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Minimal first countable spaces

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A topological space is E_0 (resp. E_1) provided every point is the countable intersection of neighborhoods (resp. closed neighborhoods). For $i = 0$ and $i = 1$, characterizations of minimal E_i spaces (E_i spaces with no strictly coarser E_i topology) and E_i-closed spaces (E_i spaces which are closed in every E_i space containing them) are given; for example, the properties of minimal E_i and minimal first countable T_{i+1} are shown to be equivalent. Minimal E_0 spaces are characterized as countable spaces with the cofinite topology, and minimal E_1 spaces are characterized as E_1-closed and semiregular spaces. E_0-closed spaces are shown to be precisely the finite discrete spaces.

1. *Introduction*

In a recent paper [2], Aull mixed the Hausdorff separation axiom with the first countable axiom to yield a new separation axiom denoted as E_1; a space is E_1 provided every point is the countable intersection of closed neighborhoods. Clearly, an E_1 space is Hausdorff, and Aull proved the interesting fact that a countably compact E_1 space is minimal E_1 (a space with a topological property P is *minimal* P provided there are no strictly coarser P-topologies).

In Section 2 of this paper, we derive several characterizations of minimal E_1 spaces. In particular we prove that a space is minimal E_1

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exactly when it is first countable Hausdorff and has no strictly coarser first countable Hausdorff topology; such spaces have been investigated by Stephenson [12]. Also, in this section we characterize E_1-closed spaces (a space with a topological property P is *P-closed* whenever it is closed in every P space containing it).

Closely associated with E_1 spaces are spaces in which every point is G_δ ; these spaces have been studied and designated as G_δ-spaces by Anderson [1] and as E_0 spaces by Aull [2]. In Section 3 we prove that a space is minimal E_0 if and only if it is countable with the cofinite topology (a space has the *cofinite topology* provided the nonvoid open sets are complements of finite sets).

We now list some definitions and facts that will be used throughout the sequel.

A filter base consisting of open sets is called an *open filter base*. A space is said to be *feebly compact* [13] or *lightly compact* [3] provided every locally finite family of nonvoid open sets is finite. A space X is *Hausdorff except* for a subset $A \subseteq X$, cf. [1], provided each pair of distinct points of which one is in $X \setminus A$ can be separated by disjoint open sets.

(1.1) The following are equivalent for a space X :

(a) X is feebly compact;

(b) every countable open filter base has at least one adherent point;

(c) every countable open cover has a finite subfamily whose closures cover X ;

(d) X is closed in every space Y which contains X as a subspace, is Hausdorff except for X, and is first countable at each point of $Y \setminus X$.

The equivalence of (a) and (c) is established in [3]. The equivalence of (b), (c) and (d) is similar to the proof in the Hausdorff case [7, pp. 145-146]. We now list two facts by Aull [2].

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(1.2) A countably compact E_1 space is minimal E_1 .

(1.3) A first countable Hausdorff space is E_1 .

The natural numbers will be denoted by $\mathbb{N}$. The symbol (X, τ) will denote a topological space whose set is X and whose topology is τ . For a space (X, τ) , the regular-open sets (sets equal to the interior of their closure) form an open base for a topology denoted by τ_s and called the *semiregularization* of τ , cf. [*7*, p. 138]. A space (X, τ) is *semiregular* provided $\tau = \tau_s$ and is *semiregular at a point* p , or p is a *semiregular point* of (X, τ) , provided the regular-open sets containing p form a base at p .

2. E_1 spaces

We first characterize E_1-closed spaces and then minimal E_1 spaces.

THEOREM 2.1. *For an* E_1 *space* (X, τ) , *the following are equivalent:*

(a) (X, τ) *is* E_1*-closed;*

(b) (X, τ) *is feebly compact;*

(c) X *is a closed set in every* E_1 *space* Y *which contains* (X, τ) *as a subspace and is first countable at each point of* $Y \setminus X$ *;*

(d) (X, τ_s) *is minimal* E_1 .

Most of the proof is straightforward; the rest is essentially the same as the proof in the Hausdorff case [*10*; *7*, pp. 145-146].

For a topological property P , let $P(1)$ denote the combined topological properties of first countable and P ; so, a Hausdorff(1) space is a first countable Hausdorff space. The next corollary extends Theorem 2.5 of [*12*].

COROLLARY 2.2. *A space is Hausdorff*(1)*-closed if and only if it is* E_1*-closed and first countable.*

Let X be the space of real numbers with the topology whose base is the usual open intervals minus countable subsets. The subspace $[0, 1]$

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of X is an example of an H-closed E_1 space that is not first countable at any point [*1*, Example 2]. The same type of topology placed on the "long line" yields an E_1-closed space that is neither H-closed nor **first** countable at any point.

DEFINITION. A point p in a space is an E_1 *point* provided p is a countable intersection of closed neighborhoods.

LEMMA 2.3. *A semiregular,* E_1 *point in a feebly compact space has a countable fundamental system of neighborhoods.*

Proof. Let X be a feebly compact space which is semiregular at an E_1 point p. There is a decreasing family $\mathcal{B} = \{B_n \mid n \in \mathrm{N}\}$ of open sets of p such that $\{p\} = \cap \bar{B}_n$. To show that $\mathcal{B}$ is a base at p, let V be a regular-open set containing p. $X \setminus V$ is the closure of an open set and by Theorem 14 in [*3*] is feebly compact. By (1.1) (b), the trace of $\mathcal{B}$ on $X \setminus V$ cannot be an open filter base on $X \setminus V$; so, there is $B \in \mathcal{B}$ such that $B \subseteq V$.

THEOREM 2.4. *The following are equivalent for a space* X :

(a) X *is minimal* E_1 *;*

(b) X *is* E_1*-closed and semiregular;*

(c) X *is feebly compact,* E_1 *, and semiregular;*

(d) X *is* E_1 *and every countable open filter base with a unique adherent point and nonvoid intersection converges;*

(e) X *is minimal Hausdorff*(1) .

Proof. By Theorem 2.1, *(b)* and *(c)* are equivalent and *(b)* implies *(a)*. The proof that *(a)* implies *(b)* and *(d)* is similar to the proof in the Hausdorff case [*10*; *7*, pp. 146-147]. Since a minimal Hausdorff(1) space is equivalent to being first countable, Hausdorff, feebly compact, and semiregular by Theorem 2.4 in [*12*], then by (1.3), *(e)* implies *(c)* and by Lemma 2.3, *(c)* implies *(e)*. The proof is completed if we show that *(d)* implies *(a)*. Suppose (X, τ) satisfies *(d)* and σ is a coarser E_1 topology on X. For each p in X, there is a decreasing family $\mathcal{B} = \{B_n \mid n \in \mathrm{N}\}$ of σ-open sets containing p with the property that

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$\{p\} = \bigcap \mathrm{Cl}_\sigma(B_n)$. Since $\sigma \subseteq \tau$, then by *(d)*, $\mathcal{B}$ converges to p in (X, τ) . This shows that (X, τ) is minimal E_1 .

COROLLARY 2.5. *A first countable, minimal Hausdorff space is minimal* E_1 .

DEFINITION. For a topological property P , a space is said to be *Katětov* P provided it has a coarser minimal P topology.

COROLLARY 2.6. *A space is Katětov* E_1 *if and only if it is Katětov Hausdorff*(1) .

By (1.2), countably compact E_1 spaces are minimal E_1 ; the converse is false as shown by Example 3.2 in [12] of a completely regular, locally compact, minimal Hausdorff(1) (and hence minimal E_1) space which is not countably compact. Also, there are first countable, minimal Hausdorff spaces [4 , Example 1.5] which are not countably compact. By Theorem 2.4, minimal E_1 spaces are E_1-closed; the converse is false as shown by the examples following Corollary 2.2. By Theorem 2.4, a necessary and sufficient condition for an E_1-closed space to be minimal E_1 is semiregularity. We now develop a necessary and sufficient condition for an E_1-closed space to be countably compact.

A regular space has the property that each pair of disjoint closed sets of which one is finite can be separated by disjoint open sets. A space which satisfies this property with finite replaced by countable is called *hyperregular*. So, every hyperregular T_1 space is T_3 . It is easy to verify the following fact.

(2.7) A countably compact T_3 space is hyperregular.

Dugundji [8] has defined a space to be *weakly normal* precisely when it is completely regular and hyperregular T_1 . There exists a countable compact T_3 space that is not completely regular (modify the space in Section 3 of [6] by replacing Z_n with $\{n\} \times \Omega' \times \Omega' \setminus \{(n, \Omega, \Omega)\}$) ; this example shows there is a hyperregular T_1 space which is not weakly normal.

THEOREM 2.8. *For hyperregular spaces, feeble compactness is equivalent to countable compactness.*

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Proof. [8, p. 232]

COROLLARY 2.9. *An E_1-closed space is countably compact if and only if it is hyperregular.*

As a consequence of Theorem 4.1 in [12] and Theorem 2.4, we have the next result.

(2.10) A product of nonvoid spaces is minimal E_1 if and only if each coordinate space is minimal E_1 and there is at most a countable number of coordinate spaces with more than one point.

(2.11) A product of nonvoid spaces is E_1-closed if and only if each coordinate space is E_1-closed and there is at most a countable number of coordinate spaces with more than one point.

Proof. The proof follows from (2.10), Theorems 2.1 and 2.4, and the easily proven fact that if $\left\{(X_\alpha, \tau_\alpha) \mid \alpha \in A\right\}$ is a family of spaces, then $(\pi\tau_\alpha)_s = \pi(\tau_\alpha)_s$ where $\pi\tau_\alpha$ denotes the product topology on πX_α .

By proofs of Theorems 5.7, 5.9, and 5.10 of [12], the next three facts follow.

(2.12) Any E_1 space X can be densely embedded in an E_1-closed space Y which is first countable at each point of $Y \setminus X$.

(2.13) An E_1 space can be densely embedded in a minimal E_1 space if and only if it is first countable and semiregular.

(2.14) An E_1 space can be embedded in a minimal E_1 space if and only if it is first countable.

We conclude this section by giving a cotopological characterization of minimal E_1 spaces. The definitions and notation used are defined in [5].

LEMMA 2.15. *Let (X, τ) be an E_1 space and $\mathcal{F}$ a countable open filter base with nonvoid intersection and with a unique adherent point p . Let $\mathcal{B} = \{B \in \tau \mid p \in \bar{B}$ or $B \subseteq X \setminus \bar{F}$ for some $F \in \mathcal{F}\}$.*

(a) $\mathcal{B}$ is a base for τ ;

(b) $\{X \setminus \bar{B} \mid B \in \mathcal{B}\}$ is a base for a topology labeled $\tau(\mathcal{B})$ and $\tau(\mathcal{B}) \subseteq \tau$;

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(c) *if* σ *is the topology defined on* X *by* $U \in \sigma$ *if and only if* $U \in \tau$ *and* $p \in U$ *implies* $F \subseteq U$ *for some* $F \in \mathcal{F}$ *, then* $\tau(\mathcal{B}) = \sigma_s$ *;*

(d) $\left(X, \tau(B)\right)$ *is* E_1 *;*

(e) *if* $\mathcal{F}$ *does not converge, then* $\tau(\mathcal{B}) \neq \tau$ *.*

The proofs of parts *(a)*, *(b)*, *(d)* and *(e)* are similar to the proof of Lemma 1 in [*14*]. The proof of part *(c)* is straightforward and left to the reader. Part *(c)* shows that the cotopological method of obtaining coarser topologies in this particular case is a combination of the filter method and semiregularization, cf. [5, Section 2].

THEOREM 2.16. *An* E_1 *space is minimal* E_1 *if and only if there are no strictly coarser* E_1 *cotopologies.*

Proof. Necessity follows from the definition of minimal E_1 . Sufficiency follows from Lemma 2.15 and Theorem 2.4.

3. E_0 spaces

Berri [4] and Hewitt [9] proved that a space is minimal T_1 if and only if the space has the cofinite topology. In [5], it is observed that a space is T_1-closed if and only if it is a finite discrete space.

THEOREM 3.1. *A space* X *is* E_0*-closed if and only if* X *is a finite discrete space.*

Proof. The sufficiency is obvious. For the necessity, it is enough to observe the following:

(i) the intersection of each countable open filter base in X is nonvoid and

(ii) each point of X is isolated.

We now proceed to characterize minimal E_0 spaces.

LEMMA 3.2. *If* (X, τ) *is a minimal* E_0 *space and* $\mathcal{F}$ *is a countable open filter base with void intersection, then* $\mathcal{F}$ *converges to each point.*

Proof. Let $p \in X$. Define a topology σ on X by $U \in \sigma$ if and only if $U \in \tau$ and $p \in U$ implies $F \subseteq U$ for some $F \in \mathcal{F}$. (X, σ) is an E_0 space and $\sigma \subseteq \tau$. Since (X, τ) is minimal E_0, then $\sigma = \tau$. This shows that $\mathcal{F}$ converges to p.

LEMMA 3.3. *If a minimal E_0 space X has one nonisolated point, then every point is nonisolated.*

Proof. Let p in X be nonisolated. There is a decreasing sequence $\{U_n | n \in \mathrm{N}\}$ of open sets containing p whose intersection is p. Now $\mathcal{F} = \{U_n \setminus \{p\} | n \in \mathrm{N}\}$ is a countable open filter base with void intersection. By Lemma 3.2, $\mathcal{F}$ converges to each point of X; this shows that each point is nonisolated.

LEMMA 3.4. *A minimal E_0 space X contains at most a finite number of isolated points.*

Proof. Assume X is a minimal E_0 space with a denumerable number of isolated points, say $\{x_n | n \in \mathrm{N}\}$ such that $x_n = x_m$ only if $n = m$. Let $U_n = \{x_m | m \geq n\}$ for each n in N. $\mathcal{F} = \{U_n | n \in \mathrm{N}\}$ is a countable open filter base with void intersection. By Lemma 3.2, $\mathcal{F}$ converges to the point x_1 which is a contradiction as x_1 is an isolated point.

THEOREM 3.5. *A space is minimal E_0 if and only if the space is countable with the cofinite topology.*

Proof. Since a countable space with the cofinite topology is E_0 and minimal T_1 [4, Theorem 2.2], then it is minimal E_0. Conversely, suppose X is a minimal E_0 space. Assume X is not countable. By Lemmata 3.3 and 3.4, X contains no isolated points. Let $p \in X$; as in the proof of Lemma 3.3, there is a countable open filter base $\mathcal{F} = \{F_n | n \in \mathrm{N}\}$ with void intersection. We now show that $U \subseteq X$ is a nonvoid open set if and only if $F_n \subseteq U$ for some $F_n \in \mathcal{F}$. If U is an open set and $q \in U$, then since $\mathcal{F}$ converges to q by Lemma 3.2, there is a F_n in $\mathcal{F}$ such that $F_n \subseteq U$. Conversely, suppose $U \subseteq X$ and $F_m \subseteq U$ for some $F_m \in \mathcal{F}$. Let $q \in U$. There is a decreasing sequence $\{U_n \mid n \in \mathbb{N}\}$ of open sets containing q whose intersection is q. By Lemma 3.2, $\{U_n \setminus \{q\} \mid n \in \mathbb{N}\}$ converges to p. There is $k \in N$ such that $U_k \setminus \{q\} \subseteq F_m \cup \{p\}$ and $p \notin U_k$. Since $q \in U$, then $U_k \subseteq U$. This shows that U is open.

Since $X \setminus \{p\} = \cup X \setminus F_n$, then for some m, $X \setminus F_m$ is infinite. Let $\{p_n \mid n \in \mathbb{N}\}$ be a subset of $X \setminus F_m$, and for each $n \in \mathbb{N}$, let $W_n = F_n \cup \{p_k \mid k \geq n\}$. Then $G = \{W_n \mid n \in \mathbb{N}\}$ is a countable open filter base with void intersection. By Lemma 3.2, G converges to p. So, $W_k \subseteq F_m \cup \{p\}$ for some $k \in \mathbb{N}$. This implies that $\{p_r \mid r \geq k\} \subset F_m \cup \{p\}$ which is a contradiction. This shows that X is countable.

Since the cofinite topology on a countable set is an E_0 topology which is coarser than any other E_0 topology on the set, then X must possess the cofinite topology. This completes the proof of the theorem.

THEOREM 3.6. *The following are equivalent for a space* X *:*

(a) X *is minimal* E_0 *;*

(b) X *is first countable and minimal* T_1 *;*

(c) X *is minimal* $T_1(1)$.

Proof. The equivalence of *(a)* and *(b)* follows from Theorem 3.5. Clearly, *(b)* implies *(c)*. The proof that *(c)* implies *(a)* is similar to the proof in the E_1 case.

References

[1] Frank W. Anderson, "A lattice characterization of completely regular G_δ-spaces", *Proc. Amer. Math. Soc.* 6 (1955), 757-765.

[2] C.E. Aull, "A certain class of topological spaces", *Prace Mat.* 11 (1967), 49-53.

[3] R.W. Bagley, E.H. Connell and J.D. McKnight, Jr, "On properties characterizing pseudocompact spaces", *Proc. Amer. Math. Soc.* 9 (1958), 500-506.

[4] Manuel P. Berri, "Minimal topological spaces", *Trans. Amer. Math. Soc.* 108 (1963), 97-105.

[5] Manuel P. Berri, Jack R. Porter and R.M. Stephenson, Jr, "A survey of minimal topological spaces", (submitted).

[6] Manuel P. Berri and R.H. Sorgenfrey, "Minimal regular spaces", *Proc. Amer. Math. Soc.* 14 (1963), 454-458.

[7] Nicolas Bourbaki, *General topology, Part 1* (Addison-Wesley, Reading, Massachussets; London, Ontario, 1966).

[8] James Dugundji, *Topology* (Allyn and Bacon, Boston, 1966).

[9] Edwin Hewitt, "A problem of set-theoretic topology", *Duke Math. J.* 10 (1943), 309-333.

[10] Miroslav Katětov, "Über H-abgeschlossene und bikompakte Räume", *Časopis Pěst. Mat. Fys.* 69 (1940), 36-49.

[11] Chen-Tung Liu, "Absolutely closed spaces", *Trans. Amer. Math. Soc.* 130 (1968), 86-104.

[12] R.M. Stephenson, Jr, "Minimal first countable topologies", *Trans. Amer. Math. Soc.* 138 (1969), 115-127.

[13] A.H. Stone, "Hereditarily compact spaces", *Amer. J. Math.* 82 (1960), 901-916.

[14] Giovanni Viglino, "A co-topological application to minimal spaces", *Pacific J. Math.* 27 (1968), 197-200.

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