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A COMMUTATIVITY RESULT FOR RINGS

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It is proved that a ring satisfying the variable identity $[[x^n,y],y] = 0$, $n = n(x,y) \geqslant 1$, has nil commutator ideal.

Herstein proved many years ago [1] that a ring satisfying $[x^n,y] = 0$, $n = n(x,y) \geqslant 1$, has nil commutator ideal. More recently he proved [2] that the same conclusion holds for a ring satisfying a variable identity of the form $[[x^m,y^n], z^q] = 0$. Herstein has also remarked in [2] that it seems that the methods of his paper can be adapted to generalize his result to the weaker variable identity $[[x^m,y^n], y^n] = 0$. In [3] this has been done in the case that $m = m(x,y)$ is bounded. The case when $n(x,y)$ is bounded is much more difficult. In this case, one may assume that $n(x,y)$ is fixed, for if $n(x,y) \leqslant k$ and $r = k!$ then $[[x^m,y^r], y^r] = 0$. We have succeeded when $r = 1$, so our result generalizes the above mentioned result of [1]. It is a special case of the general question but it is far from being trivial. To prove it we have applied most of the machinery developed in [2] together with an idea which worked very nicely in our situation. It seems that to get the result for a fixed $r > 1$, which will still be a special case of the general question, one would need other ideas.

All the results of [2] up to Lemma 5 have been obtained on the basis of the weaker hypothesis $[[x^m,y^n], y^n] = 0$, and by ([3] Theorem 3) this hypothesis implies the result of Lemma 5. In particular, those results hold for a ring R in which given $x,y \in R$ there exists a positive

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integer $m = m(x,y)$ such that $[[x^m,y],y] = 0$. By proceeding as in [2], we can reduce the problem to establishing communitivity of R under the additional hypotheses that R is prime with a non-nilpotent element c such that any non-zero ideal of R contains some power of c. Moreover, R is torsion-free, its centre is 0, every element in R is regular or nilpotent and R is the sum of its nil right ideals. Also given $x,y \in R$ there exists some $m = m(x,y) \geq 1$ such that $[[x^m,y],y] = 0$ and $[x^m,y]$ is nilpotent. We assume all these hypotheses and proceed to prove that R is commutative.

In [2], given a regular element $x \in R$, the subring $W(x) = \bigcup_{n \geq 1} C_R(x^n)$ was defined and very much effort was necessary to prove that $W(x) = C_R(x)$. In our case we can prove this immediately.

LEMMA 1. *If* $x \in R$ *is regular then* $C_R(x^n) = C_R(x)$ *for all* $n \geq 1$.

Proof. We have to prove that $[b,x^n] = 0$ implies $[b,x] = 0$. It suffices to prove that $[b,x^n] = 0$ implies $[[b,x],x] = 0$, for if $[b,x]$ commutes with x we have $0 = [b,x^n] = nx^{n-1}[b,x]$ so $[b,x] = 0$ since R is torsion-free and x is regular. We first prove $[[b,x],x] = 0$ in the case that b is nilpotent. If $b^2 = 0$ then there exists some $m \geq 1$ such that $[[(x^n+b)^m,x],x] = 0$. It follows that $mx^{n(m-1)}[[b,x],x] = 0$, so $[[b,x],x] = 0$. If the index of nilpotency of b is k then for $1 < i < k$, b^i has index of nilpotency $< k$, and since $b^i \in C_R(x^n)$ we may assume by induction that $[[b^i,x],x] = 0$. Starting with $[[(x^n+b)^m,x],x] = 0$ and using the induction hypothesis we get again $mx^{n(m-1)}[[b,x],x] = 0$ and therefore $[[b,x],x] = 0$. If $b \in C_R(x^n)$ is arbitrary then $b' = [b,x] \in C_R(x^n)$ and it is nilpotent by Theorem 1 [2] with x replaced by x^n. Hence $[[b',x],x] = 0$ so $[b',x] = 0$ and finally we get $[b,x] = 0$.

Now in [2] after the proof of Theorem 7 up to the conclusion $d_y^4 = 0$ for any regular element $y \in R$, there is no use of the stronger condition

$[[x^m,y^n],z^q] = 0$. The Corollary to Theorem 5 has been used, but its result holds in our case; it is the above Lemma. It follows that we also have $d_y^4 = 0$ for y regular in R . In [2] this fact implied $N(y)^m = 0$ for some m and the final result followed using Theorem 3, namely using the fact that $T(y)$ - the left annihilator of $N(y)$ in $N(y)$ - is 0 . This last fact was proved using the commutativity of $N(y)$ which was proved in Theorem 2. Here we prove that $N(y)$ is anticommutative and a careful consideration of the proof of Theorem 3 shows that anticommutativity of $N(y)$ suffices to yield the result $T(y) = 0$.

LEMMA 2. *If* $y \in R$ *is regular then* $N(y)$ *is anticommutative and* $N(y)^3 = 0$.

Proof. If $a \in N(y)$ and λ is an integer then $y + \lambda a$ is regular, so $d_{y+\lambda a}^4 = 0$. Since $d_{y+\lambda a} = d_y + \lambda d_a$ we have $(d_y + \lambda d_a)^4 = 0$ and a Vandermonde argument yields $d_a^4 = 0$. Now suppose $a^k = 0$, $a^{k-1} \neq 0$, then applying d_a^4 to an element $z \in R$ and multiplying on the left by a^{k-1} we get $a^{k-1}za^4 = 0$, so $a^4 = 0$ since R is prime. Since $a^4 = 0$ and $a^2d_a^4(z) = 0$, we get $-4a^3za^3 = 0$, so $a^3 = 0$ since R is prime and torsion-free. Finally $a^3 = 0$ and $ad_a^4(z) = 0$ imply $a^2 = 0$. Thus $N(y)$ is nil of index 2, so it is anticommutative and being torsion-free it follows easily that $N(y)^3 = 0$.

Now as in [2], $T(y) = 0$ and $N(y)^3 = 0$ imply $N(y) = 0$. Given x we already know that for some $m \geq 1$ we have $[x^m,y] \in N(y)$ so $[x^m,y] = 0$. The rest is as in [2]. We have proved

THEOREM. *Let* R *be a ring in which, given* $x,y \in R$ *there exists a positive integer* $m = m(x,y)$ *such that* $[[x^m,y],y] = 0$. *Then the commutator ideal of* R *is nil. Equivalently, the nilpotent elements of* R *form an ideal* N *and* R/N *is commutative.*

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