{for all } i \in \beta, j \in \alpha,$$

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where $\sigma(i,\gamma)$, for $\gamma \in I(k-1,n)$ and $i \notin \gamma$, is the sign of the permutation of (i,γ) which reorders $(i,\gamma_1,\ldots,\gamma_{k-1})$ to $\gamma+i$. The Laplace formulas can be written as

$$M^{\beta}_{\alpha}(A) = \sum_{j\in\alpha} a_{ij}(\mathrm{adj}\, A^{\beta}_{\alpha})^{i}_{j} \quad \text{for all } i \in \beta,$$

and

$$M^{\beta}_{\alpha}(A) = \sum_{i\in\beta} a_{ij}(\mathrm{adj}\, A^{\beta}_{\alpha})^{i}_{j} \quad \text{for all } j \in \alpha.$$

The Binet formulas (see [6, Lemma on page 313]) can be written as

$$M^{\beta}_{\alpha}(A+B) = \sum_{\alpha'+\alpha''=\alpha;\,\beta'+\beta''=\beta;\,|\alpha'|=|\beta'|} \sigma(\alpha',\alpha'')\sigma(\beta',\beta'')M^{\beta'}_{\alpha'}(A)M^{\beta''}_{\alpha''}(B). \tag{2.1}$$

3. The proof of Theorem 1.1

We begin with the following extension lemma which is inspired by the work of Brezis–Nguyen [2] and fine properties of minors.

LEMMA 3.1. *Let* $2 \leqslant k \leqslant n$, $\alpha \in I(k,n)$ *and* $u \in C^2(\Omega), \varphi \in C^2_c(\Omega)$. *Then*

$$\int_{\Omega} M^{\alpha}_{\alpha}(D^2u)\varphi\, dx = \sum_{i\in\alpha+(n+1)} \sum_{j\in\alpha+(n+2)} \int_{\Omega\times(0,1)^2} \Big(\mathrm{adj}\,(D^2U)^{\alpha+(n+1)}_{\alpha+(n+2)}\Big)^{i}_{j}\, \partial_{ij}\Phi\, d\widetilde{x}, \tag{3.1}$$

for any extensions $U \in C^2(\Omega\times[0,1)^2)\cap C^3(\Omega\times(0,1)^2)$ *and* $\Phi \in C^2_c(\Omega\times[0,1)^2)$ *of* u *and* φ. *Here* $\widetilde{x} = (x, x_{n+1}, x_{n+2})$ *and* $\partial_i := \partial/\partial x_i$.

PROOF. Denote $V := U|_{x_{n+2}=0}$, $\Psi := \Phi|_{x_{n+2}=0}$. Then

$$\begin{aligned}\int_{\Omega} M^{\alpha}_{\alpha}(D^2u)\varphi\, dx &= -\int_{\Omega\times(0,1)} \partial_{n+1}\Big(M^{\alpha}_{\alpha}(D^2V)\Psi\Big)\, dx\, dx_{n+1} \\ &= -\int_{\Omega\times(0,1)} \partial_{n+1}\Big(M^{\alpha}_{\alpha}(D^2V)\Big)\Psi\, dx\, dx_{n+1} - \int_{\Omega\times(0,1)} M^{\alpha}_{\alpha}(D^2V)\partial_{n+1}\Psi\, dx\, dx_{n+1}.\end{aligned}$$

On the one hand

$$\partial_{n+1}(M^{\alpha}_{\alpha}(D^2V)) = \sum_{i\in\alpha} M^{\alpha}_{\alpha}(D^2V(i)),$$

where $D^2V(i) := (a^i_{st})_{1\leqslant s,t\leqslant n+1}$ is an $(n+1)\times(n+1)$ matrix with

$$a^i_{st} = \begin{cases}\partial_s\partial_t V & \text{if } s \neq i, t = 1,\ldots,n+1,\\ \partial_{n+1}\partial_s\partial_t V & \text{if } s = i, t = 1,\ldots,n+1.\end{cases}$$

By the Laplace formulas,

$$M^{\alpha}_{\alpha}(D^2V(i)) = \sum_{j\in\alpha} \sigma(i,\alpha-i)\sigma(j,\alpha-j)\partial_{n+1}\partial_i\partial_j V M^{\alpha-i}_{\alpha-j}(D^2V(i)).$$

Then

$$\partial_{n+1}\left(M_\alpha^\alpha(D^2V)\right) = \sum_{i\in\alpha}\sum_{j\in\alpha}\partial_{n+1}\partial_i\partial_j V\sigma(i,\alpha-i)\sigma(j,\alpha-j)M_{\alpha-j}^{\alpha-i}(D^2V(i))$$
$$= \sum_{i\in\alpha}\sum_{j\in\alpha}\sigma(i,\alpha-i)\sigma(j,\alpha-j)\partial_{n+1}\partial_i\partial_j V M_{\alpha-j}^{\alpha-i}(D^2V). \tag{3.2}$$

On the other hand,

$$\sum_{i\in\alpha}\partial_i\left(\sigma(\alpha-i,i)M_\alpha^{\alpha+(n+1)-i}(D^2V)\right)$$
$$= \sum_{i\in\alpha}\sigma(\alpha-i,i)\sum_{s\in\alpha-i}\sum_{j\in\alpha}\sigma(s,\alpha+(n+1)-i-s)\sigma(j,\alpha-j)\partial_i\partial_s\partial_j V M_{\alpha-j}^{\alpha+(n+1)-i-s}(D^2V)$$
$$+ \sum_{i\in\alpha}\sigma(\alpha-i,i)\sum_{j\in\alpha}\sigma((n+1),\alpha-i)\sigma(j,\alpha-j)\partial_i\partial_{n+1}\partial_j V M_{\alpha-j}^{\alpha-i}(D^2V)$$
$$= \sum_{j\in\alpha}\sigma(j,\alpha-j)\sum_{i\in\alpha}\sum_{s\in\alpha-i}\sigma(\alpha-i,i)\sigma(s,\alpha+(n+1)-i-s)\partial_i\partial_s\partial_j V M_{\alpha-j}^{\alpha+(n+1)-i-s}(D^2V)$$
$$+ \sum_{i\in\alpha}\sum_{j\in\alpha}\sigma(i,\alpha-i)\sigma(j,\alpha-j)\partial_{n+1}\partial_i\partial_j V M_{\alpha-j}^{\alpha-i}(D^2V). \tag{3.3}$$

For any $i_1, i_2 \in \alpha$ with $i_1 \neq i_2$,

$$\sigma(\alpha-i_1,i_1)\sigma(i_2,\alpha+(n+1)-i_1-i_2)$$
$$= (-1)^{k-1}\sigma(i_1,\alpha-i_1)\sigma(i_2,\alpha-i_1-i_2)\sigma(i_2,n+1)$$
$$= (-1)^{k-1}\sigma(i_1,\alpha-i_1-i_2)\sigma(i_2,\alpha-i_1-i_2)\sigma(i_1,i_2),$$

which implies that

$$\sigma(\alpha-i_1,i_1)\sigma(i_2,\alpha+(n+1)-i_1-i_2) = -\sigma(\alpha-i_2,i_2)\sigma(i_1,\alpha+(n+1)-i_2-i_1)$$

since

$$\sigma(i_1,i_2) = \begin{cases} -1 & \text{if } i_1 > i_2, \\ 1 & \text{if } i_1 < i_2. \end{cases}$$

Combining these results, we can easily obtain

$$\partial_{n+1}(M_\alpha^\alpha(D^2V)) = \sum_{i\in\alpha}\partial_i(\sigma(\alpha-i,i)M_\alpha^{\alpha+(n+1)-i}(D^2V)). \tag{3.4}$$

Hence

$$\int_\Omega M_\alpha^\alpha(D^2u)\varphi\,dx = -\sum_{i\in\alpha+(n+1)}\int_{\Omega\times(0,1)}\sigma(\alpha+(n+1)-i,i)M_\alpha^{\alpha+(n+1)-i}(D^2V)\partial_i\Psi\,dx\,dx_{n+1}$$
$$= \sum_{i\in\alpha+(n+1)}-\sigma(\alpha+(n+1)-i,i)\int_{\Omega\times(0,1)}(M_\alpha^{\alpha+(n+1)-i}(D^2U)\partial_i\Phi)|_{x_{n+2}=0}\,dx\,dx_{n+1}.$$

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It is a well-known consequence of integration by parts identities that the right-hand side of the above identity can be written as

$$\sum_{i\in\alpha+(n+1)} -\sigma(\alpha+(n+1)-i,i)A(i), \tag{3.5}$$

where

$$\begin{aligned} A(i) :&= -\int_{\Omega\times(0,1)^2} \partial_{n+2}(M_\alpha^\beta(D^2U)\partial_i\Phi)\,d\widetilde{x} \\ &= -\int_{\Omega\times(0,1)^2} \partial_{n+2}(M_\alpha^\beta(D^2U))\partial_i\Phi\,d\widetilde{x} - \int_{\Omega\times(0,1)^2} M_\alpha^\beta(D^2U)\partial_{i,n+2}\Phi\,d\widetilde{x}. \end{aligned}$$

Here $\beta := \alpha+(n+1)-i$. An argument similar to the one used in (3.2)–(3.4) gives

$$\begin{aligned} \partial_{n+2}(M_\alpha^\beta(D^2U)) &= \sum_{t\in\beta}\sum_{j\in\alpha}\sigma(t,\beta-t)\sigma(j,\alpha-j)\partial_{n+2}\partial_t\partial_j U M_{\alpha-j}^{\beta-t}(D^2U) \\ &= \sum_{j\in\alpha}\partial_j(\sigma(\alpha-j,j)M^\beta_{\alpha-j+(n+2)}(D^2U)) \\ &= -\sum_{j\in\alpha}\partial_j(\sigma(\alpha+(n+2)-j,j)M^\beta_{\alpha+(n+2)-j}(D^2U)). \end{aligned}$$

Thus

$$A(i) = \sum_{j\in\alpha+(n+2)} -\sigma(\alpha+(n+2)-j,j)\int_{\Omega\times(0,1)^2} M^\beta_{\alpha-j+(n+2)}(D^2U)\partial_{ij}\Phi\,d\widetilde{x}.$$

Combining this with (3.5), we obtain (3.1), which completes the proof. □

From the results of [7] characterising the Hessian determinant on the space $W^{1,2}(\mathbb{R}^2)$, the 2-Hessian is well defined and continuous on $W^{1,2}(\Omega)$. More precisely, the 2-Hessian $H_2[u]$ is defined for all $u \in W^{1,2}(\Omega)$ by

$$\langle H_2[u],\varphi\rangle := \sum_{i=1}^n\sum_{j\neq i}\int_\Omega\Big(\partial_iu\partial_ju\partial_{ij}\varphi - \frac{1}{2}\partial_iu\partial_iu\partial_{jj}\varphi - \frac{1}{2}\partial_ju\partial_ju\partial_{ii}\varphi\Big)\,dx, \tag{3.6}$$

for any $\varphi \in C_c^2(\Omega)$. It is simple to show the weak continuity results by the Hölder inequality. Hence we just need to prove Theorem 1.1 for $k \geqslant 3$.

Proof of Theorem 1.1. It is well known that Theorem 1.1 can be obtained as a simple corollary by standard approximation if we prove (1.1) and (1.2) for $u, u_1, u_2 \in C^2(\overline{\Omega})$ and $\varphi \in C_c^2(\Omega)$. Let $\widetilde{u}_1$ and $\widetilde{u}_2$ be extensions of u_1 and u_2 to $\mathbb{R}^n$ such that

$$\|\widetilde{u}_1\|_{W^{2-2/k,k}(\mathbb{R}^n)} \leqslant C\|u_1\|_{W^{2-2/k,k}(\Omega)}, \quad \|\widetilde{u}_2\|_{W^{2-2/k,k}(\mathbb{R}^n)} \leqslant C\|u_2\|_{W^{2-2/k,k}(\Omega)}$$

and

$$\|\widetilde{u}_1 - \widetilde{u}_2\|_{W^{2-2/k,k}(\mathbb{R}^n)} \leqslant C\|u_1-u_2\|_{W^{2-2/k,k}(\Omega)},$$

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where C depends only on k, n and Ω. According to a well-known extension theorem of Stein in [8, 9], there is a bounded linear extension operator

$$E : W^{2-2/k,k}(\mathbb{R}^n) \to W^{2,k}(\mathbb{R}^n \times (0, +\infty)^2).$$

Let U_1, U_2 be extensions of $\widetilde{u_1}$ and $\widetilde{u_2}$ to $\mathbb{R}^n \times (0, +\infty)^2$, respectively. Hence

$$\|D^2 U_i\|_{L^k(\mathbb{R}^n\times(0,1)^2)} \leqslant C\|u_i\|_{W^{2-2/k,k}(\Omega)} \quad \text{for } i = 1, 2,$$

and

$$\|D^2 U_1 - D^2 U_2\|_{L^k(\mathbb{R}^n\times(0,1)^2)} \leqslant C\|u_1 - u_2\|_{W^{2-2/k,k}(\Omega)}.$$

Let $\Phi \in C_c^2(\Omega \times [0, 1)^2)$ be an extension of φ such that

$$\|D^2\Phi\|_{L^\infty(\Omega\times(0,1)^2)} \leqslant C\|D^2\varphi\|_{L^\infty(\Omega)}.$$

Since

$$|M_\alpha^\beta(A) - M_\alpha^\beta(B)| \leqslant C(|A| + |B|)^{k-1}|A - B|$$

for any $\alpha, \beta \in I(k, n + 2)$ and $(n + 2) \times (n + 2)$ matrices A, B, it follows from Lemma 3.1 and Hölder's inequality that

$$\begin{aligned}
\left|\int_\Omega (H_k[u_1] - H_k[u_2])\varphi\, dx\right| &\leqslant \sum_{\alpha\in I(k,n)} \left|\int_\Omega (M_\alpha^\alpha(D^2u_1) - M_\alpha^\alpha(D^2u_2))\varphi\, dx\right| \\
&\leqslant \sum_{\alpha\in I(k,n)} \sum_{i\in\alpha+(n+1)} \sum_{j\in\alpha+(n+2)} \int_{\Omega\times(0,1)^2} |M_{\alpha-j+(n+2)}^{\alpha-i+(n+1)}(D^2U_1) - M_{\alpha-j+(n+2)}^{\alpha-i+(n+1)}(D^2U_2)||\partial_{ij}\Phi|\, d\widetilde{x} \\
&\leqslant C\int_{\Omega\times(0,1)^2} (|D^2U_1| + |D^2U_2|)^{k-1}|D^2(U_1 - U_2)||D^2\Phi|\, d\widetilde{x} \\
&\leqslant C\|u_1 - u_2\|_{W^{2-2/k,k}}(\|u_1\|^{k-1}_{W^{2-2/k,k}} + \|u_2\|^{k-1}_{W^{2-2/k,k}})\|D^2\varphi\|_{L^\infty}.
\end{aligned}$$

This completes the proof of Theorem 1.1. □

Remark 3.2. There is another way to prove (1.2) using [1, Theorem 1.1] combined with the 'Fubini-type' characterisation of the space $W^{2-2/k,k}(\mathbb{R}^n)$ (see [10, Section 3] and [11, Section 2.5.13]). Fix $\alpha \in I(k, n)$ and let $W^{2-2/k,k}(\alpha; \mathbb{R}^n)$ be the subspace of $L^k(\mathbb{R}^n)$ associated to the norm

$$\|f\|_{L^k} + \Big\|\|f\|_{W^{2-2/k,k}_{x_\alpha}(\mathbb{R}^k)}\Big\|_{L^k_{x_{\overline{\alpha}}}},$$

where subscripts denote variables of integration for the fractional Sobolev and L^k norm. Then

$$\|f\|_{W^{2-2/k,k}(\mathbb{R}^n)} \sim \sum_{\alpha\in I(k,n)} \|f\|_{W^{2-2/k,k}(\alpha;\mathbb{R}^n)}. \tag{3.7}$$

Let u_1, u_2, $\widetilde{u}_1$ and $\widetilde{u}_2$ be the functions mentioned in the proof of Theorem 1.1. Fix $\alpha \in I(k, n)$ and $x_{\overline{\alpha}} \in \mathbb{R}^{n-k}$. Then [1, Theorem 1.1] implies that

$$\begin{aligned}
&\int_{\mathbb{R}^k} (M_\alpha^\alpha(D^2\widetilde{u}_1(x_\alpha, x_{\overline{\alpha}}) - M_\alpha^\alpha(D^2\widetilde{u}_2(x_\alpha, x_{\overline{\alpha}}))\varphi(x_\alpha, x_{\overline{\alpha}})\, dx_\alpha \\
&\qquad \leqslant C\|\widetilde{u}_1 - \widetilde{u}_2\|_{W^{2-2/k,k}_{x_\alpha}(\mathbb{R}^k)}(\|\widetilde{u}_2\|^{k-1}_{W^{2-2/k,k}_{x_\alpha}(\mathbb{R}^k)} + \|\widetilde{u}_1\|^{k-1}_{W^{2-2/k,k}_{x_\alpha}(\mathbb{R}^k)})\|D^2\varphi\|_{L^\infty_{x_\alpha}}.
\end{aligned}$$

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It follows from (3.7) and Holder's inequality that

$$\begin{aligned}
&\Big|\int_\Omega (H_k[u_1]-H_k[u_2])\varphi\,dx\Big| \\
&\quad = \sum_{\alpha\in I(k,n)}\int_{\mathbb{R}^{n-k}}\Big(\int_{\mathbb{R}^k}(M_\alpha^\alpha(D^2\widetilde{u}_1(x_\alpha,x_{\overline{\alpha}})-M_\alpha^\alpha(D^2\widetilde{u}_2(x_\alpha,x_{\overline{\alpha}}))\varphi(x_\alpha,x_{\overline{\alpha}})\,dx_\alpha\Big)\,dx_{\overline{\alpha}} \\
&\quad \leqslant C\sum_{\alpha\in I(k,n)}\big\|\|\widetilde{u}_1-\widetilde{u}_2\|_{W^{2-2/k,k}_{x_\alpha}(\mathbb{R}^k)}\big\|_{L^k_{x_{\overline{\alpha}}}}\Big(\big\|\|\widetilde{u}_1\|_{W^{2-2/k,k}_{x_\alpha}(\mathbb{R}^k)}\big\|^{k-1}_{L^k_{x_{\overline{\alpha}}}}+\big\|\|\widetilde{u}_2\|_{W^{2-2/k,k}_{x_\alpha}(\mathbb{R}^k)}\big\|^{k-1}_{L^k_{x_{\overline{\alpha}}}}\Big)\|D^2\varphi\|_{L^\infty} \\
&\quad \leqslant C\|u_1-u_2\|_{W^{2-2/k,k}}\Big(\|u_1\|^{k-1}_{W^{2-2/k,k}}+\|u_2\|^{k-1}_{W^{2-2/k,k}}\Big)\|D^2\varphi\|_{L^\infty},
\end{aligned}$$

which implies (1.2). We emphasise that it is hard to obtain the fundamental representation (1.1) of the distributional k-Hessian in this manner.

4. The proof of Theorem 1.2

We only need to prove Theorem 1.2 when $k > 2$, since (3.6) immediately gives the result in the case when $k = 2$. According to the embedding properties of the Sobolev spaces $W^{s,p}(\Omega)$ into the space $W^{2-2/k,k}(\Omega)$ (for more detail, see [11, page 196]), when:

(i) $s + 2/k > 2 + \max\{0, n/p - n/k\}$, the embedding $W^{s,p}(\Omega) \subset W^{2-2/k,k}(\Omega)$ holds;

(ii) $s + 2/k < 2 + \max\{0, n/p - n/k\}$, the embedding fails; and

(iii) $s + 2/k = 2 + \max\{0, n/p - n/k\}$, there are two sub-cases:

  (a) if $p \leqslant k$, then the embedding $W^{s,p}(\Omega) \subset W^{2-2/k,k}(\Omega)$ holds;

  (b) if $p > k$, the embedding fails.

In order to prove Theorem 1.2, we just consider three cases:

$$1 < p \leqslant k \text{ and } s + \frac{2}{k} < 2 + \frac{n}{p} - \frac{n}{k}; \quad k < p \text{ and } 0 < s < 2 - \frac{2}{k}; \quad k < p \text{ and } s = 2 - 2/k.$$

Lemma 4.1. *Let $g \in C_c^\infty(B(0,1))$ be given by*

$$g(x) = \int_0^{|x|} h(r)\,dr \quad \textit{for all } x \in \mathbb{R}^n, \tag{4.1}$$

where $h \in C_c^\infty((0,1))$ satisfies

$$\int_0^1 h(r)\,dr = 0, \qquad \int_0^1 h^k(r)r^{-k+n+1}\,dr \neq 0.$$

Then

$$\sum_{\alpha\in I(k,n)}\int_{B(0,1)} M_\alpha^\alpha(D^2g(x))|x|^2\,dx \neq 0.$$

Proof. According to the symmetry of the integral, it is sufficient to show that

$$\int_{B(0,1)} M_\alpha^\alpha(D^2g(x))|x|^2\,dx \neq 0 \tag{4.2}$$

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for any $\alpha \in I(k,n)$. It is simple to show that

$$D^2 g = \frac{1}{|x|^3}(A+B),$$

where $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ are $n \times n$ matrices such that

$$a_{ij} = h(|x|)|x|^2\delta_i^j, \quad b_{ij} = (h'(|x|)|x| - h(|x|))x_i x_j \quad \text{for } i,j = 1,\dots,n.$$

We make use of Binet's formula (2.1) and the fact that B has rank one to see that

$$M_\alpha^\alpha(A+B) = M_\alpha^\alpha(A) + \sum_{i\in\alpha}\sum_{j\in\alpha}\sigma(i,\alpha-i)\sigma(j,\alpha-j)b_{ij}M_{\alpha-i}^{\alpha-j}(A).$$

For any $i, j \in \alpha$ with $i \neq j$,

$$\begin{aligned} M_{\alpha-i}^{\alpha-j}(A) &= h^{k-1}(|x|)|x|^{2k-2}M_{\alpha-i}^{\alpha-j}(E) \\ &= h^{k-1}(|x|)|x|^{2k-2}\sum_{s\in\alpha-i}\delta_i^s\sigma(s,\alpha-i-s)\sigma(i,\alpha-j-i)M_{\alpha-i-s}^{\alpha-j-i}(E) \\ &= 0, \end{aligned}$$

where E is the $n \times n$ identity matrix, which implies that

$$M_\alpha^\alpha(A+B) = h^k(|x|)|x|^{2k} - h^k(|x|)|x|^{2k-2}\sum_{i\in\alpha}x_i^2 + h^{k-1}(|x|)h'(|x|)|x|^{2k-1}\sum_{i\in\alpha}x_i^2.$$

Hence

$$\int_{B(0,1)} M_\alpha^\alpha(D^2 g)|x|^2\,dx = \int_{B(0,1)}|x|^{-3k+2}M_\alpha^\alpha(A+B)\,dx = I - II + III.$$

Using polar coordinates to evaluate the integrals,

$$I := \int_{B(0,1)} h^k(|x|)|x|^{-k+2}\,dx = 2\pi\prod_{i=1}^{n-2}I(i)\int_0^1 h^k(r)r^{-k+n+1}\,dr,$$

where $I(i) = \int_0^\pi \sin^i\theta\,d\theta$. Without loss of generality, $\alpha = (n-k+1, n-k+2,\dots,n)$. So

$$\begin{aligned} II :&= \int_{B(0,1)} h^k(|x|)|x|^{-k}\sum_{i\in\alpha}x_i^2\,dx \\ &= \int_0^\pi\cdots\int_0^\pi \sin^{n-2}\theta_1\sin^{n-3}\theta_2\cdots\sin\theta_{n-2}\sin^2\theta_1\cdots\sin^2\theta_{n-k}\,d\theta_1\cdots d\theta_{n-2} \\ &\quad\cdot\int_0^{2\pi}d\theta_{n-1}\int_0^1 h^k(r)r^{-k+n+1}\,dr \\ &= 2\pi\int_0^1 h^k(r)r^{-k+n+1}\,dr\prod_{i=1}^{k-2}I(i)\prod_{i=k-1}^{n-2}I(i+2). \end{aligned}$$

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Note that $I(s) = ((s-1)/s)I(s-2)$ for $s = 2, 3, \ldots$. Hence

$$II = 2\pi \frac{k}{n} \prod_{i=1}^{n-2} I(i) \int_0^1 h^k(r) r^{-k+n+1}\, dr.$$

Similarly,

$$\begin{aligned} III := & \int_{B(0,1)} h^{k-1}(|x|) h'(|x|) |x|^{-k+1} \sum_{i\in\alpha} x_i^2\, dx \\ &= 2\pi \frac{k}{n} \prod_{i=1}^{n-2} I(i) \int_0^1 h^{k-1}(r) h'(r) r^{-k+2+n}\, dr \\ &= 2\pi \frac{k-n-2}{n} \prod_{i=1}^{n-2} I(i) \int_0^1 h^k(r) r^{-k+n+1}\, dr, \end{aligned}$$

which implies (4.2). □

Proof of Theorem 1.2. Without loss of generality, $(-8, 8) \subset \Omega$. As noted at the beginning of Section 4, we divide our proof into three cases.

Case 1: $1 < p \leqslant k$ *and* $s + 2/k < 2 + n/p - n/k$. Consider $u_\varepsilon : \Omega \to \mathbb{R}$ defined by

$$u_\varepsilon(x) = \varepsilon^\rho g\Big(\frac{x}{\varepsilon}\Big),$$

where $0 < \varepsilon < 1$, g is given by (4.1) and ρ is a constant such that

$$s - \frac{n}{p} < \rho < 2 - \frac{n}{k} - \frac{2}{k}. \tag{4.3}$$

On the one hand,

$$\|u_\varepsilon\|_{s,p} \leqslant \|u_\varepsilon\|_{L^p}^{1-s/2} \|D^2 u_\varepsilon\|_{L^p}^{s/2} \leqslant \varepsilon^{\rho - s + n/p} \|g\|_{L^p}^{1-s/2} \|D^2 g\|_{L^p}^{s/2},$$

which implies that

$$\|u_\varepsilon\|_{s,p} \to 0 \quad \text{as } \varepsilon \to \infty.$$

On the other hand, let $\varphi \in C_c^2(\Omega)$ be such that $\varphi(x) = |x|^2 + O(|x|^3)$ as $x \to 0$. Then

$$\begin{aligned} \int_\Omega H_k[u_\varepsilon]\varphi\, dx &= \sum_{\alpha\in I(k,n)} \varepsilon^{(\rho-2)k} \int_\Omega M_\alpha^\alpha\Big(D^2 g\Big(\frac{x}{\varepsilon}\Big)\Big)\varphi(x)\, dx \\ &= \sum_{\alpha\in I(k,n)} \varepsilon^{(\rho-2)k+n} \int_{B(0,1)} M_\alpha^\alpha(D^2 g)\varphi(\varepsilon x)\, dx \\ &= \varepsilon^{\rho k - 2k + n + 2} \sum_{\alpha\in I(k,n)} \int_{B(0,1)} M_\alpha^\alpha(D^2 g)|x|^2 dx + O(\varepsilon^{\rho k - 2k + n + 3}). \end{aligned}$$

From Lemma 4.1 and (4.3), it follows that

$$\Big|\int_\Omega H_k[u_\varepsilon]\varphi\, dx\Big| = C\varepsilon^{\rho k - 2k + n + 2} \to \infty \quad \text{as } \varepsilon \to 0.$$

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Case 2: $k < p$ and $0 < s < 2 - 2/k$. For $m \gg 1$, we set

$$u_m := m^{-\rho} x_k \prod_{i=1}^{k-1} \sin^2(mx_i),$$

where ρ is a constant with $s < \rho < 2 - 2/k$. Let $\varphi \in C_c^2(\Omega)$ be such that

$$\varphi(x) = \prod_{i=1}^{n} \varphi'(x_i), \tag{4.4}$$

where $\varphi' \in C_c^2((0, \pi)), \varphi' \geqslant 0$ and $\varphi' = 1$ in $(\frac{1}{4}\pi, \frac{3}{4}\pi)$. Since $\|u_m\|_{L^\infty} \leqslant Cm^{-\rho}$ and $\|D^2 u_m\|_{L^\infty} \leqslant Cm^{2-\rho}$, it follows that

$$\|u_m\|_{s,p} \leqslant C\|u_m\|_{L^p}^{1-s/2} \|u_m\|_{2,p}^{s/2} \leqslant Cm^{s-\rho}.$$

In the same way as in the proof of [1, Proposition 4.1],

$$\begin{aligned}\Big|\int_\Omega H_k[u_m]\varphi\,dx\Big| &\geqslant \Big|\sum_{\alpha\in I(k,n)} \int_{((1/4)\pi,(3/4)\pi)^n} M_\alpha^\alpha(D^2 u_m)\,dx\Big| \\ &\geqslant m^{2k-2-k\rho} 2^k \sum_{\alpha\in I(k,n)} \int_{((1/4)\pi,(3/4)\pi)^n} x_k^{k-2} \Big(\prod_{i=1}^{k-1} \sin(mx_i)\Big)^{2k-2} \Big(\sum_{j=1}^{k-1} \cos^2(mx_j)\Big)\,dx \\ &= Cm^{2k-2-k\rho}.\end{aligned}$$

Case 3: $k < p$ and $s = 2 - 2/k$. For any $m \in \mathbb{N}$ with $m \geqslant 2$, define u_m by

$$u_m(x) = \frac{1}{(\log m)^{1/(2k)}} x_k \sum_{l=1}^{m} \frac{1}{n_l^{2-2/k} l^{1/k}} \prod_{i=1}^{k-1} \sin^2(n_l x_i) \quad \text{for all } x \in \mathbb{R}^n,$$

where $n_l = m^{k^{3l}}$. Let $\varphi \in C_c^2(\Omega)$ be defined by (4.4). An argument similar to the one used in the proof of [1, Proposition 5.1] now shows that

$$\|u_m\|_{W^{s,p}(\Omega)} \leqslant C\|u_m\|_{W^{s,p}((0,2\pi)^n)} \leqslant C\frac{1}{(\ln m)^{1/(2k)}}$$

and

$$\begin{aligned}\Big|\int_\Omega H_k[u_m]\varphi\,dx\Big| &= C\Big|\sum_{\alpha\in I(k,n)} \int_{(0,2\pi)^k} M_{\alpha_0}^{\alpha_0}(D^2 u_m) \prod_{i=1}^{k} \varphi'(x_i)\,dx_1 \cdots dx_k\Big| \\ &\geqslant C(\ln m)^{1/2},\end{aligned}$$

where $\alpha_0 = (1, \ldots, k) \in I(k, n)$. Then Theorem 1.2 is completely proved. □

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