WHITEHEAD GROUPS OF SEMIDIRECT PRODUCTS OF FREE GROUPS

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1. Statements of main theorems. Let G be a group. We denote the Whitehead group of G by Wh G and the projective class group of the integral group ring $\mathbb{Z}(G)$ of G by $\tilde{K}_0\mathbb{Z}(G)$. Let α be an automorphism of G and T an infinite cyclic group. Then we denote by $G\times_\alpha T$ the semidirect product of G and T with respect to α. For undefined terminologies used in the paper, we refer to [**3**] and [**7**].

One of the problems in algebraic K-theory is to determine which classes of G will give rise to the triviality of both Wh G and $\tilde{K}_0\mathbb{Z}(G)$. In [**3, 6**] we have shown that both Wh G and $\tilde{K}_0\mathbb{Z}(G)$ are trivial for certain classes of semidirect products of free groups. The main purpose of this paper is to find a wider class of such groups which includes, in particular, those in [**3, 6**].

First we recall the following definition [**7**, p. 214]: Any group possessing only a single element will be called a group of type 0. Inductively, we define G to be a group of type $n+1$ if $G=H\times_\alpha T$ where H is a group of type n. In particular, any free abelian group of finite rank n is a group of type n. Farrell and Hsiang have shown the following:

LEMMA 1 [**7**, Theorem 29]. *If G is a group of type n, then* Wh $G=0$ *and* $\tilde{K}_0\mathbb{Z}(G)=0$.

Moreover, Farrell and Hsiang have pointed out that if G is a group of type n, then $\mathbb{Z}(G)$ is right Noetherian and of finite right global dimension, and so $\tilde{C}(\mathbb{Z}(G),\alpha)=0$, where α is any automorphism of G, by [**8**, Theorem 1.6].

Now let G be a group of type n, $F_1, F_2,\ldots$ a set of free groups, each of rank at least two, and let

$$H_k = G\times F_1\times\ldots\times F_k \tag{1}$$

be the direct product of $G, F_1,\ldots,F_k\,(k=1,2,\ldots)$. Then we extend the results of Farrell and Hsiang to the following theorems.

THEOREM 1. *For each $k=1,2,\ldots$,* Wh $H_k=0$, $\tilde{K}_0\mathbb{Z}(H_k)=0$ *and* $\tilde{C}(\mathbb{Z}(H_k),\mathrm{id})=0$.

THEOREM 2. *Let α be an automorphism of H_k which leaves all but one of the F_j $(j=1,\ldots,k)$ pointwise fixed. Then for any infinite cyclic group T,* $\mathrm{Wh}(H_k\times_\alpha T)=0$, $\tilde{K}_0\mathbb{Z}(H_k\times_\alpha T)=0$ *and* $\tilde{C}(\mathbb{Z}(H_k\times_\alpha T),\mathrm{id})=0$. *In addition,* $\tilde{C}(\mathbb{Z}(H_k),\alpha)=0$.

REMARK. The results in Theorem 1 and Theorem 2 clearly extend those of [**3**, Theorem A, Theorem B and Lemma 1].

Next, let us recall the following definition. Let G be a group, $F=\{t_\lambda\}_{\lambda\in\Lambda}$ a free group with free generators t_λ and $\mathscr{A}=\{\beta_\lambda\}_{\lambda\in\Lambda}$ a set of automorphisms β_λ of G. If $w(t_\lambda)=w(t_{\lambda_1},\ldots,t_{\lambda_j})$ is a word in $t_{\lambda_1},\ldots,t_{\lambda_j}$, defining an element in F, we denote the automorphism $w(\beta_{\lambda_1},\ldots,\beta_{\lambda_j})$ (substituting t_{λ_l} in w by β_{λ_l}, $l=1,\ldots,j$) by $w(\beta_\lambda)$. The semidirect

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product $G\times_{\mathscr{A}}F$ of G and F with respect to $\mathscr{A}$ is defined as follows: $G\times_{\mathscr{A}}F=G\times F$ as sets, and multiplication in $G\times_{\mathscr{A}}F$ is given by

$$(g, w(t_\lambda))(g', w'(t_\lambda))=(gw(\beta_\lambda)^{-1}(g'), w(t_\lambda)w'(t_\lambda)),$$

for all $(g, w(t_\lambda))$, $(g', w'(t_\lambda))$ in $G\times_{\mathscr{A}}F$. If $\beta_\lambda=\beta$ for all $\lambda\in\Lambda$, then $G\times_{\mathscr{A}}F$ is just the semidirect product $G\times_\beta F$ of G and F with respect to β(cf. [**4**]).

Now let H_k be as given by (1) and α, T as given in Theorem 2. Let $F=\{t_\lambda\}_{\lambda\in\Lambda}$ be a free group and $\mathscr{A}=\{\alpha^{n_\lambda}\times\mathrm{id}\,T\}_{\lambda\in\Lambda}$ a set of automorphisms $\alpha^{n_\lambda}\times\mathrm{id}\,T$ of $H_k\times_\alpha T$ induced by the automorphisms α^{n_λ} of H_k, where n_λ is any integer, for all $\lambda\in\Lambda$. Then we form the semidirect product

$$H'=(H_k\times_\alpha T)\times_{\mathscr{A}}F$$

of $H_k\times_\alpha T$ and F with respect to $\mathscr{A}$. Note that if $n_\lambda=1$ for all $\lambda\in\Lambda$, then H' reduces to $H=(H_k\times_\alpha T)\times_\alpha F$. Then we extend the results in [**4**] to the following theorem, the proof of which gives, in particular, a short, direct proof of that in [**4**].

THEOREM 3. Wh $H'=0$ *and* $\tilde{K}_0\mathbb{Z}(H')=0$.

2. A general theorem. In this section, we prove the following more general result.

THEOREM 4. *Let G be a group of some particular form such that G and $G\times T$ are of the same form, where T is any infinite cyclic group. Suppose that the Whitehead group of any such group is trivial. Let $H_k=G\times F_1\times\ldots\times F_k$, where $F_1,\ldots,F_k$ are free groups each of rank at least two. Then*

$$\mathrm{Wh}\,H_k=0,\ \tilde{K}_0\mathbb{Z}(H_k)=0\quad\textit{and}\quad\tilde{C}(\mathbb{Z}(H_k),\mathrm{id})=0$$

for each $k=1,2,\ldots$.

We remark that the proof of this theorem is similar to that in [**6**, Theorem] and we use the following lemma in [**6**].

LEMMA 2. *Let G be a group such that* Wh $G=0$, $\tilde{K}_0\mathbb{Z}(G)=0$ *and* $\tilde{C}(\mathbb{Z}(G),\mathrm{id})=0$. *Then for any free group F,* $\mathrm{Wh}(G\times F)=0$. [It follows from the Bass-Heller-Swan decomposition formula (cf. [**1**] or [**3**, Theorem 1]) that the hypotheses are equivalent to $\mathrm{Wh}(G\times T)=0$.]

Proof of Theorem 4. First we note that for each $k=1,2,\ldots,H_k$ and $H_k\times T$ are of the same form.

We prove the theorem by induction on the number of the free groups.

For $k=1$, we have $H_1=G\times F_1$. Since $\mathrm{Wh}(G\times T)=0$, by hypothesis, it follows from Lemma 2 that Wh $H_1=0$. Since $H_1\times T$ is of the same form as H_1, we also have $\mathrm{Wh}(H_1\times T)=0$, and so, by the the Bass-Heller-Swan decomposition formula, $\tilde{K}_0\mathbb{Z}(H_1)=0$ and $\tilde{C}(\mathbb{Z}(H_1),\mathrm{id})=0$. This starts the induction.

Now suppose inductively that the theorem holds for some $k=m$ and for any G of the given form. Then using Lemma 2 again, we get Wh $H_{m+1}=\mathrm{Wh}(H_m\times F_{m+1})=0$. By the above observation that H_k and $H_k\times T$ are of the same form for each k, we also have

$\mathrm{Wh}(H_{m+1}\times T)=0$, and we deduce, again from the Bass-Heller-Swan decomposition formula, that $\tilde{K}_0\mathbb{Z}(H_{m+1})=0$ and $\tilde{C}(\mathbb{Z}(H_{m+1}), \mathrm{id})=0$. This completes the proof.

3. Proofs of main theorems. Now we give the proofs of our theorems.

Proof of Theorem 1. We remark that if G is a group of type n, then $G\times T$ is a group of type $n+1$. Thus G and $G\times T$ are of the same form. Hence the assertions follow immediately from Theorem 4, by the results of Farrell-Hsiang (Lemma 1).

Proof of Theorem 2. As mentioned above, if G is a group of type n, then $\mathbb{Z}(G)$ is right Noetherian and of finite right global dimension. Then, as shown in [**2**] and [**9**], $\mathbb{Z}(G\times F)$ is right coherent and of finite right global dimension, where F is any free group. The same arguments as were used in the proof of [**3**, Theorem B] establish the result.

Proof of Theorem 3. Let the generator of T be t. Let

$$s_\lambda = t^{-n_\lambda}t_\lambda$$

for all $\lambda\in\Lambda$, where n_λ is given as above, and let F' be the group generated by s_λ $(\lambda\in\Lambda)$. Then clearly F' is a free group isomorphic to F. Moreover, by changing the generators in $T\times F$, the group $H'=(H_k\times_\alpha T)\times_{\mathscr{A}}F$ can be easily seen to be isomorphic to the direct product $H''=(H_k\times_\alpha T)\times F'$, which is just $(F'\times H_k)\times_\alpha T$. Thus, by Theorem 2, $\mathrm{Wh}\,H''=0=\tilde{K}_0\mathbb{Z}(H'')$. Hence $\mathrm{Wh}\,H'=0$ and $\tilde{K}_0\ (H')=0$, as required.

REMARK. Let G_1 and G_2 be as given in [**6**]. Then as shown in the proof of Theorem 3, G_1 can be seen to be canonically isomorphic to the group $(A\times_\alpha T)\times F''$, where F'' is a free group isomorphic to F. Similarly G_2 is canonically isomorphic to $(G_1\times_\beta T')\times F'''$, where F''' is a free group isomorphic to F'. Thus G_2 is of the form

$$(((A\times_\alpha T)\times F'')\times_\beta T')\times F''',$$

i.e. G_2 is of the form $H_2\times_\beta T'$, where H_2 is as given by (1). Hence the results in Theorem 3 give, in particular, the main theorem in [**6**] and hence that in [**5**].

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