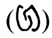
Canad. Math. Bull. Vol. **31** (3), 1988

PLANS' THEOREM FOR LINKS: AN APPLICATION OF t_k MOVES

BY
JÓZEF H. PRZYTYCKI

ABSTRACT. We use t_k moves on links to prove that the first homology group with Z_2-coefficients of an odd sheeted cyclic cover of S^3 branched over a link is even dimensional.

The Plans' theorem [3] (see also [2], [5], or [1]) says that for a knot K and an odd number s, the s-fold cyclic branched cover of S^3 branched over K ($M_K^{(s)}$) has the first homology group a direct double (i.e. $H_1(M_K^{(s)}, Z) = G \oplus G$). It is not true generally for links as the example of the Hopf link () shows, $(H_1(M^s), Z) = Z_s$, but it remains true in Z_2 coefficients.

THEOREM 1. *Let s be an odd number then $H_1(M_L^{(s)}, Z_2) = G \oplus G$ (that is $H_1(M_L^{(s)}, Z_2) = (Z_2)^m$ where m is an even number).*

To prove the theorem we use the Fox method (as presented in [4]) and t_k moves [4].

Consider diagrams of oriented links L_0, and L_k which are identical, except the parts of the diagram shown on Fig. 1.

L_0  $L_k = t_k(L_0)$

k positive half twists

FIGURE 1.

The t_k move is the elementary operation on an oriented diagram L_0 resulting in L_k (Fig. 1).

LEMMA 2. *For s odd, a t_s move does not change $H_1(M_L^{(s)}, Z_2)$.*

Now Theorem 1 follows because it is a special case of the Plans' theorem for knots and any link can be gotten from a knot using t_s moves.

To prove Lemma 2 we have to recall Corollary 2.6 of [4] namely that a t_s move preserves the Alexander module of a link modulo

$$\frac{t^s + 1}{t + 1} = 1 - t + t^2 - \ldots + t^{s-1}.$$

Received by the editors January 8, 1987, and, in revised form, July 15, 1987.
AMS Subject Classification (1980): Primary 57M25; Secondary 57M12.

But $H_1(M_L^{(s)}, Z)$ can be thought as a $Z[t, t^{-1}]$ module presented by the matrix $V^T - tV$ (where V is a Seifert matrix of L) and additional relations

$$\frac{t^s - 1}{t - 1}(w) = 0$$

for each generator w of the module ([1], Proposition 9.20).

In particular $(t^s - 1)/(t - 1)$ is an anihilator of the module. Furthermore,

$$\frac{t^s - 1}{t - 1} \equiv \frac{t^s + 1}{t + 1} \pmod 2,$$

therefore Lemma 2 holds.

REMARK 3. In fact Theorem 1 can be partially extended to s even; namely: $\dim H_1(M_L^s, Z_2) \equiv (c(L) - 1)(s - 1) \bmod 2$ where $c(L)$ is the number of components of L.

SKETCH OF A PROOF. Let V be a Seifert matrix of L (of dimension $d = 2g + c(L) - 1$) then $H_1(M_L^{(s)}, Z)$ has a presentation by the $(s - 1)d \times (s - 1)d$ matrix

$$A = \begin{bmatrix} V + V^T & -V & 0 & & 0 & 0 \\ -V^T & V + V^T & -V & \vdots & 0 & 0 \\ 0 & -V^T & V + V^T & & 0 & 0 \\ & \cdots & & & & \\ 0 & 0 & 0 & & -V^T & V + V^T \end{bmatrix}$$

and A is skew symmetric over Z_2, so the conclusion of Remark 3 follows.

I am grateful to D. Rolfsen for useful conversations.

After submitting the paper for publication I got a letter (dated April 10, 1987) from V. Turaev. He informed me that he also generalized the Plans' theorem (in 1983 or 1984) but had never written it down (except a short description in a letter to C. Weber). His method uses the result of C. Weber [6].

THEOREM (Turaev). *Let H be the first integer homology group of an s-sheeted cyclic branched cover of a* 3*-dimensional homology sphere branched over an n-component link. The covering projection induces a homomorphism i of H into the Z_s-coefficients homology of the link complement. If s is odd then the kernel of i is a (direct) double. The Plans' theorem for knots and Theorem* 1 *can be derived from the above theorem.*

I have also received a letter from Pablo Del Val (a student of J. Montesinos) informing me that he is working on the homology of cyclic coverings of S^3

branched over a link and studying with C. Weber the possibility of extending the Plans' theorem. They are writing a paper on Plans' theorem for links.

References

1. G. Burde, H. Zieschang, *Knots*, De Gruyter studies in Math. **5**, Berlin, New York 1985.

2. C. McA. Gordon, *A short proof of a theorem of Plans on the homology of the branched cyclic coverings of a knot*, Bull. Amer. Math. Soc. **168** (1971), pp. 85-87.

3. A. Plans, *Aportacion al estudio de los grupos de homologia de los recubrimientos ciclicos ramificados correspondiente a un nudo*, Rev. Real. Acad. Cienc. Exact., Fisica y Nat. Madrid, **47** (1953), pp. 161-193.

4. J. H. Przytycki, *t_k moves on links*, to appear in Proceedings of the Summer Research Conference on Artin's braid groups, Santa Cruz, 1986.

5. D. Rolfsen, *Knots and links*, Publish or Perish, Inc., Berkeley 1976, Math. Lect., Series 7.

6. C. Weber, *La demonstration de J. Levine des théorème de A. Plans*, Lect. Notes Math. **1051** (1984), pp. 315-330.

Math Department
University of British Columbia
Vancouver, British Columbia
Canada V6T 1Y4
(and Warsaw University, Poland)

https://doi.org/10.4153/CMB-1988-047-6 Published online by Cambridge University Press