10
Knot theory and physical states of quantum gravity

10.1 Introduction

In the previous two chapters we developed several aspects of the loop representation of quantum gravity. One of the main consequences of these developments is a radically new description of one of the symmetries of the theory: because of diffeomorphism invariance wavefunctions in the loop representation must be invariant under deformations of the loops, they have to be knot invariants. This statement is much more than a semantical note. Knot invariants have been studied by mathematicians for a considerable time and recently there has been a surge in interest in knot theory. Behind this surge of interest is the discovery of connections between knot theory and various areas of physics, among them topological field theories. We will see in this chapter that such connections seem to play a crucial role in the structure of the space of states of quantum gravity in the loop representation. As a consequence we will discover a link between quantum gravity and particle physics that was completely unexpected and that involves in an explicit way the non-trivial dynamics of the Einstein equation. Such a link could be an accident or could be the first hint of a complete new sets of relationships between quantum gravity, topological field theories and knot theory.

We will start this chapter with a general introduction to the ideas of knot theory. We will then develop the notions of knot polynomials and the braid group. In the third section we will discuss the connection between knot theory and topological field theories, through the Chern–Simons theory. In section 10.4 we will show how to use the previous notions to construct states of quantum gravity in the loop representation related to the Kauffman and Jones polynomials. In the last section we will present a simple explanation for the existence of the Jones polynomial state and a discussion on the possibility of generating new solutions.

https://doi.org/10.1017/9781009290203.012 Published online by Cambridge University Press

10.2 Knot theory

The study of invariants of closed curves under smooth deformations is quite old. One of the first examples was the introduction by Gauss (1820) of the linking number. The linking number is an invariant of two closed curves that measures the number of times one of the curves winds around the other. This is obviously an invariant since the only way to change that number is to cut one of the closed curves and therefore it is not a smooth deformation. Although such an invariant may appear quite trivial, we will see it plays an important role in topological field theories and quantum gravity. In particular, it admits an integral expression, as we discussed in chapter 3,

$$L(\gamma_1, \gamma_2) = \frac{1}{4\pi} \oint_{\gamma_1} dx^a \oint_{\gamma_2} dy^b \epsilon_{abc} \frac{(x-y)^c}{|x-y|^3}. \tag{10.1}$$

Such an expression was considered by Maxwell [174] in connection with electromagnetic theory. If one builds a thin solenoid with the shape of each loop the above integral measures the magnetic flux produced by each solenoid across the other [175] in appropriate units. In particular Maxwell gives a good explanation for why that expression gives one or zero as result. It measures the solid angle that one of the loops subtends from the point of view of the other as one traverses along the latter. Therefore the result is either an integer multiple of 4π or 0 depending on how the loops are linked. Notice that the expression we give for the linking number depends explicitly on a background metric and yet the result is diffeomorphism invariant.

It is evident that there is much more to knot theory than the linking number as can be illustrated by the Borromean rings which we show in figure 10.1.

This example illustrates a usual difficulty with trying to distinguish knots through the values of a particular knot invariant. Every time one introduces an invariant it is able to detect up to a certain degree of knotting. For each invariant one can construct complicated links or knots such that the invariant does not detect the linking.

The fundamental problem of knot theory is the classification of knots and links*. The main question is how to tell apart two knots that are not smoothly deformable to each other.

Historically, there was a surge of interest in knot theory towards the end

* Usually "knot" refers to a single curve and "link" to many curves. We will loosely use them indistinguishably whenever the context allows. We will also use the word "loop" in the precise sense introduced in chapter 1 whenever applicable. For instance the Gauss linking number is a genuine function of loops.

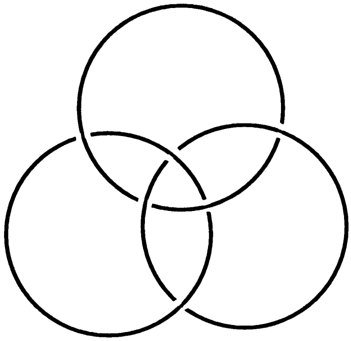

Fig. 10.1. The Borromean rings. An example of loops that have a non-trivial linking but zero linking number if taken in pairs.

of last century due to a failed theory of atoms of James Clerk Maxwell, Lord Kelvin and Peter Guthrie Tait [176]. After the discovery of the complete theory of electromagnetic phenomena, the outstanding unsolved problem in physics was the explanation of atomic spectra. In the proposed theory atoms were depicted as knotted lines of aether (this predated special relativity and quantum mechanics). The theory had several attractive features, among which it associated the stability of atoms with the topological nature of knots. The main lasting impact of it, however, was that through its development many of the central issues of modern knot theory were brought to the forefront. Among these was the classification of knots and their representations. It is remarkable that 100 years later, although the physical motivations are quite different, the interest in knot theory remains basically the same.

The typical depiction of a knot is through its projection on a plane, as we did when depicting the Borromean rings. This adds an additional complication in the sense that a single knot admits a number of different projections. Smooth deformations of knots in three-dimensional space translate themselves in a series of motions in terms of the projections. Such motions are known as Reidemeister moves. There are three types of Reidemeister moves, which are depicted in figure 10.2. If two knot projections can be mapped into each other through a finite number of Reidemeister moves, they are projections of the same knot.

In the knot theory literature two knots that are connected through a finite number of Reidemeister moves are called ambient isotopic. Strictly from our point of view, it is this kind of equivalence that we are interested

https://doi.org/10.1017/9781009290203.012 Published online by Cambridge University Press

(i)
(ii)
(iii)

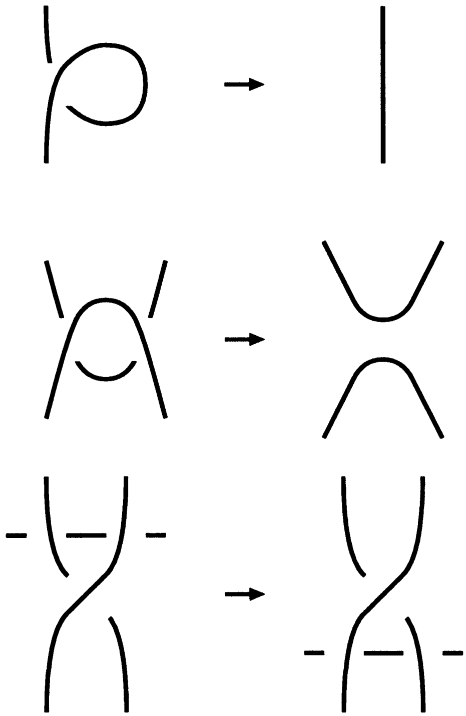

Fig. 10.2. Reidemeister moves.

in, since it corresponds to the usual diffeomorphism invariance. For several reasons that we will discuss shortly it will be useful to consider quantities invariant under a slightly different set of transformations leading to a notion called regular isotopy. Two knots are called regular isotopic to each other if they can be connected through a finite set of Reidemeister moves of types (ii) and (iii). Such an idea is important in the following context. Suppose instead of dealing with knots made of strings of zero width we were considering knots made of ribbons. It is clear that the first Reidemeister move does not correspond to a smooth deformation of a ribbon, since the elimination of a "curl" can only be attained through the introduction of a twist, as shown in figure 10.3. The justification for the consideration of regular isotopy in quantum gravity will be related to regularization issues. As we have done in previous chapters, in many contexts one needs to point-split expressions and in such splitting the resulting objects resemble ribbons rather than loops. We will give details in chapter 11.

At this point the reader may be wondering what is the connection with quantum gravity. To put it in a different way: one knows that the wavefunctions of quantum gravity are knot invariants. Which of all the possible

https://doi.org/10.1017/9781009290203.012 Published online by Cambridge University Press

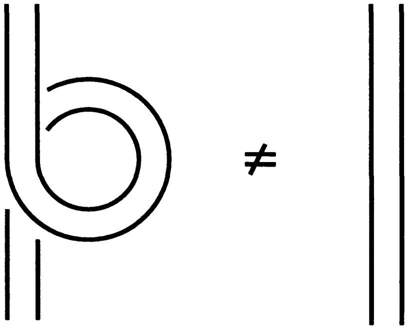

Fig. 10.3. Reidemeister moves of type (i) do not leave invariant functions of ribbons.

knot invariants are of interest for quantum gravity? At the moment the answer is quite open. We will later introduce some invariants that solve the Hamiltonian constraint. Previous to that there are three main points to be remembered: (a) any knot invariant of interest to quantum gravity has to be a function of the group of loops rather than a function of curves; (b) it should satisfy the Mandelstam identities; (c) it has to be a well defined function of intersecting loops.

Very little needs to be said about points (a) and (b); since one is interested in a loop representation that is obtained via a loop transform from the connection representation only functions of the group of loops that satisfy the Mandelstam identities should be allowed. Point (c) stems from the discussion in chapter 8. As we saw there, the Mandelstam identities related the value of the function on loops with intersections with the value on loops without. Therefore for consistency one has to consider loops with intersections. Furthermore we saw that non-intersecting loops solved the constraints for all values of the cosmological constant: they corresponded to degenerate geometries.

Intersecting knot theory is a quite novel subject. A surge of interest has arisen as a consequence of the theory of Vassiliev invariants [188, 189]. Most of the studies of knot invariants, however, were done for non-intersecting, smooth curves. It turns out several ideas can be easily generalized. We will do so in section 10.3.4.

10.3 Knot polynomials

As we mentioned above, the main problem in knot theory is to classify knots. The obvious solution to this problem is to try to generate a large number of knot invariants. The hope is that through the computation of

https://doi.org/10.1017/9781009290203.012 Published online by Cambridge University Press

their values one could distinguish knots, since knots with different values of their invariants are necessarily different. This program is at present incomplete. Though we know a large number of knot invariants they are not enough to classify knots. An important step towards the generation of knot invariants was the construction of certain polynomials associated with knots. In this section we will sketch some ideas of the theory of knot polynomials. We start with a discussion of the braid group. We then construct polynomials and their skein relations. We end with a discussion of the extension of these ideas to intersecting loops. There are many good references on the subject of knot polynomials and braids and we many passages of this chapter are modeled after these ideas. As an example we can cite the books by Kauffman [177, 178] and the review article by Guadagnini [179] and his book [180]. A more elementary but very readable treatment is given in the books by Adams [182] and Baez and Muniain [181].

10.3.1 The Artin braid group

A useful way to represent knots and links is through the braid group, B_n. Consider a set of n vertical strings starting and ending in two rows of n horizontally aligned points. The lines can cross each other an arbitrary number of times, forming a braid. Now arrange the lines in such a way that at each horizontal level there is only one crossing at the ith strand, which we denote g_i. One can describe such a braid by a sequence $g_i g_j g_k \ldots$. Such an ordered sequence states that if one follows the braid from the top to bottom (or vice-versa) one encounters a twist of the strings at the ith and $(i+1)$th positions followed by a twist of the strings at the jth and $(j+1)$th positions and so on, as shown in figure 10.4. Each twist has two possible orientations, denoted by g_i and g_i^{-1}.

The twists g_i form a group structure, called the Artin braid group. For n strings B_n has $n-1$ generators $g_1, \ldots, g_{n-1}$ that satisfy the relations

$$g_i g_j = g_j g_i, \qquad |i-j| > 1, \tag{10.2}$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \qquad i < n-1, \tag{10.3}$$

which can be easily checked by drawing n strings and applying the twists.

The strings involved in the braid group can be thought of as the space-time trajectory of particles in $2+1$ dimensions as they orbit around each other. This suggests an immediate connection between the braid group and $(2+1)$-dimensional physics. This connection has been explored in several contexts, including particle [190] and solid state physics. In particular it is the root of unusual statistics in $2+1$ dimensions connected with the idea of anyons [191, 192, 193].

https://doi.org/10.1017/9781009290203.012 Published online by Cambridge University Press

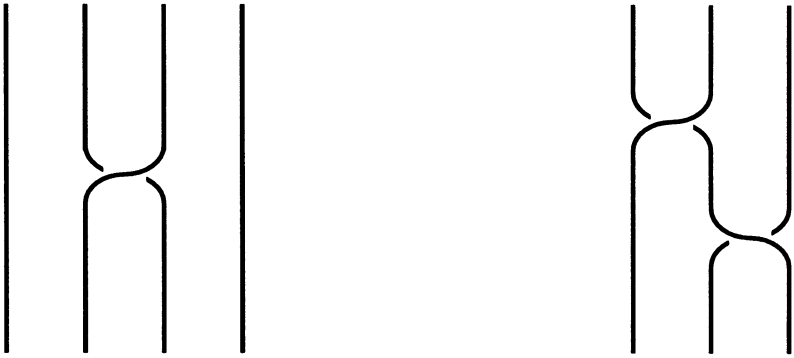

Fig. 10.4. Graphical representation of g_i and $g_1 g_2^{-1}$.

What is the relation with knots? One simply obtains a knot or a link by gluing together the ends of a braid in an order preserving manner (this is called a closure of the braid). Conversely, one can associate to each knot a braid. Therefore several properties of knots can be coded in the language of braids. The first question that arises is: given two braids, what are the conditions for their closure to yield the same knot or link up to ambient isotopy? For knot diagrams the answer is given by the Reidemeister moves. In terms of braids they translate themselves into a set of moves called the Markov moves. Reidemeister moves of type (iii) are already included in the braid group relation (10.3). Reidemeister moves of type (ii) are almost included in the relation (10.2), except for the fact that $g_1 \neq g_2 g_1 g_2^{-1}$, whereas both g_1 and $g_2 g_1 g_2^{-1}$ yield the same link under closure. The message is that to implement fully the second Reidemeister move in terms of braids, one has to identify elements that are conjugate under the adjoint action of the braid group. Two elements of the braid group that are conjugate are said to be related by a Markov move of type 1. Reidemeister moves of type (i) imply that a link diagram associated with the closure of a certain braid $b \in B_n$ and the closure of the braid $bg_n^{\pm 1} \in B_{n+1}$ are equivalent. These two elements are said to be related by a Markov move of type 2.

The advantage of the description of links in terms of braids is that one can present several properties of link diagrams in terms of algebraic notions. One can define link invariants as functionals of the elements of the braid group that are invariant under Markov moves. This implies the introduction of representations of the braid group. The closure of braids is represented by taking traces of expressions in terms of the group. We will explicitly use this kind of construction to derive expressions for some of the knot polynomials of relevance for quantum gravity. Before going into the details, we will discuss some general notions related to knot polynomials.

10.3.2 Skein relations, ambient and regular isotopies

A knot (or link) polynomial is an assignment of a finite set of numbers to a knot (or link) that is invariant under ambient or regular isotopies. Given a knot γ one gets a polynomial† $P(\gamma)_q$ in an arbitrary variable q such that all the coefficients $p_i(\gamma)$ of the polynomial are knot invariants. An important point is that for each knot the polynomial is of a finite order, but the order depends on the particular knot. An intuitive picture is that the lower coefficients of the polynomial represent "more naive" knot invariants that sense the simpler kinds of knottings whereas the higher coefficients are sensitive to more sophisticated kinds of knottings. Therefore for a simple kind of knot the lower coefficients of the polynomial are non-zero and the higher ones vanish. For more complicated knottings the lower coefficients fail to "see" the knottiness and the higher coefficients are the ones that sense it up to a certain order where again the knottiness is perceived as "trivial" by the more sophisticated higher coefficients. Therefore the order of a knot polynomial is finite and depends on the particular knot considered.

Why are these objects interesting? The reason is they are an ordered way of assigning an unlimited number of invariants to knots according to their complexity. There is therefore the expectation that they could constitute a systematic procedure for classifying knots. Moreover, some of the polynomials are defined by quite succinct recursion relations called the skein relations. The price for all this is high: there are only a handful of polynomials explicitly known at present.

The first polynomial was introduced in the 1920s by Alexander [197]. We present here a modification of that polynomial due to Conway [198] known as the Alexander–Conway polynomial $C(\gamma)_q$. It is defined by the skein relations,

$$C(U)_q = 1, \tag{10.4}$$

$$C(L_+)_q - C(L_-)_q = q\, C(L_0)_q, \tag{10.5}$$

where U is the unknot (a knot isotopic to a circle) and $L_\pm, L_0$ refer to the crossings shown in figure 10.5.

† In general they are Laurent polynomials. Sometimes it is convenient to write them as functions of a certain fractionary power of a variable, as we will see. Some polynomials may depend on several variables.

https://doi.org/10.1017/9781009290203.012 Published online by Cambridge University Press

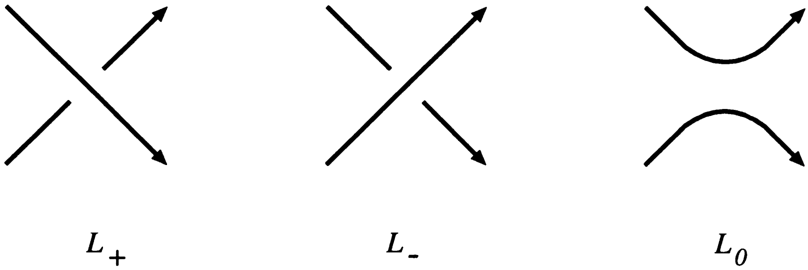

Fig. 10.5. The crossings $L_\pm$ and L_0.

The way in which the skein relations are to be interpreted is the following. The first one is simply a normalization condition that states that the polynomial evaluated for the unknot is 1. To read the second relation consider a specific knot and focus on a point where there is a line crossing. Excise a ball around the crossing so as to leave four incoming strands. The relation (10.5) states that if one evaluates the polynomial for the knot where the crossing we excised is replaced by the crossing L_+ and subtracts from it the polynomial evaluated for the same crossing replaced by L_- one gets as a result q (the polynomial variable) times the polynomial evaluated for the crossing replaced by L_0. The resulting equation is a relationship between the polynomials associated with three different knots. The strategy is to apply the relationship recursively combined with the Reidemeister moves until one gets a system of equations for the coefficients with a unique solution.

For a particular set of relations it is very difficult to prove that they determine the value of the polynomial for all knots unless one generates the skein relation in such a way as to guarantee it. The same consideration is true with respect to the diffeomorphism invariance of the objects constructed. The skein relations are relations between projections of the knots and it is quite non-trivial that the polynomial they define is independent of the projection.

Another important polynomial is the one due to Jones [199], $J(\gamma)_q$. The skein relations that define it are

$$J(U)_q = 1, \tag{10.6}$$

$$qJ(L_+)_q - q^{-1}J(L_-)_q = (q^{1/2} - q^{-1/2})J(L_0)_q. \tag{10.7}$$

The Jones polynomial is more "selective" than the Alexander–Conway one. However, there exist non-isotopic knots that have associated the same Jones polynomial, i.e., it fails to provide a classification for knots. There are other known polynomials, such as the HOMFLY [200] polynomial, which are slightly more general and contain Jones and Alexander–

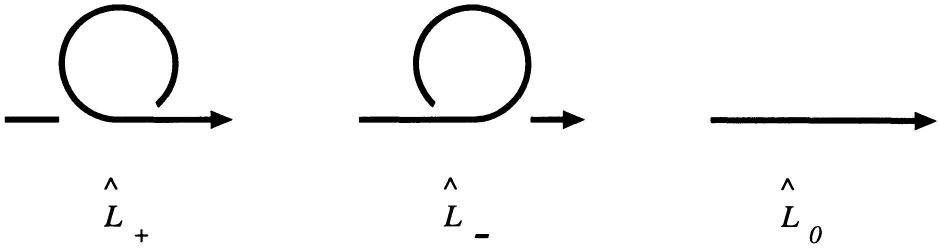

Fig. 10.6. Crossings for the skein relations of regular isotopic invariants.

Conway polynomials as particular cases. However, no polynomial known at present is sufficient to distinguish all knots.

Let us now concentrate on regular isotopy invariants. As we mentioned before, these are invariants that are sensitive to the first Reidemeister move, i.e., they "see" the additions of curls in the knots. Another way to put it is that they are invariants of (oriented) ribbons rather than of curves. Knot polynomials that are regular invariants can be defined. Their definition requires the introduction of a new set of crossings in their skein relations, $\hat{L}_{\pm,0}$, as shown in figure 10.6.

As an example of a regular invariant polynomial let us consider the Kauffman bracket, which can be viewed as a regular generalization of the Jones polynomial. The skein relations that define it are

$$K(U)_q = 1, \tag{10.8}$$

$$q^{1/4}K(L_+)_q - q^{-1/4}K(L_-)_q = (q^{1/2} - q^{-1/2})K(L_0)_q, \tag{10.9}$$

$$K(\hat{L}_+)_q = q^{3/4}K(\hat{L}_0)_q, \tag{10.10}$$

$$K(\hat{L}_-)_q = q^{-3/4}K(\hat{L}_0)_q. \tag{10.11}$$

Regular isotopic invariants of curves can be associated with ambient isotopic invariants of oriented ribbons if one gives a prescription to associate a ribbon to each curve. Such prescriptions are called "framings". Technically they correspond to an assignment of a vector to each point of the curve, such that one obtains a second curve by infinitesimally shifting the original one along the vector.

We now introduce some concepts that are useful in the discussion of regular isotopic invariants. The first of them is the writhe of a knot diagram, $w(\gamma)$, defined by

$$w(\gamma) = \sum_{\text{crossings}} \epsilon(\text{crossing}), \tag{10.12}$$

where $\epsilon(L_\pm) \equiv \pm 1$. This quantity measures the number of "curls" in the diagram. It is clearly not invariant under Reidemeister moves of type (i)

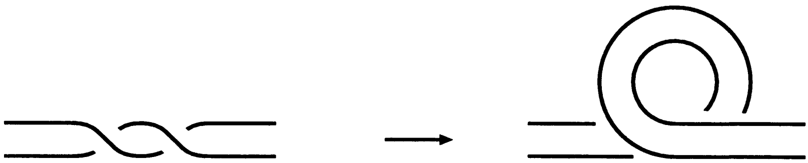

Fig. 10.7. A twist can be exchanged by a curl through a Reidemeister (i) move

but it is a regular isotopic invariant.

Another regular invariant, in this case of bands, is the twist. Assume one paints the two sides of the band with different colors. The twist measures how many times the color changes as seen from the planar projection. It can also be defined in terms of an analytic expression, but we will not discuss this here [196]. It is evident that if one performs a Reidemeister move of type (i) one can exchange a twist in a band by a curl, as shown in figure 10.7.

Using the fact that Reidemeister moves of type (i) exchange curls and twists in bands, one can combine the previous two quantities into an ambient isotopic invariant of the two curves that form the band. The resulting invariant is given by their linking number, which now can be viewed as a quantity associated with the knot diagram through a framing procedure. To reflect that association it is usually called the "self-linking" number of a knot diagram. One can summarize this result in a formula called White's theorem [194],

$$SL(\gamma) = T(\gamma) + w(\gamma), \tag{10.13}$$

where $SL(\gamma)$ stands for self-linking number of the knot diagram. Explicit expressions for all the terms in White's theorem can be given. For the Gauss linking number, apart from the integral formula we have already discussed, a definition can be introduced terms of the plane projection of two curves. This is given by

$$L(\gamma_1, \gamma_2) = \frac{1}{2} \sum_{\text{crossings}(\gamma_1, \gamma_2)} \epsilon(\text{crossing}), \tag{10.14}$$

where the summation is only over the crossings of one curve with the other. The reader can check that this expression gives the usual result for the linking of two curves. White's theorem has found important applications in biology, where one has to count the twists of DNA structures through the plane projections one gets when viewing it through a microscope [195] and also in Polyakov's description of the Fermi–Bose transmutation in the context of anyons [196].

There are many prescriptions for framing. One of them is the "vertical framing", in which the twist of the ribbon is set to zero, i.e., all twists are converted to curls. Due to White's theorem, in this framing the linking number coincides with the writhe. This is also called "blackboard framing" [178] since it corresponds to considering the projection of the knot and drawing a parallel knot along it. The resulting ribbon has no twist. Another common framing is the "standard" or "canonical" framing. This is defined by setting the self-linking number to zero. This is a "natural" framing in the sense that it does not depend on particular projections.

The value of the self-linking number in different framings differs by an integer corresponding to the number of twists introduced in the band associated with the loop by the framing procedure. The existence of the natural framing may appear as reassuring since it would seem to restore diffeomorphism invariance to the discussion. Unfortunately, the natural framing only exists in certain manifolds, e.g., S^3 manifolds, since in other cases the linking number may be ill defined or be a non-integer number [45].

The explicit relation between the Kauffman and Jones polynomials is given by

$$K(\gamma)_q = q^{\frac{3}{4}w(\gamma)} J(\gamma)_q, \tag{10.15}$$

and we will offer a proof of this in the next section. It is remarkable that all the framing dependence of the Kauffman bracket is concentrated in the prefactor involving the writhe.

10.3.3 Knot polynomials from representations of the braid group

At present a complete classification of the irreducible representations of the braid group is not known. Finding representations for the braid group is a non-trivial matter. We will present here a construction that yields the representation that gives rise to the Jones polynomial. This representation is the simplest one of the family that can be constructed with a method called the R matrix approach [201, 202].

Assume that a two-dimensional linear space V_i is associated with the ith string so that the total linear space associated with the n strings is given by the tensor product $V(n) = V_1 \otimes V_2 \otimes \cdots V_n$. In each space V_i introduce a basis e_i^A, $A = 1, 2$. Each generator is represented by a $2^n \times 2^n$ matrix of the form

$$G_i = q^{1/4}(I \otimes .... \otimes R \otimes ... \otimes I), \tag{10.16}$$

where q is an arbitrary complex number, I is the 2×2 identity matrix and the matrix R, which acts on $V_i \otimes V_{i+1}$, is given by

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-q^{-1} & q^{-1/2} & 0 \\ 0 & q^{-1/2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{10.17}$$

in the basis of $V_i \otimes V_{i+1}$ given by $\{e_i^1 e_{i+1}^1, e_i^1 e_{i+1}^2, e_i^2 e_{i+1}^1, e_i^2 e_{i+1}^2\}$.

It is a straightforward calculation to show that the relations defining the braid group (10.2),(10.3) are satisfied by the matrices G_i and their corresponding inverses. Therefore they define a representation of the braid group on the vector space $V(n)$.

In order to construct knot polynomials starting from a representation of the braid group we need to construct quantities that are invariant under Markov moves. As we discussed in section 10.3.1 by taking traces of the representation one constructs invariants under Reidemeister moves of types (ii) and (iii), i.e., regular isotopic invariants. In order to implement invariance under type (i) moves we will introduce a matrix in $V(n)$ called "enhancement matrix". This is defined by

$$\mu^n = \mu_1 \otimes \ldots \otimes \mu_n, \tag{10.18}$$

where

$$\mu_i = \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}. \tag{10.19}$$

The enhancement matrix has two main properties. First, it commutes with all the generators of the braid group G_i. To introduce the second property we recall that in a tensor product of spaces one can introduce a partial trace operation on one of the factor spaces. For instance, if one considers the trace in V_{i+1} of a tensor product $V_1 \otimes \ldots \otimes V_{i+1}$ one gets as a result an element of $V_1 \otimes \ldots \otimes V_i$. Taking this into account one can check that for the enhancement matrix

$$\mathrm{Tr}|_{V_{i+1}}(R\,\mu_{i+1}) = q^{1/2}\mathbf{1}|_{V_i}, \tag{10.20}$$

where the product $R\mu_{i+1}$ is defined in the space $V_i \otimes V_{i+1}$ as R times $\mathbf{1}|_{V_i} \otimes \mu_{i+1}$. A similar result holds for the inverse,

$$\mathrm{Tr}|_{V_{i+1}}(R^{-1}\mu_{i+1}) = q^{-1/2}\mathbf{1}|_{V_i}\,. \tag{10.21}$$

We can use this property to construct quantities that are invariant under all Reidemeister moves. Consider a matrix B representing an arbitrary element b of the braid group B_n and define the quantity

$$F(B) = q^{-\frac{3}{4}w(\hat{b})}\mathrm{Tr}|_{V(n)}(B\mu^n), \tag{10.22}$$

https://doi.org/10.1017/9781009290203.012 Published online by Cambridge University Press

where $\hat{b}$ is the link obtained as a closure of the element b and $w(\hat{b})$ is its writhe. We will now prove that this quantity is a link invariant under ambient isotopy associated with the closure $\hat{b}$ of the braid b. We prove this by showing that it is invariant under Markov moves. Since μ^n commutes with all the generators of the braid group, it is immediate to show that $F(B_2^{-1}B_1B_2) = F(B_1)$. Moreover, if B_1 and B_2 are the matrices representing the elements $b_1 \in B_n$ and $b_2 = b_1 g_n^{\pm 1} \in B_{n+1}$ in the spaces $V(n)$ and $V(n+1)$ we have

$$\begin{aligned} F(B_2) &= q^{-\frac{3}{4}w(\hat{b}_2)}\mathrm{Tr}|_{V(n+1)}(B_2\mu^{n+1}) \\ &= q^{-\frac{3}{4}w(\hat{b}_1)}q^{\mp\frac{3}{4}}\mathrm{Tr}|_{V(n)\otimes V_{n+1}}(B_1 G_n^{\pm 1}\mu^n \otimes \mu_{n+1}) \\ &= q^{-\frac{3}{4}w(\hat{b}_1)}\mathrm{Tr}|_{V(n)}(B_1\mu^n) \\ &= F(B_1) \end{aligned} \tag{10.23}$$

which can be straightforwardly checked relating the trace operation in $V(n+1)$ with that in $V(n)$. The writhes of b_1 and b_2 differ by a factor ± 1 since $G_n^{\pm 1}$ introduces an additional curl in the loop. The extra power of q this introduces exactly cancels a factor that arises when relating the traces in $V(n+1)$ and $V(n)$.

Therefore F is associated with an ambient isotopic invariant. To see which invariant it is we compute its skein relations. One can check that the matrix G_i satisfies the relation

$$q^{1/4}G_i - q^{-1/4}G_i^{-1} - (q^{1/2} - q^{-1/2})I_i = 0, \tag{10.24}$$

which combined with the definition of the invariant F gives

$$qF(BG_i) - q^{-1}F(BG_i^{-1}) = (q^{1/2} - q^{-1/2})F(B), \tag{10.25}$$

which is the skein relation for the Jones polynomial.

Equation (10.24) yields, multiplying by μ^n and taking traces, the skein relation for the Kauffman bracket polynomial. As a consequence we immediately have that the Kauffman bracket polynomial is a regular isotopy invariant and is related to the Jones polynomial by expression (10.15) which we introduced in the previous section (they only differ by a factor depending on the writhe). This will have important consequences in quantum gravity.

10.3.4 Intersecting knots

Up to now we have studied the construction of knot polynomials based on smooth loops without intersections. As we have argued before, in the case of gravity we need to consider knots with intersections, because the Mandelstam identities naturally introduce them and because they are associated with non-degenerate metrics. There is no fundamental

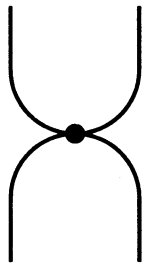

Fig. 10.8. The additional element needed in the braid group to generate invariants of links with double intersections.

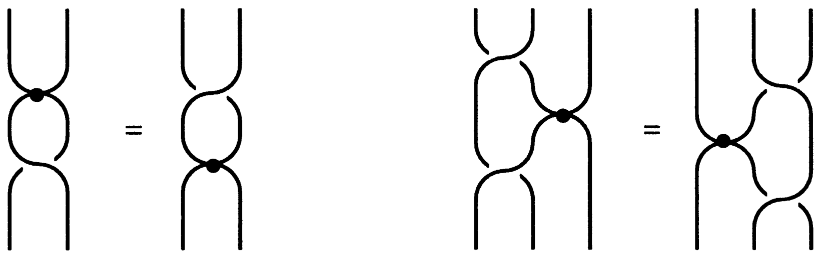

Fig. 10.9. The relations satisfied by the intersecting element of the braid group.

difficulty in adding intersections to the constructions of the braid group and the Jones polynomial we introduced.

The main idea is to extend the braid group with the introduction of an additional element that represents the crossing of two strands. The resulting structure is not a group but an algebra. If one wants to consider intersections of more than two lines additional elements need to be added. Though technically more complicated, the generalization is straightforward [189, 183]. The additional element needed to include double intersections is denoted a_i and we depict it in figure 10.8. It satisfies the relations

$$a_i g_i = g_i a_i, \tag{10.26}$$
$$g_i^{-1}\, a_{i+1}\, g_i = g_{i+1}\, a_i\, g_{i+1}^{-1}, \tag{10.27}$$

and

$$[g_i, a_j] = 0 \qquad [a_i, a_j] = 0 \qquad |i-j| > 1, \tag{10.28}$$

and the graphical representation of equations (10.26),(10.27) is given in figure 10.9.

The element a_i has no inverse (one cannot remove intersections) and that is the reason why the resulting structure of extending the braid group to intersections is not a group but an algebra.

A matrix representation including the intersecting elements is given by the $2^n \times 2^n$ matrices [183],

$$A_i = I \otimes \ldots \otimes A \otimes \ldots \otimes I, \tag{10.29}$$

where A is given by the matrix acting on $V_i \otimes V_{i+1}$:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & (1-a)q^{1/2} & 0 \\ 0 & (1-a)q^{1/2} & 1-(1-a)q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{10.30}$$

where a is another complex parameter. We see that the generalization of a polynomial to (double) intersections requires the introduction of a new variable in the polynomial. The skein relations for the new matrix are

$$A_i = q^{1/4}(1-a)G_i^{-1} + aI_i, \tag{10.31}$$

and we see that the extension of a polynomial to intersecting loops preserves the usual skein relations for the polynomial but requires additional skein relations that involve intersections. For the Kauffman bracket polynomial the additional skein relation derived from the above expression is

$$K(L_I)_{q,a} = q^{1/4}(1-a)K(L_-)_q + aK(L_0)_q\,. \tag{10.32}$$

For triple intersections a generalization of the braid group can also be given in terms of an algebra. There are three new added elements corresponding to triple intersections since different (unrelated by diffeomorphisms) spatial orientations of the incoming strands are possible. A generalization of the HOMFLY polynomial to this case was given by Armand-Ugon, Gambini and Mora [183] and it coincides with the construction we gave for the doubly intersecting case. The generalized polynomials depend on a number of extra variables due to the presence of intersections.

It should be emphasized that there exist *many* non-equivalent extensions of a given polynomial to intersecting knots. General expressions taking into account this fact are present in reference [183]. The extension that we presented above is a particular one, corresponding to the use of R matrix techniques. It is remarkable that this particular extension turns out to be connected with the knot polynomials that appear in topological field theories, as we will discuss in the next section.

10.4 Topological field theories and knots

The previous derivations concerning the Artin braid group and the knot polynomials, as attractive as they may appear in their own right, seem to have little connection with the rest of this book. Throughout this book we have always considered functions of curves defined through explicit analytic expressions. In this chapter dependences on loops have up to this point been implicit and the resulting formulation seems ill suited to be mixed with the loop calculus developed in chapter 1. The missing link is provided in this section.

Topological field theories are field theories that do not require the introduction of a background structure (in particular, no background metric) for their definition. They are therefore naturally diffeomorphism invariant. If one formulates these theories in terms of loops, the resulting quantities should be knot invariants. This may not appear to be a great surprise or advantage: after all the wavefunctions of quantum gravity are diffeomorphism invariant as well. There is, however, an important difference in the case of topological field theories. Most of these theories have only a finite number of topological degrees of freedom and as a consequence are exactly solvable. As a result they provide concrete computable expressions that are invariants of knots.

This was precisely the insight of Witten [45] who noticed that computing expectation values of loop dependent quantities in Chern–Simons and other topological theories one could come up with explicit, analytic, expressions for knot invariants. In the following section we will exploit these results to construct explicit quantum states of the gravitational field. Here we discuss the connection between Chern–Simons theory and the Jones polynomial in some detail.

10.4.1 Chern–Simons theory and the skein relations of the Jones polynomial

A Chern–Simons theory is a gauge theory in $2+1$ dimensions where the action is given by the Chern–Simons form of a connection,

$$S_{CS} = \frac{k}{4\pi}\int d^3x\, \tilde{\epsilon}^{abc}\mathrm{Tr}(\mathbf{A}_a\partial_b\mathbf{A}_c + \tfrac{2i}{3}\mathbf{A}_a\mathbf{A}_b\mathbf{A}_c), \tag{10.33}$$

where k is the coupling constant of the theory.

In contrast to the usual Yang–Mills action, the Chern–Simons action does not require the introduction of a metric or any other background structure for its definition. The Chern–Simons action is invariant under diffeomorphisms and (small) gauge transformations [59]. It is not invariant under large gauge transformations (not connected with the identity). Moreover, the integral is crucial in providing the gauge invariance: the integrand itself is not invariant. The classical equations of motion of this action require that the connection be flat and the theory be gauge invariant. The Chern–Simons action can be written for an arbitrary compact simple gauge group; however, we will restrict our attention to the $SU(2)$

case.

We can now proceed and perform a computation similar to the one we did for Yang–Mills theory in chapter 5, where we computed the value of the Wilson loop average. We recall that the Wilson loop average was identified as the generating functional of the Green functions of the theory. The main difference is that in the present case we will be able to perform the computation explicitly. Because the system does not depend on any external structure the result for the Wilson loop average will be a topological invariant. Notice that we are talking about a Euclidean formulation and the loops will exist in three dimensions. The expectation value of a Wilson loop is given by

$$< W(\gamma) >= \int DA \exp{(iS_{CS})} \, W_A(\gamma). \tag{10.34}$$

This quantity is a knot invariant since S_{CS} is invariant under diffeomorphisms and we assume the measure DA has been chosen to be invariant as well. Which knot invariant is it? We will show it satisfies the skein relations of the Kauffman bracket polynomial. The proof goes along the same lines as in the Makeenko–Migdal formulation of gauge theories that we introduced in chapter 5.

In order to check the skein relations satisfied by the expectation value of the Wilson loop we will consider its change under the addition of an infinitesimal loop. If one considers a straight strand $\hat{L}_0$ in the notation of the previous section and one adds a loop one obtains a crossing $\hat{L}_\pm$, the plus or minus sign being determined by the orientation of the loop added. Similar considerations apply to the other types of crossings; upper and under crossings are related through the addition of a loop to an intersection. We are well equipped to study the change of expressions that are functions of loops under the addition of an infinitesimal loop, so we will do the calculation in this limit. If one wants to consider the addition of a finite loop, a resummation of all orders of perturbation can be formally done, as is discussed in reference [184].

The change of the expectation value of a loop under the addition of a small loop can be computed simply by evaluating the loop derivative. A derivation along these lines was first introduced (for the non-intersecting case) by Cotta-Ramusino, Guadagnini, Martellini and Mintchev [185]. For the intersecting case it was generalized in reference [137]. Smolin [186] introduced a slightly different perturbative derivation. The first proof of the skein relation was introduced by Witten [45] using rational conformal field theory techniques.

We now consider the variation of the expectation value of a Wilson loop when a small loop of area σ^{ab} is appended to the loop γ. Let us first

https://doi.org/10.1017/9781009290203.012 Published online by Cambridge University Press

consider the case without intersections. We get

$$\sigma^{ab}\Delta_{ab}(x) < W(\gamma) >= i \int dA \quad \sigma^{ab}F^k_{ab}(x)\mathrm{Tr}(\tau^k U(\gamma^x_x))\exp(iS_{CS}), \tag{10.35}$$

where Δ_{ab} is the loop derivative and we have used

$$\Delta_{ab}(x)\mathrm{Tr}(U(\gamma)) = iF^k_{ab}(x)\mathrm{Tr}(\tau^k U(\gamma^x_x)), \tag{10.36}$$

in which γ^x_x is a loop with origin at the point x.

The exponential of the Chern–Simons action has the property that the quantum electric field acting on it is equal to the magnetic field,

$$\hat{\tilde{E}}^a_k \exp(iS_{CS}) = -i\frac{\delta}{\delta A^k_a}\exp(iS_{CS}) = \frac{k}{4\pi}\hat{B}^a_k \exp(iS_{CS}). \tag{10.37}$$

Using this relation and integrating by parts, one obtains

$$-\frac{4i\pi}{k}\int dA\,\sigma^{ab}\underset{\sim}{\epsilon}_{abc}\int dy^c\delta(x-y)\mathrm{Tr}(\tau^k U(\gamma^y_x)\tau^k U(\gamma^x_y))\exp(iS_{CS}). \tag{10.38}$$

The integral is proportional to the volume factor

$$\sigma^{ab}\epsilon_{abc}dy^c\delta(x-y), \tag{10.39}$$

which, depending on the relative orientation of the two-surface Σ^{ab} and the differential dy^c (which is tangent to γ), can lead to ±1 or zero. (This expression is only formal, a regularization is needed. We have absorbed appropriate divergent factors in the definition of the coupling constant in order to normalize the volume to ±1, see reference [184] for details.) Consequently, depending on the value of the volume, there are three possibilities

$$\delta < W(\gamma) > = 0, \tag{10.40}$$

$$\delta < W(\gamma) > = \mp\frac{3\pi i}{k} < W(\gamma) > . \tag{10.41}$$

These equations can be interpreted diagrammatically in the following way,

$$< W(\hat{L}_\pm) > - < W(\hat{L}_0) >= \mp\frac{3\pi i}{k} < W(\hat{L}_0) >, \tag{10.42}$$

and when the volume element vanishes it corresponds to a variation that does not change the topology of the crossing.

We therefore see that, to first order in the area of the added loop, the expectation value of a loop in Chern–Simons theory satisfies one of the skein relations of the Kauffman bracket polynomial. This is a quite non-trivial result that is the root of the renewed interest in knot theory in the past decade.

What about intersections? We introduced in the previous section skein relations for knot polynomials with intersections. Is Chern–Simons theory

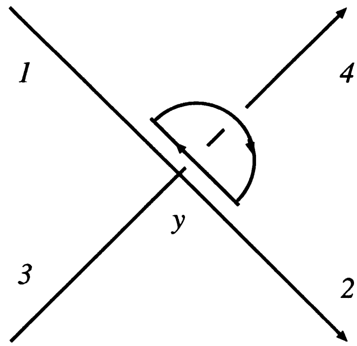

Fig. 10.10. The addition of a small loop at an intersection in the derivation of the skein relation

associated with knot invariants for intersections as well? The answer is yes. It is quite remarkable that of the many possible extensions of knot invariants to intersecting loops, the one that is most naturally picked by Chern–Simons theory coincides with the one we introduced in the previous section.

In order to derive the skein relation for intersections we consider as before an infinitesimal deformation of the loop consisting of the addition of a small closed loop, in this case at the point of intersection (see figure 10.10),

$$\sigma^{ab}\Delta_{ab}(y) < W(\gamma) >= \frac{4\pi}{k}\int dA\sigma^{ab}\epsilon_{dab}\mathrm{Tr}(\tau^k U_{23}(\gamma_y^y)U_{41}(\gamma_y^y)) \\ \times\frac{\delta}{\delta A_d^k(y)}\exp(iS_{CS}). \qquad (10.43)$$

Again, integrating by parts and choosing the element of area σ^{ab} parallel to the segment 1–2 so that the contribution of the functional derivative corresponding to the action on the segment 1–2 vanishes (since the volume element is zero) we get

$$\sigma^{ab}\Delta_{ab} < W(\gamma) > = -\frac{4i\pi}{k}\int dA\sigma^{ab}\epsilon_{abc}\int dv^c\delta(y-v) \\ \times \mathrm{Tr}(\tau^k U_{23}(\gamma_y^y)\tau^k U_{41}(\gamma_y^y))\exp(iS_{CS}). \quad (10.44)$$

Making use of the Fierz identity for the usual $SU(2)$ matrices (the convention for τ differs by a factor $i/\sqrt{2}$ from the ones considered in

https://doi.org/10.1017/9781009290203.012 Published online by Cambridge University Press

chapter 8),

$$\tau^{kA}{}_B\tau^{kC}{}_D = \frac{1}{2}\delta^A_D\delta^C_B - \frac{1}{4}\delta^A_B\delta^C_D, \tag{10.45}$$

one finally gets

$$\begin{aligned}
&\sigma^{ab}\Delta_{ab} < W(\gamma) >= \\
&-\frac{2i\pi}{k}\int dA\ \sigma^{ab}\epsilon_{abc}\int dv^c\delta(y-v)\mathrm{Tr}(U_{23}(\gamma^y_y))\mathrm{Tr}(U_{41}(\gamma^y_y))\exp(iS_{CS}) \\
&+\frac{i\pi}{k}\int dA\ \sigma^{ab}\epsilon_{abc}\int dv^c\delta(y-v)\mathrm{Tr}(U_{23}(\gamma^y_y)U_{41}(\gamma^y_y))\exp(iS_{CS}),
\end{aligned} \tag{10.46}$$

where we have called $U_{ij}(\gamma_{x_1}^{x_2})$ the holonomy from point x_1 to x_2 traversing through lines i and j.

These relations can be interpreted as the following skein relation for the intersection:

$$< W(L_\pm) > = \left(1 \pm \frac{i\pi}{k}\right) < W(L_I) > \mp \frac{2i\pi}{k} < W(L_0) >, \tag{10.47}$$

$$< W(\hat{L}_\pm) > = \left(1 \mp \frac{3i\pi}{k}\right) < W(\hat{L}_0) > . \tag{10.48}$$

In order to make a comparison with the link polynomials we must first notice that the results we have obtained correspond to a linear approximation, since we have only considered an infinitesimal deformation of the link. In order to consider a finite deformation we would have to consider higher order derivatives of the wavefunction.

It is convenient to rewrite the relations obtained in such a way that the correspondence with those of the Kauffman bracket polynomial in the intersecting case is manifest. To do this we notice that the factor $(1-3\pi i/k)$ plays the role of $q^{3/4}$ in the usual skein relation and therefore in the linearized case if we define q as $q = \exp(-4\pi i/k)$. Inverting relation (10.47) we get

$$< W(L_I) >= \left(1 \mp \frac{i\pi}{k}\right) < W(L_\pm) > \pm \frac{2i\pi}{k} < W(L_0) >, \tag{10.49}$$

which allows us to recognize that the value of the variable a of the generalized Kauffman bracket polynomial is up to first order $a = -2\pi i/k$.

The expression relating $< W(L_+) >$ and $< W(L_-) >$ can be obtained in this case by combining equations (10.49). Again we emphasize that the above proofs are only to first order in the area of the loop; in order to prove the skein relations for the addition of a finite loop one can formally sum the perturbative series and confirm for the finite case the result we found infinitesimally. A detailed discussion of this is presented in the paper by Brügmann [184].

So we see that the generalized Kauffman bracket, introduced in the last section for loops with double self-intersections from the R matrix representation of the braid group, is actually the loop transforms of a physical non-degenerate quantum state of the gravitational field defined by values of q and a that to first order in perturbation theory coincide with the ones presented above.

It should be noticed that in order to recover exactly the expression for the polynomials introduced in the previous section we should normalize our results in such a way as to ensure that the value of the polynomials for the unknot is equal to one. This can easily be accomplished by dividing the above expressions by $< W(\text{unknot}) >$. This does not affect the skein relations and ensures the normalization condition.

At this point the reader may be confused. Our promise was to produce via Chern–Simons theory explicit expressions for knot invariants. As a result of our construction we almost obtained this objective, except for the fact that the resulting polynomial is not a genuine knot invariant, but rather a regular knot invariant. Why is the resulting expression not invariant under Reidemeister moves of type (i)?

The difficulty already arises if one considers the expectation value of a Wilson loop in the case of a $U(1)$ Chern–Simons theory. In that case the integral is a Gaussian and the result is the exponential of the self-linking number. The self-linking number is a quantity that involves a $0/0$ indeterminacy, which can be removed by considering a limit. The problem is that the limit is metric dependent. A way to view this is that the limit is a (metric dependent) regularization procedure and the result of it is not metric independent. Another way of viewing it is to consider a point-splitting regularization of the loop. In that case the final result is metric independent (it is the linking number of the split components of the loop) but depends on the particular way the loop is split.

Another difficulty is added in the non-Abelian case. Since the Chern–Simons form is not invariant under large gauge transformations and the Wilson loop is, the resulting integral is not expected to be invariant under large gauge transformations. Therefore, strictly speaking it cannot be a function only of a loop. How this problem relates to the framing ambiguity is not clear. However, it should be stressed that this problem does not arise in the Abelian case (in which all the transformations are small) but the framing ambiguity still persists. The fact that the non-Abelian Chern–Simons form is not invariant under large gauge transformations poses difficulties to doing computation in the non-Abelian case using the rigorous integration techniques of Ashtekar and collaborators [203].

The framing ambiguity issue completely disappears in the extended loop representation, since the extended holonomy is not invariant under large gauge transformations. This issue lies at the crux of the problem of how much is it needed to extend the group of loops to account for these kinds of issues. Is the extension to framed loops enough, as the Chern–Simons integral seems to suggest or does one really need to consider the full extended group of loops? These issues are at present not settled.

10.4.2 Perturbative calculation and explicit expressions for the coefficients

The original intention in connecting knot theory and topological field theories was that in this way one would obtain explicit expressions for knot invariants. Through the calculations of the last section we now know that there is an explicit connection between the expectation value of the Wilson loop in a Chern–Simons theory and the Kauffman bracket. Because Chern–Simons theories are perturbatively renormalizable, one can compute an explicit expression for the expectation value of the Wilson loop in terms of Feynman diagrams. Such an expression we know is equal to the Kauffman bracket. This equality will allow us to give explicit expressions for each of the coefficients of the Kauffman bracket.

We therefore consider the expression of the expectation value of the Wilson loop in a Chern–Simons theory,

$$< W(\gamma) >= \int DA \exp(iS_{CS}) W_A(\gamma), \tag{10.50}$$

and expand it in powers of the coupling constant k. In order to do this, we write the Wilson loop explicitly,

$$W_A(\gamma) = \sum_{i=0}^{\infty} X^{a_1\,x_1\,\ldots a_i\,x_i}(\gamma)\mathrm{Tr}(A_{a_1\,x_1} \cdots A_{a_i\,x_i}), \tag{10.51}$$

and get as the result,

$$< W(\gamma) >= \sum_{i=0}^{\infty} X^{a_1\,x_1\,\ldots a_i\,x_i}(\gamma) < \mathrm{Tr}(A_{a_1\,x_1} \cdots A_{a_i\,x_i}) > . \tag{10.52}$$

Therefore by evaluating the n-point functions $< \mathrm{Tr}(A_{a_1\,x_1} \cdots A_{a_i\,x_i}) >$ perturbatively we can get the expression we were seeking. In order to perform the perturbative expansion one needs to introduce a background metric in order to fix the gauge‡.

The expression for the propagator is finally given by [187]

$$< A_a^i(x) A_b^j(y) >= \frac{i}{k}\delta^{ij}\epsilon_{abc}\frac{(x-y)^c}{|x-y|^3} + O(1/k^4), \tag{10.53}$$

‡ It can be seen that the background metric enters into the gauge fixed action as a commutator of an arbitrary gauge fixing function with the BRST charge and therefore drops out from expressions involving physical states since the BRST charge annihilates such states.

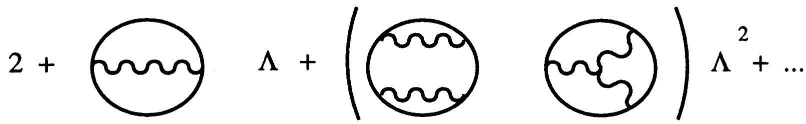

Fig. 10.11. The diagrammatic expansion of the expectation value of the Wilson loop. The circles with insertions correspond to the multitangents of order equal to the number of insertions. The wavy lines are the Chern–Simons propagators, which may be joined in triple vertices. The constant $\Lambda = 3\pi i/k$ is related in the gravitational case to the cosmological constant

where $O(1/k^4)$ may be vanishing but has not been carefully studied. We will not need explicit expressions at that order for our calculations. From it we define the quantity

$$g_{ax\ by} \equiv \frac{ik}{12\pi} < A_a^i(x) A_b^i(y) >, \tag{10.54}$$

which we have already encountered in chapter 2 as the coordinate expression of the naturally defined metric in the space of transverse vector densities.

The vertex for the theory is given by

$$\frac{ik}{4\pi}\epsilon^{abc}\epsilon_{ijk}, \tag{10.55}$$

which contracted with three propagators gives rise to the quantity,

$$\left(\frac{4\pi}{ik}\right)^2 h_{ax\ by\ cz} = \int d^3w g_{ax\ dw} g_{by\ ew} g_{cz\ fw} \epsilon^{def} + O(1/k^2). \tag{10.56}$$

We can now proceed to write perturbatively an expansion for the polynomial (shown diagrammatically in figure 10.11),

$$\Psi_k(\gamma) = a_0(\gamma) + a_1(\gamma)\frac{3\pi i}{k} - a_2(\gamma)\frac{9\pi^2}{k^2} - a_3(\gamma)\frac{27\pi^3 i}{k^3} + O(1/k^4), \tag{10.57}$$

where

$$a_0(\gamma) = 2, \tag{10.58}$$
$$a_1(\gamma) = X^{ax\ by} g_{ax\ by}, \tag{10.59}$$
$$a_2(\gamma) = \tfrac{1}{2}a_1(\gamma)^2 - \tfrac{2}{3}\mathcal{A}_2(\gamma), \tag{10.60}$$
$$a_3(\gamma) = \tfrac{1}{6}a_1(\gamma)^3 + \tfrac{2}{3}a_1(\gamma)\mathcal{A}_2(\gamma) + \tfrac{8}{9}\mathcal{A}_3(\gamma), \tag{10.61}$$

and

$$\mathcal{A}_2(\gamma) = h_{ax\ by\ cz} X^{ax\ by\ cz} + g_{ax\ cz} g_{by\ dw} X^{ax\ by\ cz\ dw}, \tag{10.62}$$

https://doi.org/10.1017/9781009290203.012 Published online by Cambridge University Press

$$\begin{aligned}\mathcal{A}_3(\gamma) = -2 \Big[& (h_{\mu_1\mu_2\alpha} g^{\alpha\beta} h_{\mu_3\mu_4\beta} - h_{\mu_1\mu_4\alpha} g^{\alpha\beta} h_{\mu_2\mu_3\beta})\, X^{\mu_1\mu_2\mu_3\mu_4} \\ & + g_{(\mu_1\mu_3} h_{\mu_2\mu_4\mu_5)_c}\, X^{\mu_1\mu_2\mu_3\mu_4\mu_5} \\ & + (2 g_{\mu_1\mu_4} g_{\mu_2\mu_5} g_{\mu_3\mu_6} + \tfrac{1}{2} g_{(\mu_1\mu_3} g_{\mu_2\mu_5} g_{\mu_4\mu_6)_c})\, X^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6} \Big],\end{aligned} \tag{10.63}$$

where as usual greek indices correspond to a pair of spatial index and a point in the manifold. Actually, if γ were a multiloop, $a_0(\gamma)$ would be two raised to the number of connected components of the loop. $a_1(\gamma)$ is the self-linking number of the loop that we have already discussed. $\mathcal{A}_2(\gamma)$ is an ambient isotopic invariant associated with the second coefficient of the Alexander–Conway knot polynomial (the precise expression is given by $\frac{1}{2}(\mathcal{A}_2 + \frac{1}{12})$) and is also related to the classical Arf and Casson knot invariants. This explicit expression was first obtained by Guadagnini, Martellini and Mintchev [205]. The third contribution has been obtained by Di Bartolo and Griego [47]. Central to finding the explicit form of the third order contribution has been the clear identification of the relations satisfied by the loop multitangents (algebraic constraints) which we discussed in chapter 2.

One could continue giving explicit expressions for higher order coefficients. However, one would need refined expressions for the propagators which consider the higher order contributions of ghosts in the diagrammatic expansion.

To summarize, we see that the use of the diagrammatic expansions allows us to construct explicit analytic expressions for the coefficients of the knot polynomials. These expressions provide the completion of the ideas we introduced in chapter 2 in which we suggested that the use of the loop coordinates was good for discussing knot invariants. At that point we were not able to construct the invariants explicitly due to the lack of a natural metric in the space of multitangents (the only natural structure was the kernel used to construct the linking number). We see that through the use of Chern–Simons theory we can construct quantities that contracted with the multitangents yield the knot invariants that we were intending to construct. We will see in the next section how to make use of these invariants to construct physical states of quantum gravity.

Let us end this section with a discussion of framing in the context of the perturbative expansions. In the previous section we showed that the expectation value of the Wilson loop gave rise to the Kauffman bracket. We also saw that the Kauffman bracket was related to the Jones polynomial through a framing dependent prefactor that condensed all the framing dependence of the Kauffman bracket. The prefactor was equal to the exponential of the writhe. Recall that in the vertical framing the writhe coincides with the self-linking number. In the perturbative context, we see that the self-linking number arises in all the coefficients of the expansion of the Kauffman bracket. From the few coefficients we have computed we can get a glimpse of how the different contributions precisely combine to give the prefactor we found in the previous subsection. Explicitly, if one writes

$$\begin{aligned}K(\gamma)_q &= q^{3/4a_1(\gamma)} J(\gamma)_q \\ &= (1 + \frac{3\pi i}{k} a_1(\gamma) - \frac{1}{2}\frac{9\pi^2}{k^2} a_1(\gamma)^2 - \frac{1}{6}\frac{27\pi^3}{k^3} a_1(\gamma)^3 + \ldots) \\ &\times (1 + J_2(\gamma)(\frac{4\pi i}{k})^2 + J_3(\gamma)(\frac{4\pi i}{k})^3 + \ldots), \qquad (10.64)\end{aligned}$$

where we have expanded the exponential of the self-linking number in powers of k and we have introduced an infinite expansion of the Jones polynomial (this corresponds to considering $q = \exp(4\pi i/k)$ as the variable in the polynomial and writing it as a Laurent expansion in powers of k). We have used the fact that the first coefficient of the Jones polynomial vanishes [177]. From this expression, and comparing with the explicit expansions we introduced before, we see that $J_2(\gamma)$, the second coefficient of the infinite expansion of the Jones polynomial, is proportional to the $\mathcal{A}_2(\gamma)$ invariant we introduced before. We also see that the presence of the terms involving the self-linking number in all the coefficients of the expansion just corresponds to the expansion of the prefactor introduced in the last subsection.

Notice that we get an expression for the coefficients of the polynomial in a particular framing (vertical). This is quite reasonable, the polynomials are defined in a framing independent manner by the skein relations but if one wants a concrete analytic expression for their coefficients one has to give it in a definite framing. The particular framing that appears is determined by the details of the regularization procedure (recall that when we computed the skein relations for the expectation value of the Wilson loop we absorbed divergent factors; the correspondence between that regularization and the one chosen for the perturbative expansion determines the particular framing).

It is not obvious to see explicitly from the expressions we introduced for $\mathcal{A}_2(\gamma)$ that it is an ambient isotopic quantity, as it should be if it is to represent the second coefficient of the Jones polynomial. The issue has been discussed (for the non-intersecting case) by Guadagnini, Martellini and Mintchev [205] and they reach the conclusion that the second coefficient is framing independent. Similar reasonings apply to the third coefficient, though the issue has not been studied in detail.

Do these analytic expressions apply for intersecting loops? Almost all of the expressions are ill defined if the loop has intersections. In order for them to be valid one has to add a prescription (for instance, a point-splitting regularization) at the intersections. The analytic expressions coincide with the coefficients of the extension of the polynomials to the intersecting case which we introduced through the extension of the braid group for some particular prescription for regularization at the intersections. This has only been analyzed for some simple cases and the issue deserves further study.

10.5 States of quantum gravity in terms of knot polynomials

We are now prepared to apply the notions of knot theory derived in the previous sections to the construction of quantum states of the gravitational field.

10.5.1 The Kauffman bracket as a solution of the constraints with cosmological constant

As we noticed in chapter 8, in the factor ordering in which triads appear to the left there exists a solution to all the constraints of quantum gravity with a cosmological constant given by the exponential of the Chern–Simons form of the Ashtekar connection,

$$\Psi_{CS}[A] = \exp\left(-\frac{6}{\Lambda}\int d^3x\tilde{\epsilon}^{abc}\mathrm{Tr}(A_a\partial_b A_c + \tfrac{2}{3}A_aA_bA_c)\right). \tag{10.65}$$

If one considers the loop transform of such a state one gets,

$$\Psi_{CS}(\gamma) = \int DAW_A(\gamma)\Psi_{CS}[A] = \int DAW_A(\gamma)\exp\left(-\frac{6}{\Lambda}S_{CS}[A]\right), \tag{10.66}$$

where with the conventions for the gravitational case

$$S_{CS} = \int d^3x\tilde{\epsilon}^{abc}\mathrm{Tr}(A_a\partial_b A_c + \tfrac{2}{3}A_aA_bA_c). \tag{10.67}$$

But this expression is precisely the same as the one we encountered when computing $< W(\gamma) >$ in the context of a Chern–Simons theory. The cosmological constant plays the role of the coupling constant k of the theory. We therefore know what the result is, it is given by the Kauffman bracket knot polynomial in the variable Λ. Therefore the implication is that the Kauffman bracket solves in the loop representation all the constraints of quantum gravity with a cosmological constant.

This suggestion appears as very striking and beautiful, since it allows us instantly to apply in quantum gravity elaborate results from Chern–Simons theory. Before becoming too enthusiastic about this result, we should point out several things that make the proof of the above statement far from solid. First of all, recall that in the Ashtekar formulation

https://doi.org/10.1017/9781009290203.012 Published online by Cambridge University Press

of quantum gravity the variables involved are complex. In Chern–Simons theory the connection is real. Therefore the analogy of the expressions presented is only formal. For instance, the expression of the propagator of the theory diverges if the connection is complex. There is no result supporting the existence of the path integral defining the expectation value of the Wilson loop if the connection is complex. The only expectation one can have is that whenever there is a well defined understanding of the complex loop transform, the final calculation will reduce to an analytic continuation of the real result of Chern–Simons theory. If that were the case, we would be justified in using the analogy. Another problem is that the state produced in the loop representation is not a genuine diffeomorphism invariant state, since a framing is required for its definition. At present, due to these difficulties, the results we present can only be taken as purely heuristic in terms of loops. The present attitude towards these problems is that loops may be insufficient to characterize all possible states in the quantum theory. The presence of a framing suggests that a formulation in terms of ribbons or thickened loops could be better suited to the treatment of these issues. At present, however, the only explored context in which they can be given some level of consistency is in terms of extended loops, where all quantities are regularized and the framing ambiguities disappear. We will devote the next chapter to the study of the extended representation and we will find that all the heuristic results that we introduced in this chapter will be mirrored — in a regularized context — in terms of extended loops.

Why should one pursue this avenue at all? Why not simply admit that the transform of the Chern–Simons state is ill defined and forget it as a means of constructing states in quantum gravity? The answer will be given by the next sections. We will see that in spite of the difficulties of putting these results in a rigorous setting a quite non-trivial number of consistent results can be achieved. In particular we will see that the action of the constraints we found in the loop representation on the transform of the Chern–Simons state yield a series of remarkable results that confirm that there is a certain amount of truth behind the formal manipulations we perform.

10.5.2 The Jones polynomial and a state with $\Lambda = 0$

One may have an unsatisfactory feeling about the result introduced in the last section. After all it depended on an arguably vague analogy of the loop transform of the Chern–Simons state and the expectation value of the Wilson loop in a Chern–Simons theory. However, given the developments of chapter 8 we are in a good position to check that the Kauffman bracket is a state of quantum gravity directly in the loop representation.

We have explicit expressions of the polynomial and of the constraints in terms of loops and it is a matter of applying the constraints and checking that the result holds. This will not be a rigorous proof either since the expression for the constraints in the loop representation was obtained through formal manipulations of either the loop transform or the elements of the T algebra. It is, however, quite reassuring that all these formal manipulations yield the same results. Moreover, we will find a remarkable surprise while doing this computational check: we will discover that some of the coefficients of the Jones polynomial must be annihilated by the Hamiltonian constraint of general relativity with $\Lambda = 0$.

The calculation will proceed order by order in the cosmological constant,

$$\hat{H}_\Lambda K(\gamma)_\Lambda = (\hat{H}_0 + \Lambda \hat{\det q}) K(\gamma)_\Lambda. \tag{10.68}$$

The above expression is a polynomial in Λ. If it is to vanish, it has to do so order by order in Λ. To compute the different orders we substitute the expansion for the Kauffman bracket of the previous section. The result is

Order Λ^0:

$$\hat{H}_0 \, 2(\gamma) = 0, \tag{10.69}$$

Order Λ^1:

$$\tfrac{1}{4}\hat{H}_0 \, a_1(\gamma) + \hat{\det q} \, 2(\gamma) = 0, \tag{10.70}$$

Order Λ^2:

$$\hat{H}_0 \, (\tfrac{1}{8} a_1(\gamma)^2 - \tfrac{1}{6}\mathcal{A}_2(\gamma)) + \hat{\det q} \, a_1(\gamma) = 0 \tag{10.71}$$

and so on for higher orders. To obtain these formulae in the conventions we are using for gravity one should replace $ik/4\pi$ by $-6/\Lambda$ in the expressions derived in section 10.4.2.

Notice that we have written $2(\gamma)$ for the number 2 that appears as leading order of the perturbative expansion of the Wilson loop. This is to emphasize that this constant is to be viewed as a constant function in loop space. What we mean by this is that operators like the determinant of the metric, which is a multiplicative operator in loop space will have a non-trivial action on it.

Let us summarize the results we will find. We will mainly prove two things:

(a) One can check by straightforward calculation that the contributions to the three orders in Λ that we listed all vanish.

(b) We will see that in the contribution to order Λ^2, the quantity

$$\hat{H}_0 \, \mathcal{A}_2(\gamma) \tag{10.72}$$

https://doi.org/10.1017/9781009290203.012 Published online by Cambridge University Press

vanishes independently and therefore the second coefficient of the expansion of the Jones polynomial is annihilated by the Wheeler–DeWitt equation for vacuum general relativity with cosmological constant equal to zero.

This last fact is one of the most remarkable results that arise from the loop representation. We find a new non-trivial, non-degenerate state of quantum gravity which we only know in terms of loops. We do not at present know its expression in terms of connections. We will see that its annihilation is the product of a very elaborate cancellation of terms. It may therefore be the manifestation of a very deep relationship between knot theory and the dynamics of quantum gravity of which we are unaware. There was no *a priori* reason to expect this coefficient to be a state and there is no simple explanation of why it is so. We will attempt an explanation in the next section.

Let us now proceed to show these results explicitly. We start with the order Λ^0. In that case we have the action of the Hamiltonian constraint with vanishing cosmological constant on the constant function in loop space $2(\gamma)$. The Hamiltonian constraint trivially annihilates this function since the loop derivative involved in its definition does, due to the fact that it is a constant function. Notice that the determinant of the metric does *not* annihilate this function. We have found the first solution ever of all the constraints of quantum gravity that is only a solution for $\Lambda = 0$ and therefore can be interpreted as associated with a non-degenerate metric. The function is just a constant in loop space. We do not know its form in the connection representation, though we can intuitively picture it as a "delta function" in connection space, requiring the connection to be flat. This would automatically be annihilated by the constraints in the connection representation if one ignores regularization issues.

In order to check that the other orders cancel we need to digress and consider in some detail the action of the constraints introduced in chapter 8. Let us start with the expression of the Hamiltonian constraint of the vacuum theory. As we saw, such an expression acts non-trivially only on the intersections of loops. We have no problem considering intersections in the expressions for the coefficients introduced in the previous section, since we have generalized the polynomials appropriately to the case of intersecting knots. In order to simplify the treatment we will consider the explicit action of the constraints for the case of a triple self-intersecting knot. We saw in chapter 8 that this is the minimal number of intersections one needs in order to produce states of quantum gravity that are not annihilated by the constraints for an arbitrary value of the cosmological constant. This is due to the fact that the definition of the determinant of the metric requires a loop with a triple tangent vector at (at least) one point in order to be non-vanishing.

The expressions for the constraints we introduced in chapter 8 are completely general, we only need to particularize them to the case of interest. As we have argued before it is, in general, incorrect to introduce limitations in the space of loops to consider states with loops with a certain number of intersections. This is not what we are doing here. We are just exhibiting the triple self-intersecting calculation for the sake of clarity but the calculation for an intersection of arbitrary order is done in exactly the same way, only additional terms arise. In particular, we will consider the calculation in the next chapter in terms of the extended representation (which includes all kinds of intersections, even non-isolated ones) and the result is the same.

Let us now consider the expression for the Hamiltonian constraint introduced in chapter 8,

$$H(x)\Psi(\gamma) = 2\oint_\gamma dy^{[b}\oint_\gamma dz^{a]}\delta(x-y)\delta(x-z)\Delta_{ab}(\gamma_o^x)\Psi(\gamma_y^z \circ \gamma_{y\,o}^{\,z}). \quad (10.73)$$

We consider a state that is a function of a loop with a triple self-intersection $\gamma = \gamma_1 \circ \gamma_2 \circ \gamma_3$, where γ_i are the petals forming the loop joined at the intersection point. The above expression particularizes to

$$\begin{aligned}\hat{H}_0(x)\ \ &\Psi(\gamma_1 \circ \gamma_2 \circ \gamma_3) = 2\{2X^{bx}(\gamma_1)X^{ax}(\gamma_2)\Delta_{ab}(\gamma_{3\,o}^{\,x})\Psi(\gamma_1 \circ \bar{\gamma}_3 \circ \bar{\gamma}_2)\\ &+2X^{bx}(\gamma_1)X^{ax}(\gamma_3)\Delta_{ab}(\gamma_{3\,o}^{\,x})\Psi(\gamma_1 \circ \gamma_2 \circ \bar{\gamma}_3)\\ &+2X^{bx}(\gamma_2)X^{ax}(\gamma_3)\Delta_{ab}((\gamma_3 \circ \gamma_1)_o^x)\Psi(\gamma_2 \circ \bar{\gamma}_1 \circ \bar{\gamma}_3)\},\end{aligned} \quad (10.74)$$

where $\bar{\gamma}_i = \gamma_i^{-1}$.

The above particularization is obtained as follows. First notice that the action of the constraint is only non-trivial at the intersection point, which we label x. The point x arises several times when one traverses the loop from beginning to end and there are three different tangent vectors at it (we assume the loop has no kinks at the intersection, i.e., all lines go "straight through", as we discussed in chapter 8). The three non-trivial contributions arise when the loop derivative is contracted with the tangent vectors $1,2$, $1,3$ and $2,3$. Each of these possibilities arises twice but it is easy to see that their contributions are the same as the ones we list here so we account for them by an overall factor of 2. We therefore start traversing the loop with the two integrals that appear in the constraint and compute the non-trivial contributions. The origin of the loop can be taken at an arbitrary point, which we fix at some point of the loop γ_3. The first contribution appears when the integral in y has traversed from the origin to the point x along γ_3 and therefore is at the origin of the loop γ_1, and the integral in z has traversed the first petal of the loop, γ_1, completely and is at the beginning of the loop γ_2. The contribution then has a multitangent corresponding to the origin of γ_1, one corresponding to the origin of γ_2 and the argument of the loop derivative is the portion of the loop γ_3 going from the origin to the intersection point. The second contribution is analogous to the first one but the integral in z has moved to the beginning of the third loop, γ_3. The last contribution has the integral in y moved to the beginning of the loop γ_2. The integral in z can only give a non-trivial contribution when reaching the beginning of γ_3 (we have already counted the possibility that it could be in γ_1). Since the variable in y is now at the beginning of γ_2 (or the end of γ_1) we denote so in the path dependence of the loop derivative. Since we are taking care explicitly of the ordering along the loop of the integrals, we denote the tangent vectors of the loops (and the associated distributions) simply through the first order multitangents evaluated at the corresponding loops.

We did not present in chapter 8 an explicit expression for the determinant of the metric, but it can be computed straightforwardly using the same techniques used for the Hamiltonian. The result is [206]

$$\begin{aligned}\hat{\det} q\Psi(\gamma) = &-4\epsilon_{abc}X^{ax}(\gamma_1)X^{bx}(\gamma_2)X^{cx}(\gamma_3)\\ &\times\left(\Psi(\gamma_1\gamma_3\bar{\gamma}_2)+\Psi(\gamma_2\gamma_1\bar{\gamma}_3)+\Psi(\gamma_2\gamma_3\bar{\gamma}_1)\right). \quad (10.75)\end{aligned}$$

Both the expression for the Hamiltonian and the determinant of the metric are cyclic expressions in terms of the three petals of the loop, in spite of the fact that their immediate appearance is that they are not.

Let us now consider the expression to order Λ^1. First let us concentrate on the action of the determinant of the metric on $1(\gamma)$. As we argued, it is non-vanishing and immediately we can see it is equal to $\epsilon_{abc}\dot{\gamma}_1^a\dot{\gamma}_2^b\dot{\gamma}_3^c\,1(\gamma)$.

To compute the action of the Hamiltonian constraint on $a_1(\gamma)$ we consider the explicit form of the wavefunction, the linking number, for a triple self-intersection. This is given by

$$\begin{aligned}a_1(\gamma_1\circ\gamma_2\circ\gamma_3) &= g_{\mu\nu}X^\mu(\gamma_1\circ\gamma_2\circ\gamma_3)X^\nu(\gamma_1\circ\gamma_2\circ\gamma_3)\\ &= g_{\mu\nu}(X^\mu(\gamma_1)+X^\mu(\gamma_2)+X^\mu(\gamma_3))(X^\nu(\gamma_1)+X^\nu(\gamma_2)+X^\nu(\gamma_3)),\end{aligned} \quad (10.76)$$

and as usual greek indices refer to a pair of spatial index and spatial point $\mu_1 = a_1\,x_1$.

We now recall the techniques that we used in the calculation in chapter 4 of the action of the Hamiltonian of Maxwell theory on the vacuum state. The loop derivative acts on each first order multitangent producing the derivative of a delta function. Explicitly,

$$\Delta_{ab}(\gamma_o^x)X^{cy}(\gamma) = \delta^c_{[b}\partial_{a]}\delta(x-y). \quad (10.77)$$

Care should be exercised when one considers the particularization of this expression for the petals of the loop. For instance, $\Delta_{ab}(\gamma_3{}_o^x)\Psi(\gamma_1)$ is non-vanishing for the loop considered since the deformation introduced by the loop derivative acts at the beginning of the petal γ_1. As a consequence

$\Delta_{ab}(\gamma 3_o^x)\Psi(\gamma_{2,3}) = 0$, and similarly for the other petals.

We can therefore write the action of the loop derivative in the first term of the Hamiltonian,

$$\begin{aligned}\Delta_{ab}(\gamma 3_o^x)\ & a_1(\gamma_1 \circ \bar{\gamma}_3 \circ \bar{\gamma}_2) = \\ & \Delta_{ab}(\gamma 3_o^x)[g_{\mu_1\,\mu_2}(X^{\mu_1}(\gamma_1) - X^{\mu_1}(\gamma_2) - X^{\mu_1}(\gamma_3)) \\ & \times(X^{\mu_2}(\gamma_1) - X^{\mu_2}(\gamma_2) - X^{\mu_2}(\gamma_3))] = \\ & 2\delta^{a_1}_{[b}\partial^x_{a]}\delta(x - x_1)g_{a_1\,x_1\,\mu_2}X^{\mu_2}(\gamma_1 \circ \bar{\gamma}_2 \circ \bar{\gamma}_3),\end{aligned} \tag{10.78}$$

where we have used $X^\mu(\gamma) = -X^\mu(\bar{\gamma})$ (as discussed in chapter 2).

We can now integrate by parts the derivative of the delta function. In order to do this, it is useful to introduce the following relation, which can be directly obtained from the definition of the propagator g:

$$\partial^x_{[a}g_{b]x\,cy} = \delta(x - y)\epsilon_{abc} - g_{ax\,by}\partial^x_c \tag{10.79}$$

which together with the transverse character of the first order multitangents implies

$$\begin{aligned}2\delta^{a_1}_{[b}\partial^x_{a]}\delta(x - x_1)g_{a_1x_1\,a_2x_2}X^{a_2x_2}(\gamma_1 \circ \bar{\gamma}_2 \circ \bar{\gamma}_3) = \\ -2\epsilon_{a_2ba}\delta(x - x_2)X^{a_2x_2}(\gamma_1 \circ \bar{\gamma}_2 \circ \bar{\gamma}_3).\end{aligned} \tag{10.80}$$

Similar contributions are obtained from the other terms in the Hamiltonian, which combined with the multitangents that multiply the loop derivative yield

$$\hat{H}_0(x)a_1(\gamma_1 \circ \gamma_2 \circ \gamma_3) = 24\epsilon_{abc}X^{ax}(\gamma_1)X^{bx}(\gamma_2)X^{cx}(\gamma_3). \tag{10.81}$$

This expression exactly cancels out the contribution from the determinant of the metric on $2(\gamma)$, which implies that the contribution to order Λ^1 vanishes.

We now consider the Λ^2 contribution. The determinant of the metric on the linking number produces a contribution of five first order multitangents contracted with an ϵ_{abc} and a propagator of Chern–Simons theory. If one considers the action of the Hamiltonian on the linking number squared the loop derivative acts on the linking number and produces ϵ_{abc} contracted with three multitangents, as in the contribution of order Λ^1, times a linking number. The two contributions cancel each other and the Λ^2 contribution vanishes if and only if

$$\hat{H}_0\mathcal{A}_2(\gamma) = 0. \tag{10.82}$$

This calculation can be checked explicitly in exactly the same way as the others. The whole calculation is just more tedious since the different reroutings affect $\mathcal{A}_2(\gamma)$ in a less trivial fashion and the loop derivative acts in various points. There also appear loop derivatives of higher order multitangents, which we presented in chapter 2. Many terms are generated by the action of the Hamiltonian, involving multitangents of order three, four and five. In the end they all cancel [209]. We will present an explicit proof of this in the next chapter since in terms of the extended loop coordinates the resulting expressions are more concise.

The remarkable fact is that in order for the expression of order Λ^2 to vanish we see that $\mathcal{A}_2(\gamma)$, which was the second coefficient of the infinite expansion of the Jones polynomial, has to be annihilated by the Hamiltonian constraint with vanishing cosmological constant. It can easily be seen that it is not annihilated by the determinant of the metric and therefore is the second solution we find to all the constraints of quantum gravity that is non-degenerate in the sense that we discussed in chapter 8. It is the first non-trivial one, in the sense that the previous one we found was just a constant. It is quite remarkable that this highly non-trivial expression is annihilated by the Wheeler–DeWitt equation in loop space.

If one continues this analysis to higher order one checks that at third order the contribution also vanishes, but the "miracle" that happens at the second order is not repeated: the different contributions cancel among themselves but one cannot identify any portion that is annihilated alone by the vacuum Hamiltonian constraint. The reason why something "special" happens at order two will be discussed in the next section. It is possible that it repeats at higher orders, but this has not yet been checked. An important point to be stressed is that any candidate to solution of the Hamiltonian constraint should also be compatible with the Mandelstam identities. This happens to second order, it does not happen at third and is yet to be investigated at higher orders.

We will see in the section 10.5.3 why the second order coefficient seems to play a special role and we will see that it is related to the role that the Gauss linking number plays in the theory.

It is interesting to notice that the above calculations have been performed for a loop with a triple self-intersection but they actually work for any loop. In particular for loops with double self-intersections, one can check the calculations very rapidly: any expression involving ϵ_{abc} contracted with three tangents automatically vanishes, and therefore all the terms that canceled among themselves in the above proof vanish independently.

We have therefore checked perturbatively that the Kauffman bracket is a solution of the constraints of quantum gravity with cosmological constant, as the conjunction of the loop transform and the Witten argument had suggested. The verification has been order by order for only the first four orders, but we see that even at that level several non-trivial cancellations had to occur. Remarkably, we found as a by-product a completely new solution to the vacuum constraints that we did not know *a priori* and which at present we cannot connect with any known expression in terms of connections. We can therefore see the power of working in the loop representations from the point of view of generating solutions of the constraints.

The new solution generated is given by the second coefficient of an infinite expansion of the Jones polynomial. Since the first coefficient $(2(\gamma))$ is also a solution, this led to the conjecture [52] that maybe the whole polynomial was a solution of the constraints with $\Lambda = 0$. It seems at present that this is not the case. Detailed calculations [210] for the third order show that the third coefficient of the expansion is not a solution and a generic argument shows that if Kauffman being a solution with Λ had to imply that Jones was a solution with $\Lambda = 0$, Jones should satisfy several relations it is known not to satisfy. It seems therefore that the construction singles out the second coefficient as a very special quantity. We will show in section 10.5.3 an argument as to why the second coefficient vplays such a singular role.

10.5.3 The Gauss linking number as the key to the new solution

As we have seen, there is evidence that the Kauffman bracket is a solution of the Hamiltonian constraint of quantum gravity with cosmological constant. The Kauffman bracket is given by the loop transform of the exponential of the Chern–Simons form,

$$K(\gamma)_\Lambda = \int DA \exp\left(-\frac{6}{\Lambda}S_{CS}\right) W_\gamma[A]. \tag{10.83}$$

As we argued, due to the results of Witten and others we know how to compute this quantity explicitly for any gauge group. It is interesting to notice that if the group is $U(1)$ [196, 45],

$$\exp\left(-\frac{\Lambda}{24}a_1(\gamma)\right) = \int DA \exp\left(-\frac{6}{\Lambda}S_{CS}\right) W_\gamma[A], \tag{10.84}$$

and $S_{CS} = \int d^3x \tilde{\epsilon}^{abc} A_a \partial_b A_c$ and the convention for the Abelian Wilson loop is $W_\gamma[A] = \exp(i \oint_\gamma dy^a A_a)$.

So we see that the prefactor that relates the Kauffman and Jones polynomials arises like the "Abelian limit" of the Kauffman bracket. (There is a difference in the numerical factor 24 due to the fact that conventions are slightly different and the Abelian limit of an $SU(2)$ theory yields three $U(1)$ contributions). In particular, it is easy to see that in the perturbative expansion if the group is Abelian all the vertex terms drop out and one gets a resummation of the exponential of the linking number.

Now, the Kauffman bracket solves the Wheeler–DeWitt equation with a cosmological constant. Is there any sense in which one could take the

https://doi.org/10.1017/9781009290203.012 Published online by Cambridge University Press

Abelian limit of this fact and argue that the exponential of the linking number does too? The quick answer to this question is no. There is no systematic way of considering "Abelian limits" in terms of the loop representation, since the non-Abelian nature of the group is embodied from the beginning in the kinematic structure of the theory. Moreover, the expressions for the Hamiltonian constraint and the determinant of the metric collapse in the Abelian limit in terms of connections. However, this idea of exploring the Abelian limit of the Kauffman bracket will lead us to a new solution of the constraints of quantum gravity.

Consider the action of the Hamiltonian constraint on the exponential of the self-linking number. The calculation can be immediately done based on the experience of section 10.5.2. Due to the Abelian nature of the self-linking number, the reroutings have a trivial action and the loop derivative has the effect we discussed when acting on the self-linking number. It is not difficult to see that the total action of the vacuum Hamiltonian constraint on the exponential of the self-linking number is equal to the action of the determinant of the metric [206]. We therefore have the remarkable fact

$$(\hat{H}_0 + \Lambda \widehat{\mathrm{det}} \mathrm{q}) \exp\left(-\frac{\Lambda}{4} a_1(\gamma)\right) = 0. \tag{10.85}$$

We have therefore found another non-trivial solution of all the constraints of quantum gravity in the loop representation. This solution is completely novel: we do not know its counterpart in the connection representation. It can be loosely understood in terms of the Abelian limit ideas that we introduced, which have no apparent counterpart in the connection representation. It is unfortunate that these ideas cannot be given a more concrete implementation, since they could possibly serve as a basis to construct other solutions to the constraints by considering "expansions in terms of Abelianness".

The remarkable fact is that this solution can be viewed as the root of the results we introduced in section 10.5.2. Since the exponential of the Gauss linking number is a solution with cosmological constant and so is the Kauffman bracket, we could consider their difference, divided by Λ^2,

$$D(\gamma)_\Lambda = \frac{K(\gamma)_\Lambda - \exp(\Lambda a_1(\gamma))}{\Lambda^2}, \tag{10.86}$$

and this quantity solves the Hamiltonian constraint with cosmological constant.

Each polynomial solution with a cosmological constant corresponds, in the limit $\Lambda \to 0$, to a solution of the constraint $\hat{H}_0$. For instance, the Kauffman bracket produces in that limit $2(\gamma)$, which we showed was a solution of $\hat{H}_0$. In the case of D we have

$$\mathcal{A}_2(\gamma) = \lim_{\Lambda \to 0} D(\gamma). \tag{10.87}$$

So we see that the fact that the exponential of the self-linking number is a solution of the Hamiltonian constraint with a cosmological constant has the direct consequence that $\mathcal{A}_2(\gamma)$ has to be a solution of $\hat{H}_0$.

Unfortunately, there is no simple way of constructing a similar argument for the higher coefficients. The root of this difficulty is that the motivation for finding this solution, based on notions of Abelian limit, was quite vague and cannot be embodied in an approximation scheme. Our lack of understanding of the Abelian limit in the loop representation also prevents us from making a clear connection with expansions of the theory in terms of Newton's constant ("weak" [207] and "strong" [208] limits) and should be studied more carefully.

10.6 Conclusions

We have seen that the developments in knot theory, in particular the ideas of knot polynomials, can be successfully extended to the case of intersecting loops and be used in practice to construct quantum states of gravity. We have succeeded in constructing two different states with cosmological constant and two states of the vacuum Hamiltonian constraint. They all solve the constraints in very non-trivial fashion and several of them have no simple counterpart in terms of the connection representation that we know of at present. In a sense this chapter has unleashed the full power of the loop representation in that it allows us to make effective use of the notions of knot theory to solve the constraints. All the solutions that we have discussed here were presented in a formal fashion and only exhibited explicitly for the case of a triply self-intersecting loop. One could try to regularize them using point-splitting or loop-thickening techniques such as the ones we introduced in chapter 8 for the non-intersecting solutions and also generalize the results to loops with more intersections. It is intriguing that all solutions with cosmological constant are regular isotopic invariants whereas the solutions with $\Lambda = 0$ are ambient isotopic. We will postpone the discussion of all these issues to the next chapter where we will discuss these solutions in terms of the extended loop representation in which all regularization issues can be analyzed in a clear fashion. We will see that the solutions survive the scrutiny of a careful regularization.