Appendix A

A Primer on Topological Vector Spaces and Locally Convex Spaces

This section contains some auxiliary results on topological vector spaces and locally convex spaces in particular. Most of the results are standard and can be found in textbooks such as Meise and Vogt (1997), as well as Jarchow (1981). Note that for some of the results (e.g. Proposition A.4) in this appendix, it is essential that we only consider Hausdorff topological vector spaces. Since we are only working with real vector spaces, some of the proofs simplify substantially (compare to the general proofs for $\mathbb{R}$ and $\mathbb{C}$; see Rudin (1991, Chapter 1)).

A.1 Basic Material on Topological Vector Spaces

A vector space with a Hausdorff topology making vector addition and scalar multiplication continuous is called a topological vector space or TVS (see Definition 1.1). Note that a morphism of TVS is a continuous linear map. In particular, two TVS are isomorphic (as TVS) if they are isomorphic as vector spaces and the isomorphism is a homeomorphism.

Conventions Let U, V be subsets of a (topological) vector space E, $s \in \mathbb{R}$ and $I \subseteq \mathbb{R}$. Then we define

$$U + V := \{z = u + v \mid u \in U, v \in V\}, \quad sU := \{z = su \mid u \in U\}, \quad I \cdot U := \bigcup_{s \in I} sU.$$

A.1 Definition Let $(E, \mathcal{T})$ be a TVS and U a subset of E. We say that U is

(a) a 0-*neighbourhood* if U is a neighbourhood of 0;
(b) *bounded* if for every 0-neighbourhood V there is $s > 0$ with $U \subseteq sV$.

In general, topological vector spaces sequences are not sufficient to test, for example, continuity. Instead one would need nets to test for continuity and a

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complete topological vector space should be defined in terms of convergence of Cauchy nets (see Meise and Vogt, 1997, p. 258). However, for our calculus we usually do not need complete spaces and the limits we consider can always be described in terms of sequential limits. Thus we do not go into details here and stay in the realm of the more familiar sequences. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq E$ is called

(c) a *Cauchy sequence* if for every 0-neighbourhood $V \subseteq E$ there exists $N \in \mathbb{N}$ such that
$$x_n - x_m \in V \quad \text{for all } n,m \geq N;$$

(d) a *Mackey–Cauchy sequence* if there exists a bounded subset $B \subseteq E$ and a family $m_{k,l} \in \mathbb{N}$ for $k,l \in \mathbb{N}$ such that
$$m_{k,l}(x_k - x_l) \in B, \quad \text{for all } k,l \in \mathbb{N}$$
and such that for every $R > 0$ there is $N \in \mathbb{N}$ with $m_{k,l} > R$ if $k,l > N$ (i.e. $m_{k,l} \to \infty$). Note that every Mackey–Cauchy sequence is a Cauchy sequence.

Now we say that the topological vector space $(E,\mathcal{T})$ is

(e) *sequentially complete* if every Cauchy sequence in E converges;
(f) *Mackey complete* if every Mackey–Cauchy sequence in E converges.

Mackey completeness as per (f) can be shown (see Kriegl and Michor, 1997, Theorem 2.14) to be equivalent to the notion from Definition 1.12.

A.2 Lemma *Let E be a topological vector space and $U \subseteq E$ a 0-neighbourhood. Then the following holds:*

(a) For each $x \in E$ the translation $\lambda_x \colon E \to E$, $y \mapsto x + y$ is a homeomorphism.
(b) For each $r \in \mathbb{R} \setminus \{0\}$, scaling $s_r \colon E \to E$, $x \mapsto rx$ is a homeomorphism.
(c) U contains a balanced *0-neighbourhood V, that is, $tV \subseteq V$ for each $|t| \leq 1$.*
(d) U contains a 0-neighbourhood W such that $W + W \subseteq U$.
(e) If $\mathcal{B}$ is a basis of 0-neighbourhoods, then for each $x \in E$ the set $\{x + W \mid W \in \mathcal{B}\}$ is a basis of x-neighbourhoods.
(f) Each 0-neighbourhood contains a closed 0-neighbourhood.
(g) If $K \subseteq E$ is compact and $U \Subset E$ with $K \subseteq U$, then there exists $0 \in W \Subset E$ such that $K + W \subseteq U$.

Proof **(a–b) and (e)**. The maps λ_x and s_r have inverses λ_{-x} and $s_{1/r}$. Thus the claim is clear from the definition of topological vector spaces. Since translations are homeomorphisms (a) implies (e).

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(c). By continuity of the scalar multiplication $\mu\colon \mathbb{R} \times E \to E$, the set $\mu^{-1}(U)$ is open. Thus we can find $(-\varepsilon, \varepsilon) \times W \subseteq \mu^{-1}(U)$ and thus $V := (-\varepsilon, \varepsilon)W = \mu((-\varepsilon, \varepsilon) \times W) \subseteq U$. Then $[-1,1]V = V$ and V is a balanced 0-neighbourhood contained in U.

(d). As addition $\alpha\colon E \times E \to E$ is continuous with $\alpha(0,0) = 0$, the preimage $\alpha^{-1}(U \times U)$ is a $(0,0)$-neighbourhood. We can thus find $W_1, W_2 \subseteq E$ 0-neighbourhoods in E such that $W_1 \times W_2 \subseteq \alpha^{-1}(U)$. Then $W := W_1 \cap W_2 \subseteq U$ satisfies $W + W \subseteq U$.

(f). We conclude from (c) and (d) that there is a 0-neighbourhood V with $V - V \subseteq U$. For $w \in \overline{V}$ (closure), the set $w + V$ is a w-neighbourhood (by (e)). Hence we can pick $v_1 \in V$ such that $v_1 \in w + V$, that is, $v_1 = w + v_2$ for some $v_2 \in V$. But then $w = v_1 - v_2 \in U$, and so $\overline{V} \subseteq U$.

(g). For every $x \in K$ we can pick by (e) a 0-neighbourhood V_x such that $x + V_x \subseteq U$. By (d) there is $0 \in W_x \Subset E$ with $W_x + W_x \subseteq V_x$. Then $(x + W_x)_{x \in K}$ is an open cover of K and by compactness we can choose a finite subset $F \subseteq K$ with $K \subseteq \bigcup_{x \in F}(x + W_x)$. Then $W := \bigcap_{x \in F} W_x$ is an open 0-neighbourhood. For $y \in K$, there exists $x \in F$ such that $y \in x + W_x$. Then $y + W \subseteq x + W_x + w \subseteq x + V_x \subseteq U$. As y was arbitrary $K + W \subseteq U$. □

A.3 Proposition (Rudin, 1991, I Theorem 1.22) *If E is a topological vector space which contains a compact 0-neighbourhood, then E is finite dimensional.*

A.4 Proposition (Uniqueness of topology (Treves, 2006, Theorem 9.1)) *If E is a finite-dimensional topological vector space of dimension d, then $E \cong \mathbb{R}^d$ as topological vector spaces, where $\mathbb{R}^d$ carries the usual norm topology.*

A.5 Lemma *Let $f\colon E \to F$ be a linear map between topological vector spaces. Then f is continuous (open) if and only if it is continuous (open) in 0.*

Proof Clearly the conditions are necessary. To prove sufficiency, we assume that f is continuous in 0. Pick $x \in E$ and observe that the translations $\tau_{-x}(y) = y - x$ and $\tau_{f(x)}(z) = z + f(x)$ are continuous and even homeomorphisms. Thus $f(y) = \tau_{f(x)} \circ f \circ \tau_{-x}$ is continuous in x. Since x was arbitrary, f is continuous in every point.

Now let f be open in 0 and U an open set. For $x \in U$, $\tau_{-x}(U)$ is an open 0-neighbourhood, whence $\tau_{f(x)} \circ f \circ \tau_{-x}(U) = f(U)$ is open. □

En route towards locally convex spaces let us first recall some results on convex sets.

A.6 Definition A subset S of a topological vector space E is said to be

- *absorbent* if $E = \bigcup_{t>0} tS$;

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- *convex* if for all $x, y \in S$ and $t \in [0,1]$, the linear combination $tx + (1-t)y \in S$;
- a *disc* if it is convex and balanced, that is, for all $x \in S$, $|\lambda| \leq 1, \lambda \in \mathbb{R}$, $\lambda x \in S$.

A.7 Lemma *Every* 0*-neighbourhood of a topological vector space is absorbent.*

Proof Let U be a 0-neighbourhood of the topological vector space E, and let $x_0 \in E$. Since scalar multiplication is continuous there exists some neighbourhood V of x_0 and $\delta > 0$ such that for all $x \in V$, $\lambda x \in U$ when $|\lambda| < \delta$, $\lambda \in \mathbb{R}$. Especially $\lambda x_0 \in U$, and thus $x_0 \in \lambda^{-1}U$ for $|\lambda^{-1}| > 1/\delta$. □

A.8 Example The ball $B_r(x) := \{y \in E \mid \|x - y\| < r\}$ in a normed space $(E, \|\cdot\|)$ is convex (and a disc if $x = 0$).

A.9 Lemma *The interior A° of a convex set A is convex.*

Proof Let $x, y \in A^\circ$. By Lemma A.2(c) there is some balanced 0-neighbourhood U such that $x + U \Subset A^\circ$ and $y + U \Subset A^\circ$ are neighbourhoods of x and y contained in A. For any $z = tx + (1-t)y$, $t \in [0,1]$ and $u \in U$ we have

$$z + u = tx + (1-t)y + tu + (1-t)u = t(x+u) + (1-t)(y+u).$$

As A is convex, $z + u \in A$ and thus $z + U \Subset A$. Hence $tx + (1-t)y \in A^\circ$, for all $t \in [0,1]$. □

A.10 Lemma *If N is a convex* 0*-neighbourhood in a topological vector space, then N contains an open disc.*

Proof Consider first the set $M := -N \cap N$. If $|\lambda| \leq 1$ we see that $\lambda M = (-\lambda)N \cap \lambda N$. Now, as $0 \in N$ and N is convex, we have that $-\lambda N, \lambda N \subseteq N$. In particular, $\lambda M \subseteq M$ for all $|\lambda| \leq 1$, that is, M is balanced. By Lemma A.2(c), we can find a balanced 0-neighbourhood $U \subseteq N$. As U is balanced we see that $U \subseteq -N \cap N$. Hence the interior V of $M = -N \cap N$ is a convex 0-neighbourhood (by Lemma A.9 as it is the interior of an intersection of convex sets; Exercise A.1.3). Now the interior of a balanced set is again balanced (Exercise A.1.1), whence V is a disc. □

Exercises

A.1.1 Let B be a balanced subset of a topological vector space. Show that then also the interior of B is balanced.

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A.1.2 Let $U \Subset E$ be a bounded 0-neighbourhood in a topological vector space E. Show that every 0-neighbourhood contains a set of the form $\{rU\}_{r\in]0,\infty[}$.
Hint: Use Lemma A.7 together with the fact that U is bounded.

A.1.3 Show that the intersection $C \cap D$ of two convex sets C and D is convex.

A.2 Seminorms and Convex Sets

In the main text we have defined locally convex spaces using seminorms. In this section we shall review seminorms and, in particular, their connection to convex sets (thus justifying the name 'locally convex space').

A.11 Definition A family $\mathcal{P}$ of seminorms on a vector space E is said to be *separating* if for each $x \in E$, $p(x) \neq 0$ for at least one $p \in \mathcal{P}$.

A.12 Proposition *Let E be a vector space and $(p_i)_{i\in I}$ a separating family of seminorms on E. Then a Hausdorff vector topology is generated by the subbase*

$$B_{i,\epsilon}(x_0) := \{x \in E \mid p_i(x - x_0) < \epsilon\}, \quad i \in I, \epsilon > 0, x_0 \in E. \tag{A.1}$$

Thus $(E, \{p_i\}_I)$ is a locally convex space and the topology contains a 0-neighbourhood basis of convex sets. Finally, each p_i is continuous with respect to the locally convex topology.

Proof Let us first note that the subbase (A.1) generates the initial topology induced by the family $\{q_i : E \to E/\ker p_i\}_{i\in I}$, where the right-hand side is endowed with the normed topology induced by p_i (see Exercise A.2.1). Let U be a 0-neighbourhood. Then U contains some finite intersection $\bigcap B_{i,\epsilon}(x)$, which is convex since the seminorm balls are convex and intersections preserve convexity; Exercise A.1.3. Thus every 0-neighbourhood contains a convex 0-neighbourhood. For the Hausdorff property we choose for $x_1, x_2 \in E$ a seminorm p_i such that $0 < p_i(x_1 - x_2)$. Set $\delta = p_i(x_1 - x_2)/3$. Now if $z \in B_{i,\delta}(x_2) \cap B_{i,\delta}(x_1)$ were non-empty, we must have

$$0 < \delta = p_i(x_1 - x_2) \leq p_i(x_1 - z) + p_i(z - x_2) \leq \frac{2}{3}\delta,$$

which is absurd. Therefore the intersection must be empty and E is Hausdorff.

We have continuity of addition since for each

$$U_{x+y} = B_{i_1,\epsilon_1}(x_0 + y_0) \cap B_{i_2,\epsilon_2}(x_0 + y_0) \cap \cdots \cap B_{i_n,\epsilon_n}(x_0 + y_0),$$

the neighbourhoods

$$U_{x_0} = B_{i_1,\epsilon_1/2}(x_0) \cap B_{i_2,\epsilon_2/2}(x_0) \cap \cdots \cap B_{i_n,\epsilon_n/2}(x_0)$$

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and

$$U_{y_0} = B_{i_1,\epsilon_1/2}(y_0) \cap B_{i_2,\epsilon_2/2}(y_0) \cap \cdots \cap B_{i_n,\epsilon_n/2}(y_0)$$

satisfy $U_x + U_y \subset U_{x+y}$ since for any $x + y \in B_{i,\epsilon/2}(x) + B_{i,\epsilon/2}(y)$,

$$p_i(x+y) \leq p_i(x) + p_i(y) < \epsilon/2 + \epsilon/2 = \epsilon.$$

For the continuity of scalar multiplication, consider the neighbourhood

$$U_{\lambda_0 x_0} = B_{i_1,\epsilon_1}(\lambda_0 x_0) \cap B_{i_2,\epsilon_2}(\lambda_0 x_0) \cap \cdots \cap B_{i_n,\epsilon_n}(\lambda_0 x_0).$$

Let $|\lambda - \lambda_0| < \delta$ and $x \in B_{i,\delta}(x_0)$, then

$$\begin{aligned} p_i(\lambda x - \lambda_0 x_o) &= p_i((\lambda - \lambda_0)x + \lambda_0(x - x_0)) \\ &\leq |\lambda - \lambda_0| p_i(x) + |\lambda_0| p_i(x - x_0) \\ &< \delta(p_i(x - x_0) + p_i(x_0)) + |\lambda_0| p_i(x - x_0) \\ &< \delta(\delta + p_i(x_0) + |\lambda_0|) < \epsilon \end{aligned}$$

if δ is small enough. So we can find $\delta_1, \ldots, \delta_n$ and $\delta = \min\{\delta_1, \ldots, \delta_n\}$ such that

$$(\lambda, x) \in]\lambda_0 - \delta, \lambda_0 + \delta[\times U_{x_0} \implies \lambda x \in U_{\lambda_0 x_0}.$$

To see that p_i is continuous, let $]a, b[$ be an open interval in $[0, \infty[$, then $p_i^{-1}(]a, b[) = (E \setminus \overline{B}_{i,a}(0)) \cap B_{i,b}(0)$ is open being a finite intersection of open sets, when $\overline{B}_{i,a}(0) := \{x \in E \mid p_i(x) \leq a\}$. Since $p_i^{-1}([0, b[) = B_{i,b}(0)$, we deduce that p_i is continuous. □

A.13 Definition Let $(p_i)_{i \in I}$ be a family of seminorms. We say the family

- satisfies the *basis condition* if for each two seminorms p_i and p_j, there exists a third seminorm p_k and $C > 0$ such that

$$\max\{p_i(x), p_j(x)\} \leq C p_k(x), \quad \text{for all } x \in E;$$

- is called a *fundamental system* of seminorms, if it generates the topology on E and satisfies the basis condition.

Note that for a fundamental system of seminorms, the subbase (A.1) is a basis for the topology it generates.

A.14 Example Consider again the space of smooth functions $C^\infty([0,1], \mathbb{R})$ with the Fréchet topology induced by the seminorms

$$\|f\|_n := \sup_{0 \leq k \leq n} \sup_{x \in [0,1]} \left| \frac{\mathrm{d}^k}{\mathrm{d}x^k} f(x) \right|.$$

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For two of these seminorms we obviously have

$$\max\{\|f\|_n, \|f\|_m\} \le \|f\|_{\max\{n,m\}} \text{ for all } f \in C^\infty([0,1],\mathbb{R}).$$

Hence these seminorms form a fundamental system of seminorms and their r-balls form a basis of the topology called the compact open C^∞-topology.

Let us associate now to every disc a seminorm. The upshot will be that one can equivalently define a locally convex space as a topological vector space with a 0-neighbourhood base consisting of convex sets.

A.15 Definition For a vector space E and a disc A in E, define the *Minkowski functional*, $p_A : E \to \mathbb{R}, p_A(x) := \inf\{t > 0. \mid x \in tA\}$, where $\inf \emptyset := \infty$.

A.16 Lemma *If U is a disc 0-neighbourhood in the locally convex space E, then the Minkowski functional p_U is a continuous seminorm on E.*

Proof By Lemma A.7, $p_U(x) \in [0,\infty[$ for all $x \in E$. For the triangle inequality, let $x, y \in E$, then if $x \in tU$ and $y \in sU$,

$$\frac{1}{t+s}(x+y) = \frac{t}{t+s}\frac{x}{t} + \frac{s}{t+s}\frac{y}{s} \in U$$

or rather $x + y \in (t+s)U$. Therefore, $p_U(x+y) \le t + s = p_U(x) + p_U(y)$. Scalar factors can be taken out of the seminorm due to U being balanced: If $\lambda \in \mathbb{R} \setminus \{0\}$, $\lambda x = \frac{\lambda}{|\lambda|}|\lambda| x \in tU$ if and only if $|\lambda| x = \frac{|\lambda|}{\lambda}\lambda x \in tU$. Therefore, $p_U(\lambda x) = \inf\{t > 0 : \lambda x \in tU\} = \inf\{t > 0 : |\lambda| x \in tU\} = |\lambda| p_U(x)$. Continuity of the seminorm follows from $p_U^{-1}(]a,b[) = (E \setminus (\overline{aU})) \cap \bigcup_{0 \le t \le b} tU$ is open for all $a, b \in [0,\infty[$. □

As Lemma A.10 shows, every convex 0-neighbourhood gives rise to a disc and these give rise to seminorms by Lemma A.16. Thus an equivalent definition of a locally convex space (fitting the name better; see Rudin, 1991, Theorem 1.34 and Remark 1.38) is the following.

A.17 Definition A Hausdorff topological vector space is a locally convex space if it contains a 0-neighbourhood basis of convex sets.

Finally, let us recall Kolmogorov's normability criterion, which gives a (necessary and sufficient!) condition for a topological vector space to be normable, that is, the vector topology coincides with the topology induced by some norm.

A.18 Theorem (Kolmogorov's normability criterion) *A topological vector space E is normable if and only if E has a bounded and convex 0-neighbourhood.*

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Proof The criterion is necessary as in a normed space, the unit ball is a convex bounded 0-neighbourhood.

Now let E be a topological vector space with a bounded and convex 0-neighbourhood N. By Lemma A.10 we can pick an open disc $U \subseteq N$. Note that U is also bounded and define $\|x\| := p_U(x)$ for $x \in E$, where p_U is the Minkowski functional associated to U. Now, thanks to Exercise A.1.2, the sets sU, $s \in]0, \infty[$ form a neighbourhood base of 0 in E. If $x \neq 0$ is an element of E, the Hausdorff property implies that there exists $s > 0$ with $x \notin sU$, that is, $\|x\| \geq s > 0$. We deduce from Lemma A.16 that $\|\cdot\|$ is a (continuous) norm on E. Hence the norm topology induced by $\|\cdot\|$ is coarser than the original topology. Conversely, recall that since U is open, we have $\{x \in E \mid \|x\| < s\} = sU$. As the sU form a neighbourhood base, this shows that the norm topology coincides with the original topology, whence E is normable. □

The Kolmogorov normability criterion allows us to describe the pathology occurring for dual space of topological vector spaces beyond Banach spaces.

A.19 Proposition *Let E be a locally convex space and let*

$$E' := \{\lambda\colon E \to \mathbb{R} \mid \lambda \text{ is continuous and linear}\}$$

be its dual space. If E' is a topological vector space such that the evaluation map $\mathrm{ev}\colon E \times E' \to \mathbb{R}$, $(x, \lambda) \mapsto \lambda(x)$ *is continuous, then E is normable.*

Proof Assume that E' is a topological vector space such that ev is continuous. Then there are 0-neighbourhoods $U \Subset E$ and $V \Subset E'$ such that $\mathrm{ev}(U \times V) \subseteq [-1, 1]$. Since V is absorbent by Lemma A.7 this implies that every continuous linear functional is bounded on U. Now Theorem 3.18 of Rudin (1991) yields that U is already bounded. Shrinking U, we may assume that U is convex and bounded. Hence Kolmogorov's criterion, Theorem A.18, shows that E must be normable. □

Recall from Exercise 1.4.1 that if E is normable, the evaluation map on the dual space is indeed continuous with respect to the operator norm on the dual space.

Exercises

A.2.1 Let E be a vector space and p a seminorm on E.

(a) Show that $\ker p := \{x \in E \mid p(x) = 0\}$ is a vector subspace of E.

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(b) Prove that $\|q(x)\| := \inf_{y \in \ker p} p(x+y)$ defines a norm on the quotient, where q is the (surjective!) quotient map $q\colon E \to E/\ker p$.

A.2.2 Let E be a locally convex space whose topology $\mathcal{T}$ is generated by a family $\mathcal{P}$ of seminorms. Show that if q is a continuous seminorm on E, then the topology generated by $\mathcal{P} \cup \{q\}$ is equal to $\mathcal{T}$.

A.2.3 Show that every locally convex space admits a fundamental system of seminorms.
Hint: Show that $\max\{p,q\}$ is a continuous seminorm and use the previous exercise.

A.3 Subspaces of Locally Convex Spaces

In this section we recall some material on subspaces of locally convex spaces.

A.20 Definition A vector subspace $F \subseteq E$ of a locally convex space is called *complemented* if there exists a locally convex space X such that $E \cong E \times X$ (isomorphic as locally convex spaces).

A.21 Lemma *A subspace $F \subseteq E$ is complemented if and only if there exists a continuous linear map $\pi\colon E \to E$ with $\pi(E) = F$ and $\pi \circ \pi = \pi$. Further, a complemented subspace is always closed. We call π a* continuous projection.

Proof Let F be complemented with isomorphism $\varphi\colon E \to F \times X$. Then $\pi := \varphi^{-1} \circ p_1 \circ \varphi$ with $p_1\colon F \times X \to F \times X$, $(f,x) \mapsto (f,0)$ is a continuous projection.

Conversely, let $\pi\colon E \to E$ be a continuous projection with $\pi(E) = F$. Then $X := \ker \pi$ is a closed subspace of E and $F \times X \to E$, $(f,x) \mapsto f + x$ is a continuous linear map with continuous inverse $e \mapsto (\pi(e), e - \pi(e))$.

If F is complemented, we have an associated continuous projection and see that $F = \ker(\mathrm{id}_E - \pi)$ is closed. □

A.22 Example Finite-dimensional subspaces of locally convex spaces are always complemented (Rudin, 1991, Lemma 4.21): Thus all subspaces of a finite-dimensional locally convex space are complemented. More generally, every closed subspace of a Hilbert space is complemented (Rudin, 1991, Theorem 12.4).

Note, however, that complemented subspaces (e.g. of Banach spaces) may be rare. Indeed one can prove that a Banach space for which every closed subspace is complemented must already be a Hilbert space (see Lindenstrauss and

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Tzafriri, 1971). Moreover, there are examples of infinite-dimensional Banach spaces whose only complemented subspaces are finite dimensional.

A.23 Example Consider the Banach space c_0 of all (real) sequences converging to 0 as a subspace of the Banach space ℓ^∞ of all bounded real sequences, with the norm

$$\|(x_1, x_2, \ldots)\|_\infty = \sup_{n \in \mathbb{N}} |x_n|.$$

Then c_0 is not complemented in ℓ^∞. The proof is, however, more involved, and so we defer to Werner (2000, Satz IV.6.5).

Exercises

A.3.1 Let $(E_i)_{i \in I}$ be a family of locally convex spaces. Prove that $\prod_{i \in I} E_i := \{(x_i)_I \mid x_i \in E_i\}$ with componentwise addition and scalar multiplication and endowed with the product topology is a locally convex space.

A.3.2 Show that $F \subseteq E$ is complemented if and only if the projection $q\colon E \to E/F$ has a continuous linear right inverse $\sigma\colon E/F \to E$ (i.e. $q \circ \sigma = \mathrm{id}_{E/F}$).

A.4 On Smooth Bump Functions

In finite-dimensional differential geometry, one uses commonly local-to-global arguments employing smooth bump functions (also sometimes called cut-off functions) and partitions of unity. This strategy fails, in general, due to a lack of bump functions. We briefly discuss the problem and refer to Kriegl and Michor (1997, Chapter III) for more information.

A.24 Definition For a map $f\colon V \to F$ with $V \subseteq_{o} E$ and E, F locally convex spaces the *carrier* of f is the set $\mathrm{carr}(f) := \{x \in V \mid f(x) \neq 0\}$. As usual the *support* of f is defined to be the closure of the carrier.

A.25 Definition Let E be a locally convex space, $x \in E$ and $U \subseteq E$ an x-neighbourhood.

(a) A C^k-map $f\colon E \to [0, \infty[$ for $k \in \mathbb{N}_0 \cup \{\infty\}$ is a C^k*-bump function* with carrier in U if $\mathrm{carr}(f) \subseteq U$ and $f(x) = 1$.

(b) If $(U_i)_{i \in I}$ is an open cover of E, we say that a family $(f_i\colon E \to [0, \infty[)_{i \in I}$ is a C^k*-partition of unity* (subordinate to the cover) if every f_i is C^k with carrier in U_i and $\sum_i f_i(x) = 1$, for all $x \in E$.

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A.26 Definition Let E be a locally convex space and $k \in \mathbb{N}_0 \cup \{\infty\}$. We say that E is

(a) C^k*-regular*, if for any neighbourhood U of a point x there exists a C^k-bump function with carrier in U and $f(x) = 1$.
(b) C^k*-paracompact*,[1] if every open cover admits a C^k-partition of unity.

Obviously similar definitions make sense if we consider instead of a locally convex space a manifold modelled on locally convex spaces. While the existence of partitions of unity hinges on topological properties of the manifold (paracompactness!), the smoothness of these partitions depends only on the availability of bump functions on the model space.

A.27 (Typical local-to-global argument) Let M be a manifold which admits smooth partitions of unity subordinate to open covers. Assume we have an object defined for every chart (U,φ) in an atlas $\mathcal{A}$, and to illustrate this we choose smooth Riemannian metrics on $TU \cong U\times H$, that is, $g_U : U\times H\times H \to \mathbb{R}$, $(u,h,k) \mapsto \langle h,k\rangle_u$. Using the chart we transport it back to the manifold, that is, $g_\varphi : TU \oplus TU \to \mathbb{R}$, $(v,w) \mapsto \langle T\varphi(v), T\varphi(w)\rangle_{\varphi(\pi(v))}$. Now choose a smooth partition of unity p_φ subordinate to the open covering $(U,\varphi)_{\varphi\in\mathcal{A}}$. Then

$$g : TM \oplus TM \to \mathbb{R}, \quad (v,w) \mapsto \sum_{\varphi\in\mathcal{A}} p_\varphi(\pi_M(v))g_\varphi(v,w)$$

is a Riemannian metric on M. Note that if $g_\varphi(v,w)$ is not defined, $p_\varphi(\pi_M(v))$ is zero so the definition makes sense.

Recall (e.g. from Hirsch, 1994, Section 2.2) that every finite-dimensional space is C^∞-regular. It is also C^∞-paracompact, as a result by Toruńczyk (see Kriegl and Michor, 1997, Corollary 16.16) shows that every Hilbert space is C^∞-paracompact. We shall now recall some results about C^k-regularity of locally convex (and, in particular, Banach) spaces.

A.28 Proposition (Bonic and Frampton, 1966) *A locally convex space is* C^k*-regular if and only if the topology is initial with respect to the functions* $C^k(E,\mathbb{R})$.

Proof The initial topology with respect to the C^k-functions is generated by the subbase

$$f^{-1}(]a,b[), \qquad f \in C^k(E,\mathbb{R}), a,b \in \mathbb{R}\cup\{\pm\infty\}.$$

[1] C^0-paracompact is equivalent to the usual notion of paracompactness due to Engelking (1989, Theorem 5.1.9).

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Hence it is clear that if E is C^k-regular the topology is initial with respect to the C^k functions. For the converse, consider $x \in U$, where U is open in the initial topology. Then we find $a_1, \ldots, a_n, b_1, \ldots, b_n$ and $f_1, \ldots, f_n \in C^k(E, \mathbb{R})$ for $n \in \mathbb{N}$ such that $x \in \bigcap_{1 \leq i \leq n} f_i^{-1}(]a_i, b_i[) \subseteq U$. Adjusting choices of the f_i, we may assume without loss of generality that $a_i = -\varepsilon_i$ and $b_i = \varepsilon_i$ for some $\varepsilon_i > 0$. Since $\mathbb{R}$ is C^∞-regular, we pick $h \in C^\infty(\mathbb{R}, \mathbb{R})$ with $h(0) = 1$ and $h(t) = 0$, for all $|t| \geq 1$. Then $f \colon E \to \mathbb{R}$, $y \mapsto \prod_{1 \leq i \leq n} h(f_i(x)/\varepsilon_i)$ is a bump function with carrier in U. □

For a Banach space the existence of C^k-bump functions is tied to differentiability of the norm.

A.29 Definition Let $(E, \|\cdot\|)$ be a normed space. The norm $\|\cdot\| \colon E \to \mathbb{R}$ is *rough* if there exists an $\varepsilon > 0$ such that for every $x \in E$ with $\|x\| = 1$, there exists $v \in E$ with $\|v\| = 1$ and

$$\limsup_{t \searrow 0} \frac{\|x + tv\| + \|x - tv\| - 2}{t} \geq \varepsilon.$$

If a Banach space is C^1-regular then it does not admit a rough norm (see Kriegl and Michor, 1997, 14.11).

One can prove that the Banach spaces $(C([0,1], \mathbb{R}), \|\cdot\|_\infty)$ (i.e. the continuous functions with the compact open topology) and $(\ell_1, \|\cdot\|_1)$ (see Kriegl and Michor, 1997, 13.11 and 13.12) have rough norms, whence they are not even C^1-regular. On the other hand, since nuclear Fréchet spaces are C^∞-regular, the space $C^\infty([0,1], \mathbb{R})$ is C^∞-regular. Similar statements then hold for spaces of smooth sections into bundles. Again we refer to Kriegl and Michor (1997).

A.5 Inverse Function Theorem beyond Banach Spaces

Before we conclude this appendix on locally convex spaces, let us briefly discuss (the lack of) an important tool from calculus which is driving many basic results in (finite-dimensional) differential geometry. Many basic existence results and constructions in finite-dimensional differential geometry are more or less direct consequences of the inverse function theorem, the constant rank theorem and its cousin the implicit function theorem. Note that the inverse function and the implicit function theorems still hold in Banach spaces (Lang, 1999, I, §5), but the constant rank theorem is already more delicate; see Margalef-Roig and Domínguez (1992). Beyond Banach spaces, the situation breaks down as the following example, due to Hamilton (1982, I. 5.5.1), shows (see also 1.55).

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A.30 Example Consider the Fréchet space $C^\infty([-1,1],\mathbb{R})$ of all smooth functions from $[-1,1]$ to $\mathbb{R}$,[2] together with the differential operator

$$P\colon C^\infty([-1,1],\mathbb{R}) \to C^\infty([-1,1],\mathbb{R}), \quad P(c)(x) \coloneqq c(x) - xc(x)c'(x).$$

A computation shows that P is C^∞ with derivative $dP(c,g)(x) = g(x) - xg(x)c'(x) - xc(x)g'(x)$, that is, for $c \equiv 0$ the derivative is the identity. However, the image of P is no 0-neighbourhood in $C^\infty([-1,1],\mathbb{R})$ as it does not contain any of the functions $g_n(x) \coloneqq \frac{1}{n} + \frac{x^n}{n!}$ for $n \in \mathbb{N}$ (but $\lim_{n\to\infty} g_n = 0$ in $C^\infty([-1,1],\mathbb{R})$). In conclusion, the inverse function theorem does not hold for P.

To give a more geometric example, the exponential map of a Lie group (modelled e.g. on a Fréchet space) might fail to even be a local diffeomorphism around the identity. For example, this happens for diffeomorphism groups; see Example 3.42.

A.31 (How to recover an inverse function theorem) The calculi discussed so far are too weak to provide an inverse function theorem on their own. If one has more information (such as metric estimates in the Fréchet setting) there are inverse function theorems that can apply in more general situations. The most famous one is certainly the Nash–Moser inverse function theorem (Hamilton, 1982) which works with so-called tame maps on tame spaces. Further generalised theorems are available in the framework of Müller's bounded geometry (Müller, 2008) and Glöckner's inverse function theorems; see Glöckner (2006b, 2007) and the references therein. To keep the exposition short we do not provide details here. Note, though, that the generalisations mentioned require specific settings or certain estimates which are often hard to check in applications.

A consequence of the lack of an inverse function theorem is that in infinite-dimensional differential geometry one needs to be careful when considering the notions of immersion and submersion (see §1.7). Further, there is no general solution theory for ordinary differential equations beyond Banach spaces (even for linear differential equations!).

Exercises

A.5.1 Fill in the missing details for Example A.30: Show that the differential operator P is differentiable and compute its derivative. For $g_n(x) =$

[2] The Fréchet space structure is given pointwise operations with the compact open C^∞-topology. It is defined via the metric $d(f,g) \coloneqq \sum_{i=0}^\infty \frac{\|f-g\|_i}{2^i}$, where $\|f\|_i$ is the supremum norm of the ith derivative.

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$\frac{1}{n} + \frac{x^n}{n!}$ show that $g_n \to 0$ in the compact open C^∞-topology. Let f be a smooth map on $[-1,1]$. Develop f and $P(f)$ into their Taylor series at 0. Then show that $P(f) = g_n$ is impossible.

A.6 Differential Equations beyond Banach Spaces

Here we exhibit several examples of ill-posed differential and integral equation on locally convex spaces.

A.32 Example (No solutions in incomplete spaces (Dahmen)) Consider the mapping

$$h\colon [0,1] \to C^\infty([1,2],\mathbb{R}), \quad t \mapsto (x \mapsto x^t).$$

We apply the exponential law to h. Note, however, that Theorem 2.12 is not quite sufficient as $[0,1]$ and $[1,2]$ are manifolds with boundary, whence we refer to the more general exponential law (Alzaareer and Schmeding, 2015, Theorem A). Now h is smooth if and only if $h^\wedge\colon [0,1]\times[1,2] \to \mathbb{R}$, $(t,x) \mapsto x^t = \exp(\ln(x)t)$ is smooth, that is, $h^\wedge$ is smooth on the interior of the square $[0,1]\times[1,2]$ such that the derivatives extend continuously onto the boundary. This is a trivial calculation. Since the derivative of h corresponds via the exponential law to the partial derivative of $h^\wedge$ (see Lemma 2.10), we see that $dh(t;y)(x) = (d_1 h^\wedge(t,\cdot;y))^\vee(x) = y\ln(x)x^t$. Let us define now two subspaces of $C^\infty([1,2],\mathbb{R})$ as the locally convex spaces generated by the image of h and $h' = dh(\cdot;1)$:

$$E := \operatorname{span}\{h(t) \mid t \in [0,1]\}, \qquad F := \operatorname{span}\{h'(t) \mid t \in [0,1]\}.$$

By construction $h(t) \notin F$ and $h'(t) \notin E$. This entails that $h|^E\colon [0,1] \to E \subseteq C^\infty([1,2],\mathbb{R})$ is not differentiable and shows that sequential closedness is indispensable in Lemma 1.25. As a consequence, both subspaces are not closed and, in particular, not (Mackey) complete. We see that the (trivial) differential equation $\gamma'(t) = h'(t)$ or equivalently the (weak) integral equation $\gamma(t) = \int_0^1 h'(t)\mathrm{d}t$ does not admit a solution in F.

It should not come as a surprise that in the absence of suitable completeness properties differential and integral equations may be ill posed. However, even in complete spaces relative benign (e.g. linear), differential equations do not admit solutions.

A.33 Example (Hamilton, 1982, I.5.6.1) Consider the Fréchet space $C^\infty([0,1],\mathbb{R})$ of smooth functions endowed with the compact open C^∞-topology.

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Recall that the differential operator $D\colon C^\infty([0,1],\mathbb{R}) \to C^\infty([0,1],\mathbb{R})$, $f \mapsto f' = df(\cdot;1)$ is continuous and linear. Consider a solution $f\colon]-\varepsilon,\varepsilon[\to C^\infty([0,1],\mathbb{R})$ of the linear differential equation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} f(t) &= D(f) = f', \\ f(0) &= f_0. \end{cases} \tag{A.2}$$

Apply the exponential law (Alzaareer and Schmeding, 2015, Theorem A) to the smooth map f. We obtain a smooth map $f^\wedge\colon]\varepsilon,\varepsilon[\times[0,1] \to \mathbb{R}$, $(t,x) \mapsto f(t)(x)$ such that (A.2) is equivalent to the partial differential equation

$$\partial_t f^\wedge(t,x) = \partial_x f^\wedge(t,x), \quad f(0,x) = f_0(x). \tag{A.3}$$

By the Whitney extension theorem (Whitney, 1934) there is a (non-unique) extension $F_0 \in C^\infty([0,2],\mathbb{R})$ of f_0 and it is easy to see that the function $f(t,x) := F_0(x+t), t,x \in [0,1]$ solves (A.3). Since the extension F_0 is non-unique, the solution to (A.2) is non-unique (albeit we study a linear ordinary differential equation with smooth right-hand side!).

A related example is given by the heat equation on the circle.

A.34 Example (Heat equation on $\mathbb{S}^1$ (Milnor, 1982, Example 6.3)) The heat equation on the circle $\mathbb{S}^1$ is given by

$$\partial_t f(t,\theta) = \partial_\theta^2 f(t,\theta),$$

where ∂_θ is the derivative on $\mathbb{S}^1$. Again the derivative induces a continuous linear derivative operator $D\colon C^\infty(\mathbb{S}^1,\mathbb{R}) \to C^\infty(\mathbb{S}^1,\mathbb{R})$, $f \mapsto \frac{\mathrm{d}}{\mathrm{d}\theta} f$, whence the heat equation can be understood as an ordinary differential equation on $C^\infty(\mathbb{S}^1,\mathbb{R})$. We do not go into the details concerning solutions of this equation. However, the reader may want to refer to Chapter 7 for examples of partial differential equations which are treated using similar techniques for ordinary differential equations on infinite-dimensional manifolds.

The following examples are due to Milnor (see Milnor, 1982, Examples 6.1 and 6.2).

A.35 Example (Too many solutions) Let $\mathbb{R}^\mathbb{N}$ be the Fréchet space of real-valued sequences (with the topology induced by identifying $\mathbb{R}^\mathbb{N} \cong \prod_{n\in\mathbb{N}} \mathbb{R}$) and define the *left shift*

$$\Lambda\colon \mathbb{R}^\mathbb{N} \to \mathbb{R}^\mathbb{N}, \quad (x_1,x_2,x_3,\ldots) \mapsto (x_2,x_3,\ldots).$$

Then Λ is continuous and linear and the differential equation $\mathbf{y}'(t) = \Lambda(\mathbf{y})(t)$ reduces to the system of equations $y_i'(t) = y_{i+1}(t), i \in \mathbb{N}$. For every initial value this system has infinitely many solutions.

To see this consider the initial condition $\mathbf{y}(0) = \mathbf{0} \in \mathbb{R}^{\mathbb{N}}$. If $\varphi \in C^\infty(\mathbb{R},\mathbb{R})$ vanishes in a neighbourhood of 0, then we set $y_1(t) = \varphi(t)$ and $y_n(t) = \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}\varphi(t)$ for $n \geq 2$ to obtain a solution $\mathbf{y}(t) = (y_1(t), y_2(t), \ldots)$ of the equation with $\mathbf{y}(0) = \mathbf{0}$.

A.36 Example (No solutions) The space $E := \{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \mid \text{ almost all } x_n = 0\}$ is locally convex with respect to the box topology (i.e. the topology whose basis is given by the sets $\prod_{n\in\mathbb{N}} U_n, U_n \Subset \mathbb{R}$; see Exercise A.6.2). Define the *right shift*

$$R\colon E \to E, \quad (x_1, x_2, x_3, \ldots) \mapsto (0, x_1, x_2, x_3, \ldots).$$

Then it is not hard to see that R is continuous linear and we consider the initial value problem

$$\begin{cases} \mathbf{y}' &= R(\mathbf{y})(t), \\ \mathbf{y}(0) &= (1,0,0\ldots). \end{cases}$$

Note that for a prospective solution $\mathbf{y}(t) = (y_1(t), y_2(t), \ldots)$, the differential equation yields $y_1'(t) = 0$, whence $y_1(t) \equiv 1$ by the initial condition. Then $y_2'(t) = y_1(t) = 1$. Integrating, we see that inductively y is a solution if $y_i(t) = \frac{t^{i-1}}{(i-1)!}$. However, for $t \neq 0$ this sequence has infinitely many terms not equal to 0 and thus does not exist in E, that is, the initial value problem (given by a linear differential equation with smooth right-hand side!) does not have any solution in E.

Exercises

A.6.1 Review §2.2 and Theorem 2.12 to work out the details of the identification of the Partial Differential Equations heat equation with the Ordinary Differential Equations on $C^\infty(\mathbb{S}^1,\mathbb{R})$ in Example A.34.

A.6.2 Consider the space $E := \{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \mid \text{ almost all } x_n = 0\}$ with the box topology (i.e. the topology induced by $E \cap \prod_{n\in\mathbb{N}} U_i$, where $U_i \Subset \mathbb{R}$). Show that:

(a) E is a locally convex space and the box topology is properly finer than the subspace topology induced by $E \subseteq \mathbb{R}^{\mathbb{N}}$. Then show that R is continuous.

(b) Every base of 0-neighbourhoods in E is necessarily uncountable, so E cannot be a metrisable space.

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A.7 Another Approach to Calculus: Convenient Calculus

Convenient calculus was introduced by Frölicher and Kriegl (1988) (see Kriegl and Michor (1997) for an introduction). To understand the basic idea, we recall a theorem by Boman.

A.37 Theorem (Boman's theorem (Boman, 1967)) *A map $f \colon \mathbb{R}^d \to \mathbb{R}$, $d \in \mathbb{N}$ is smooth if and only if for each smooth curve $c \colon \mathbb{R} \to \mathbb{R}^d$ the curve $f \circ c$ is smooth.*[3]

Now smoothness of curves $c \colon \mathbb{R} \to E$ with values in a locally convex space E is canonically defined (e.g. via Definition 1.3). Hence smooth curves can be used to define (conveniently) smooth maps on locally convex spaces which are Mackey complete.[4] Following the usual lingo of convenient calculus, a locally convex space which is Mackey complete is called *convenient vector space*.

A.38 Definition Let E, F be convenient vector spaces.

(a) Write $c^\infty E$ for E endowed with the final topology with respect to all smooth curves $\mathbb{R} \to E$. We call this topology the c^∞*-topology* and its open sets c^∞*-open*.
(b) Let $U \subseteq E$ be c^∞-open, $f \colon U \to F$ a map. Then f is *convenient smooth* or C^∞_{conv} if $f \circ c \colon \mathbb{R} \to F$ is a smooth curve for every smooth curve $c \colon \mathbb{R} \to U$.

Obviously, the chain rule holds for conveniently smooth mappings and one can define and study derivatives. Once again manifolds, tangent spaces and so on make sense. Further, Boman's theorem asserts that between finite-dimensional spaces convenient smooth coincides with Fréchet smooth (more on this in A.41).

A.39 Remark (Bornology vs. topology) By now, the reader will have wondered why one defines C^∞_{conv}-maps on c^∞-open subsets instead of using the native topology on E. The reason for this is that differentiability of curves into a locally convex space does not depend on the topology of E, but rather on the bounded sets, the *bornology* of E. One can show that smoothness of curves is a bornological concept, and this is captured by the c^∞-topology (which is finer than the native topology but induces the same bornology).

The c^∞-topology is somewhat delicate to handle. For example, $c^\infty(E \times F) \neq c^\infty E \times c^\infty F$ (in general) and $c^\infty E$ will not be a topological vector space. However, it can be shown that for a Fréchet space the c^∞-topology coincides with the Fréchet space topology; see Kriegl and Michor (1997, Section I.4).

[3] We emphasise here that smoothness can be tested against smooth curves, while this becomes false for finite orders of differentiability.
[4] See Definition 1.12. Mackey completeness is weaker than sequential completeness.

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By definition, a conveniently smooth map is continuous with respect to the c^∞-topology on E. However, as the c^∞-topology is finer than the native locally convex topology, conveniently smooth maps may fail to be continuous with respect to the native topologies. To stress it once more: **Differentiability in the convenient calculus is *not built on top of continuity***. Having now defined two notions of calculus we will clarify their relation in the next section and discuss some of their properties.

Bastiani versus Convenient Calculus

We have already seen in the last sections that both the convenient and the Bastiani calculus yield the well-known concept of (Fréchet) smooth maps between finite-dimensional vector spaces. To clarify the relation between Bastiani and convenient calculus, observe that the definition of smooth curves into a locally convex space coincides in both calculi. The Bastiani chain rule, Proposition 1.23, yields the following.

A.40 Lemma *Let E, F be convenient vector spaces, $U \overset{\circ}{\subseteq} E$ and $f\colon E \supseteq U \to F$ a C^∞-map. Then f is C^∞_{conv}.*

Thus (completeness properties aside) Bastiani smoothness is the stronger and more restrictive concept, which enforces continuity with respect to the native topologies. However, on Fréchet spaces both calculi coincide (Bertram et al., 2004, Theorem 12.4).

A.41 Let E, F be convenient spaces, $U \overset{\circ}{\subseteq} E$. Then the differentiability classes of a map $f\colon U \to F$ are related as follows:

$$C^\infty_{\mathrm{conv}} \underset{E\,\text{Fréchet}}{\overset{\Longleftarrow}{\Longrightarrow}} C^\infty \underset{E,F\,\text{normed}}{\Longleftrightarrow} FC^\infty. \tag{A.4}$$

The dividing line between convenient calculus and Bastiani calculus is continuity; see Glöckner (2006a) for examples of discontinuous conveniently smooth maps. Also see Kriegl and Michor (1997, Theorem 4.11) for more information on spaces on which the concepts coincide.

One may ask oneself now if there is one calculus which is preferable over the other.

A.42 (Bastiani calculus is more convenient than convenient calculus) A major difference between Bastiani and convenient calculus is continuity. Arguably continuity with respect to the native topologies, as in the Bastiani calculus, is desirable for smooth maps. In particular, the infinite-dimensional spaces and

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manifolds often have an intrinsic topology one would rather like to preserve instead of having to deal with the somewhat delicate c^∞-topology. This is one reason why in infinite-dimensional Lie theory, Bastiani calculus is prevalent (continuity allows one to use techniques from topological group theory, such as the local-to-global result for Lie groups: Proposition 3.45).

In addition, one can often conveniently establish Bastiani smoothness using induction arguments over the order of differentiability[5] and one can interpret this (together with the continuity) as an argument for the naturality of Bastiani calculus.

A.43 (Convenient calculus is more convenient than Bastiani calculus) Discarding continuity for smooth maps might not be as exotic after all since smoothness of curves is a bornological and not a topological property. Further, it leads to a convenient category of spaces with smooth maps.

To explain this, consider sets X, Y, Z and denote by Z^X the set of all maps from X to Z. Then $f \in Z^{X\times Y}$ induces a map $f^\vee \in (Z^Y)^X$ via $f^\vee(x)(y) = f(x,y)$. The resulting bijection $Z^{X\times Y} \cong (Z^Y)^X$ is known as the exponential law.

Exponential Law for Convenient Smooth Maps *Let E, F, G be locally convex vector spaces, $U \subseteq E, V \subseteq F$ c^∞-open. Then the spaces of convenient smooth maps admit a locally convex topology such that there is a linear convenient smooth diffeomorphism*

$$C^\infty_{\text{conv}}(U \times V, G) \cong C^\infty_{\text{conv}}(U, C^\infty_{\text{conv}}(V,G)), \quad f \mapsto f^\vee.$$

In particular, $f^\vee : U \to C^\infty_{\text{conv}}(V,G)$ is C^∞_{conv} if and only if $f : U \times V \to G$ is C^∞_{conv}.

The exponential law is an immensely important tool, simplifying many proofs (e.g. that the diffeomorphism group is a Lie group). It also establishes cartesian closedness of the category of convenient vector spaces with convenient smooth maps.[6] Note that the exponential law and cartesian closedness do not hold in the Bastiani setting.[7] We remark, though, that cartesian closedness in the convenient setting is a statement about the category of convenient vector spaces. The result does not carry over to the category of manifolds modelled on convenient vector spaces which turns out to be *not cartesian closed*. To

[5] Here one should mention that a similar but somewhat more involved notion of k-times Lipschitz differentiable mappings exists in the convenient setting.

[6] The 'convenient' in convenient calculus references Steenrod's 'A convenient category of topological spaces' Steenrod (1967) in which a cartesian closed category of topological spaces is built.

[7] Albeit rudiments of an exponential law exist in the Bastiani setting as §2.2 shows. See also Alzaareer and Schmeding (2015) for a stronger version.

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obtain a cartesian closed category of manifolds it seems to be unavoidable to pass to even more general concepts such as diffeological spaces (see Iglesias-Zemmour, 2013).

Summing up, there is no clear-cut answer to the question of which calculus is preferable. It will depend on the application or use one has in mind whether one goes with the stronger notion of Bastiani calculus or the weaker convenient calculus (which often has more convenient tools).