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On the number of periodic reflecting rays in generic domains

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Abstract. We prove that for generic domains $\Omega \subset \mathbb{R}^n$ with smooth boundary X for every integer $s \geq 2$ there is at most a finite number of periodic reflecting rays with just s reflections on X.

1. *Introduction*

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with C^∞ boundary $\partial\Omega = X$. We proved in [**9**] (see also [**7**]) that for generic domains Ω there exists at most a countable number of periodic multiple reflecting rays with reflections on X. In this work we strengthen this result showing that for generic X and every $s \in \mathbb{N}$, $s \geq 2$, we have

$$P_X(s) < \infty. \tag{1.1}$$

Here $P_X(s)$ denotes the number of periodic reflecting rays having just s reflections on X.

The growth of the number $P(T)$ of periodic geodesics of length less than or equal to T has been studied for Riemannian manifolds without boundary. In particular, for manifolds with metric of negative curvature $\lim \log_{T\to\infty} P(T)/T$ exists and is equal to the topological entropy of the geodesic flow (see [**5**], [**2**]). For some domains $\Omega \subset \mathbb{R}^2$ with boundary some results concerning the growth of $P(T)$ can be obtained from the estimate from above of metric entropy ([**3**], [**4**]).

In this work we study generic domains Ω without any restrictions on the geometry of the boundary X and on the dimension n. On the other hand, we establish only (1.1) and we are not able to obtain any information on the behavior of $P_X(s)$ as $s \to \infty$. It should be mentioned that some domains admit periodic generalized geodesics (cf. [**6**] for the definition) which contain geodesics lying on the boundary X and linear segments passing through inflection points of X. Moreover, a periodic reflecting ray could have segments tangent to X.

Our analysis is based essentially on two results. First, we apply a Kupka-Smale type theorem proved in our previous work [**10**] (see also [11]). This theorem says that generically the spectrum of the linear Poincaré map related to every periodic reflecting ray does not contain roots of unity. Secondly, we show below that for generic X there are no periodic reflecting rays having segment tangent to X. For $n = 2$ this result is contained in [**12**].

Let $C^\infty(X, \mathbb{R}^n)$ be the space of all C^∞ maps $X \to \mathbb{R}^n$ endowed with the Whitney topology (see Ch. II in [**1**]). The subspace $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ of all C^∞ embeddings is open in $C^\infty(X, \mathbb{R}^n)$, hence it is a Baire space. Recall that a subset A of a topological space B is called residual in B if A is a countable intersection of open dense subsets of B. A precise definition of a periodic reflecting ray is given in § 2.

Our main result is the following:

THEOREM 1.1. *Let X be a smooth compact $(n-1)$-dimensional submanifold of $\mathbb{R}^n$, $n \geq 2$, and $\mathcal{F}$ be the set of those $f \in C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ such that $P_{f(X)}(s) < \infty$ for every $s \geq 2$. Then $\mathcal{F}$ contains a residual subset of $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$.*

The paper is organized as follows. In § 2 we give some definitions and preliminary facts. It is shown in § 3 that generic X do not admit period reflecting rays having segments tangent to X. Theorem 1.1 is proved in § 4. Finally, in § 5 we prove similar results for so called (ω, θ)-rays. Here ω and θ are two fixed vectors of S^{n-1}, and an (ω, θ)-ray contains two straight line rays with directions respectively ω and θ and a finite number of linear segments reflecting on X. The results concerning (ω, θ)-rays are important for applications in scattering theory (**[8]**, **[11]**).

2. *Preliminaries*

(2.1) By a segment in $\mathbb{R}^n$ we mean a finite segment $l = [x, y] = \{z \in \mathbb{R}^n\colon\ z = px + (1-p)y, 0 \leq p \leq 1\}$ or an infinite segment, that is a straight line ray starting at some point x and having a given direction v. If l_1 and l_2 are two segments with a common end $x \in X$, X being a smooth $(n-1)$-dimensional submanifold $\mathbb{R}^n$, we say that l_1 and l_2 *satisfy the reflection law at* x (with respect to X) if l_1 and l_2 make equal acute angles with a normal vector $N_x \neq 0$ to X at x, and l_1, l_2 and N_x lie in a common two-dimensional plane.

Let γ be a closed curve in $\mathbb{R}^n$ of the form $\gamma = \bigcup_{i=1}^k l_i$, where $l_i = [x_i, x_{i+1}]$, $x_i \in X$, $i = 1, \ldots, k$, $x_{k+1} = x_1$. The curve γ will be called a periodic reflecting ray on X if the following conditions hold:

(i) the open segments $\mathring{l}_i$ do not intersect X transversally;

(ii) setting $l_{k+1} = l_1$, for every $i = 1, \ldots, k$ the segments l_i and l_{i+1} satisfy the reflection law at x_{i+1}.

The points $x_1, \ldots, x_k$ (some of which may coincide) are called *reflection points* of γ. Note that some segment l_i of γ could be tangent to X at some interior point of l_i. If some l_i is orthogonal to X at x_i or x_{i+1}, then γ will be called a symmetric ray, and in the opposite case γ is said to be a non-symmetric ray.

(2.2) Denote by $\mathcal{A}$ the set of those $f \in C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ so that for any non-symmetric (symmetric) periodic reflecting ray γ on $f(X)$ there exist different points $y_1, \ldots, y_s$ such that $y_1, \ldots, y_s, y_1$ (resp. $y_1, \ldots, y_{s-1}, y_s, y_{s-1}, \ldots, y_1$) are all the successive reflection points of γ. According to theorem A in **[13]**, the set $\mathcal{A}$ contains a residual subset of $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$.

A ray γ on X will be called *tangent to* X if it has a segment tangent to X. If γ is not tangent to X, it will be called an ordinary reflecting ray.

Let Q be a countable set of non-zero complex numbers and denote by T_Q the set of those $f \in C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ so that the spectrum of the linear Poincaré map related to every ordinary periodic reflecting ray on $f(X)$ does not contain elements of Q. It is proved in **[10]** (see also **[11]**) that T_Q contains a residual subset of $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$.

(2.3) Fix an integer $s \geq 2$ and consider the bundle $J^1(X, \mathbb{R}^n)$ of 1-jets. We will recall some standard notation (see Ch. II in [1] for details). Let $\alpha : J^1(X, \mathbb{R}^n) \to X$ and $\beta : J^1(X, \mathbb{R}^n) \to \mathbb{R}^n$ be the maps defined by

$$\alpha(j^1 f(x)) = x \quad \text{and} \quad \beta(j^1 f(x)) = f(x)$$

for every element $j^1 f(x)$ of $J^1(X, \mathbb{R}^n)$. The s-fold bundle of 1-jets, $J^1_s(X, \mathbb{R}^n)$, is given by $J^1_s(X, \mathbb{R}^n) = (\alpha^s)^{-1}(X^{(s)})$. Here we use the notation

$$A^{(s)} = \{(a_1, \dots, a_s) \in A^s : a_i \neq a_j \quad \text{whenever} \quad i \neq j\},$$

and having a map $g : A \to B$ we define $g^s : A^s \to B^s$ by $g^s(a_1, \dots, a_s) = (g(a_1), \dots, g(a_s))$.

Clearly, the set

$$V = \{j^1 f(x) \in J^1(X, \mathbb{R}^n) : \text{rank } df(x) = n - 1\} \tag{2.1}$$

is open in $J^1(X, \mathbb{R}^n)$. Therefore, if U is an open subset of $(\mathbb{R}^n)^{(s)}$, then the set

$$M = (\alpha^s)^{-1}(X^{(s)}) \cap (\beta^s)^{-1}(U) \cap V^s \tag{2.2}$$

becomes an open submanifold of $J^1_s(X, \mathbb{R}^n)$.

(2.4) We will describe an atlas on M. Take coordinate neighbourhoods $V_1, \dots, V_s$ in X with $V_i \cap V_j = \varnothing$ for $i \neq j$, and fix charts $\varphi_i : V_i \to \mathbb{R}^{n-1}$. Consider

$$\Omega = M \cap \prod_{i=1}^{s} J^1(V_i, \mathbb{R}^n) \tag{2.3}$$

and define the chart φ on Ω by

$$\varphi(j^1 f_1(x_1), \dots, j^1 f_s(x_s)) = (u; v; a) \in (\mathbb{R}^{n-1})^{(s)} \times (\mathbb{R}^n)^{(s)} \times \mathbb{R}^{ns(n-1)}, \tag{2.4}$$

where

$$u = (u_1, \dots, u_s), \quad v = (v_1, \dots, v_s), \quad a = (a_{ij}^{(t)})_{\substack{1 \leq i \leq s,\, 1 \leq j \leq n-1 \\ 1 \leq t \leq n}} \tag{2.5}$$

and

$$u_i = \varphi_i(x_i), \qquad v_i = f_i(x_i), \tag{2.6}$$

$$a_{ij}^{(t)} = \frac{\partial (f_i^{(t)} \circ \varphi_i^{-1})}{\partial u_i^{(j)}}(u_i) \tag{2.7}$$

for $1 \leq i \leq s$, $1 \leq j \leq n-1$, $1 \leq t \leq n$. Here $f_i = (f_i^{(1)}, \dots, f_i^{(n)})$, $u_i = (u_i^{(1)}, \dots, u_i^{(n-1)}) \in \mathbb{R}^{n-1}$, $v_i = (v_i^{(1)}, \dots, v_i^{(n)}) \in \mathbb{R}^n$. Notice that the vector $N_i = (N_i^{(1)}, \dots, N_i^{(n)})$ given by

$$N_i^{(t)} = (-1)^t \det \begin{pmatrix} a_{i1}^{(1)} & \cdots & a_{i1}^{(t-1)} & a_{i1}^{(t+1)} & \cdots & a_{i1}^{(n)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{in-1}^{(1)} & \cdots & a_{in-1}^{(t-1)} & a_{in-1}^{(t+1)} & \cdots & a_{in-1}^{(n)} \end{pmatrix} \tag{2.8}$$

is a normal vector to $f_i(X)$ at $f_i(x_i)$.

(2.5) Fix an integer $s \geq 2$ and set for convenience $y_{s+1} = y_1$ and $y_{-1} = y_s$. Define

$$U_s = \{(y_1, \dots, y_s) \in (\mathbb{R}^n)^{(s)} : y_i \notin [y_{i-1}, y_{i+1}] \text{ for } i = 1, \dots, s\} \tag{2.9}$$

and the function $F : U_s \to \mathbb{R}$ by

$$F(y_1, \dots, y_s) = \sum_{i=1}^{s} \|y_i - y_{i+1}\|. \tag{2.10}$$

Obviously, U_s is open in $(\mathbb{R}^n)^{(s)}$ and F is smooth on U_s. Moreover, if γ is a non-symmetric periodic reflecting ray with successive reflection points $y_1, \ldots, y_s$, then $y=(y_1, \ldots, y_s) \in U_s$ and $F(y)$ is equal to the length (period) of γ.

Next, set

$$U'_s = \{(y_1, \ldots, y_s) \in (\mathbb{R}^n)^{(s)}: y_i \notin [y_{i-1}, y_{i+1}], i=2, \ldots, s-1\} \tag{2.11}$$

and define $F_1 : U'_s \to \mathbb{R}$ *by*

$$F_1(y_1, \ldots, y_s) = \sum_{i=1}^{s-1} \|y_i - y_{i+1}\|. \tag{2.12}$$

If γ is a symmetric periodic reflecting ray on $Y=f(X)$ with successive reflection points $y_1, \ldots, y_s$, where $y_{s-1}-y_s$ is orthogonal to Y at y_s, then we have $y=(y_1, \ldots, y_s) \in U'_s$, and $2F_1(y)$ equals the length (period) of γ.

(2.6) Let M be a smooth manifold and Σ be an arbitrary subset of M. Denote by $\operatorname{Dim}\Sigma$ the smallest integer $r=0, 1, \ldots, \dim M$ such that there is a finite or countable number smooth submanifolds W_m of M with $\Sigma \subset \bigcup_m W_m$ and $\dim W_m \le r$ for every m. We set

$$\operatorname{Codim}\Sigma = \dim M - \operatorname{Dim}\Sigma.$$

3. *Reflecting rays with tangent segments*

Our aim in this section is to prove the following result.

THEOREM 3.1. *Let X be as in theorem* 1.1 *and $\mathcal{T}$ be the set of those $f \in C^\infty_{\mathrm{emb}}(X, \mathbb{R}^n)$ such that there are no periodic reflecting rays on $f(X)$ which are tangent to $f(X)$. Then $\mathcal{T}$ contains a residual subset of $C^\infty_{\mathrm{emb}}(X, \mathbb{R}^n)$.*

A non-symmetric (resp. symmetric) periodic reflecting ray γ will be called simple if there exist different points $y_1, \ldots, y_s$ such that

$$y_1, \ldots, y_s \text{ (resp. } y_1, \ldots, y_s, y_{s-1}, \ldots, y_1)$$

are all the successive reflection points of γ. Denote by $\mathcal{T}_1$ (resp. $\mathcal{T}_2$) the set of those $f \in C^\infty_{\mathrm{emb}}(X, \mathbb{R}^n)$ so that there are no simple non-symmetric (resp. symmetric) periodic reflecting rays on $f(X)$ which are tangent to $f(X)$. According to [**13**] (see 2.2) we have

$$\mathcal{T}_1 \cap \mathcal{T}_2 \cap \mathcal{A} \subset \mathcal{T},$$

where $\mathcal{A}$ is the set introduced in 2.2. Thus theorem 3.1 will be proved if we show both $\mathcal{T}_1$ and $\mathcal{T}_2$ contain residual subsets of $C^\infty_{\mathrm{emb}}(X, \mathbb{R}^n)$.

First we consider the set $\mathcal{T}_1$. Fix an integer $s \ge 2$ and let U_s and F be given by (2.9) and (2.10). Denote by $\mathcal{B}_0^{(s)}$ the set of those $f \in C^\infty_{\mathrm{emb}}(X, R^n)$ for which there are no points $(x_0, x_1, \ldots, x_s)$ in $X^{(s+1)}$ such that $x=(x_1, \ldots, x_s)$ is a critical point of $F \circ f^s$ with $f^s(x) \in U_s$ and the segment $[f(x_1), f(x_2)]$ is tangent to $f(X)$ at $f(x_0)$. The last condition is equivalent to

$$\frac{f(x_1)-f(x_0)}{\|f(x_1)-f(x_0)\|} + \frac{f(x_2)-f(x_0)}{\|f(x_2)-f(x_0)\|} = 0$$

and $\langle f(x_1)-f(x_2), N\rangle = 0$, N being a non-zero normal vector to $f(X)$ at $f(x_0)$.

Clearly, $f \in \mathscr{B}_0^{(s)}$ expresses the fact that every simple non-symmetric periodic reflecting ray γ on $f(X)$ with exactly s reflection points has no segments tangent to $f(X)$ at points which are not reflection points of γ.

To prove $\mathscr{B}_0^{(s)}$ contains a residual set we need the following:

LEMMA 3.2. *Let $p = 1, \ldots, n$ and let Q_p be the set of those $v = (v_0, v_1, v_2) \in (\mathbb{R}^n)^{(3)}$ such that $v_1^{(p)} \neq v_0^{(p)}$. Introduce the functions $d^{(m)}: Q_p \to \mathbb{R}$ given by*

$$d^{(m)}(v) = (v_1^{(m)} - v_0^{(m)})/\|v_1 - v_0\| + (v_2^{(m)} - v_0^{(m)})/\|v_2 - v_0\|$$

for $m = 1, \ldots, n$, $m \neq p$. Then the vectors grad $d^{(m)}(v)$, $m = 1, \ldots, n$, $m \neq p$, *are linearly independent over $\mathbb{R}$ for every $v \in Q_p$.*

Proof. Suppose $v \in Q_p$ and

$$\sum_{\substack{m=1\\ m\neq p}}^{n} D_m \operatorname{grad} d^{(m)}(v) = 0 \tag{3.1}$$

for some constants D_m. Set $D_p = 0$ and $D = (D_1, \ldots, D_n) \in \mathbb{R}^n$. It is convenient to introduce the notation $w_1 = (v_1 - v_0)/\|v_1 - v_0\|$, $w_2 = (v_2 - v_0)/\|v_2 - v_0\|$, $z_1 = 1/\|v_1 - v_0\|$, $z_2 = 1/\|v_2 - v_0\|$. It is easy to see now that for every $t = 1, \ldots, n$ we have

$$\frac{\partial d^{(m)}}{\partial v_0^{(t)}}(v) = z_1 w_1^{(m)} w_1^{(t)} + z_2 w_2^{(m)} w_2^{(t)}, \qquad (t \neq m)$$

and

$$\frac{\partial d^{(t)}}{\partial v_0^{(t)}}(v) = -(z_1 + z_2) + z_1 (w_1^{(t)})^2 + z_2 (w_2^{(t)})^2.$$

Therefore, by (3.1) for every t we obtain

$$0 = \sum_{m=1}^{n} D_m \frac{\partial d^{(m)}}{\partial v_0^{(t)}}(v) = \sum_{m=1}^{n} D_m (z_1 w_1^{(m)} w_1^{(t)} + z_2 w_2^{(m)} w_2^{(t)}) - D_t (z_1 + z_2)$$

which is equivalent to

$$(z_1 + z_2) D_t = z_1 \langle D, w_1 \rangle w_1^{(t)} + z_2 \langle D, w_2 \rangle w_2^{(t)}, \qquad t = 1, \ldots, n.$$

Thus we get

$$(z_1 + z_2) D = z_1 \langle D, w_1 \rangle w_1 + z_2 \langle D, w_2 \rangle w_2. \tag{3.2}$$

Taking the inner product of both sides of (3.2) by w_1, we find $\langle D, w_1 \rangle = \langle D, w_2 \rangle \langle w_1, w_2 \rangle$. In a similar way one can see that $\langle D, w_2 \rangle = \langle D, w_1 \rangle \langle w_1, w_2 \rangle$. The last two equalities imply

$$\langle D, w_1 \rangle (1 - \langle w_1, w_2 \rangle^2) = 0 \quad \text{and} \quad \langle D, w_2 \rangle (1 - \langle w_1, w_2 \rangle^2) = 0. \tag{3.3}$$

Assume first that $\langle w_1, w_2 \rangle^2 \neq 1$. Then by (3.3) we conclude that $\langle D, w_1 \rangle = \langle D, w_2 \rangle = 0$ and (3.2) yields $D = 0$. Secondly, assume $\langle w_1, w_2 \rangle^2 = 1$. Then $w_2 = \varepsilon w_1$ with $\varepsilon = \pm 1$ and (3.2) becomes

$$(z_1 + z_2) D = z_1 \langle D, w_1 \rangle w_1 + z_2 \varepsilon^2 \langle D, w_1 \rangle w_1 = (z_1 + z_2) \langle D, w_1 \rangle w_1.$$

Therefore, $D = \langle D, w_1 \rangle w_1$, and comparing the p-components of these vectors, we get $0 = D_p = \langle D, w_1 \rangle w_1^{(p)}$. Since $v \in Q_p$, $w_1^{(p)} \neq 0$, and we obtain $\langle D, w_1 \rangle = 0$. This leads again to $D = 0$ and lemma 3.2 is proved.

LEMMA 3.3. *$\mathscr{B}_0^{(s)}$ contains a residual subset of $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$.*

Proof. Introduce the open submanifold $\hat{M}$ of $J^1_{s+1}(X, \mathbb{R}^n)$ by

$$\hat{M} = J^1_{s+1}(X, \mathbb{R}^n) \cap (V \times M), \tag{3.4}$$

where V is defined by (2.1), while M is given by (2.2) for $U = U_s$. Denote by Σ the set of those $\sigma = (j^1 f_0(x_0), j^1 f_1(x_1), \ldots, j^1 f_s(x_s)) \in \hat{M}$ such that $x = (x_1, \ldots, x_s)$ is a critical point of the map $F \circ (f_1 \times \cdots \times f_s)$,

$$\frac{f_1(x) - f_0(x_0)}{\|f_1(x_1) - f_0(x_0)\|} + \frac{f_2(x_2) - f_0(x_0)}{\|f_2(x_2) - f_0(x_0)\|} = 0$$

and $\langle f_1(x_1) - f_2(x_2), N_0 \rangle = 0$, N_0 being a non-zero normal vector to $f_0(X)$ at $f_0(x_0)$. It is straightfoward to see that

$$\mathscr{B}_0^{(s)} = \{f \in C^\infty_{\text{emb}}(X, \mathbb{R}^n) : j^1_{s+1} f(X^{(s+1)}) \cap \Sigma = \varnothing\}$$

hence, according to the multijet transversality theorem (cf. Ch. II, theorem 4.13 in [**1**]), it is sufficient to show that

$$\operatorname{Codim} \Sigma > (s+1)(n-1) = \dim X^{(s+1)}.$$

Define a chart (φ, Ω) on $\hat{M}$ as we have done this for M in (2.4). More precisely, set $\Omega = \hat{M} \cap \prod_{i=0}^s J^1(V_i, \mathbb{R}^n)$ and for $\sigma = (j^1 f_0(x_0), \ldots, j^1 f_s(x_s)) \in \Omega$ put $\varphi(\sigma) = (u; v; a)$, where $u = (u_0, u_1, \ldots, u_s)$, $v = (v_0, v_1, \ldots, v_s)$, $a = (a_{ij}^{(t)})_{0 \le i \le s, 1 \le j \le n-1, 1 \le t \le n}$ and u_i, v_i and $a_{ij}^{(t)}$ are given by (2.6) and (2.7) for $0 \le i \le s$, $1 \le j \le n-1$, $1 \le t \le n$. We have to prove that $\varphi(\Omega \cap \Sigma)$ is contained in the union of a finite number smooth submanifolds of $\varphi(\Omega)$ of codimension $(s+1)(n-1)+1$ in $\varphi(\Omega)$.

The elements of $\varphi(\Omega)$ have the form

$$\xi = (u; v; a) \in (\mathbb{R}^{n-1})^{(s+1)} \times (\mathbb{R}^n)^{(s+1)} \times \mathbb{R}^{n(n-1)(s+1)}.$$

For $p = 1, \ldots, n$ consider the open subset $G_p = \{\xi \in \varphi(\Omega) : v_1^{(p)} \neq v_0^{(p)}\}$ of $\varphi(\Omega)$. Since $\varphi(\Omega) = \bigcup_{p=1}^k G_p$, it is sufficient to check that $G_p \cap \varphi(\Omega \cap \Sigma)$ is contained in a smooth submanifold of G_p of codimension $(s+1)(n-1)+1$.

Fix p and for every $\xi \in \varphi(\Omega)$ define $N_0(\xi) = (N_0^{(1)}(\xi), \ldots, N_0^{(n)}(\xi))$, where the components $N_0^{(t)}(\xi) = N_0^{(t)}$ are given by (2.8) for $i = 0$. By (3.4) and (2.2), $\Omega \subset \hat{M} \subset V^{s+1}$, and (2.1) implies $N_0(\xi) \neq 0$ for every $\xi \in \varphi(\Omega)$. Set

$$c_{ij}(\xi) = \sum_{t=1}^n \frac{\partial F}{\partial y_i^{(t)}}(v) \cdot a_{ij}^{(t)}$$

for $\xi \in \varphi(\Omega)$, $1 \le i \le s$ and $1 \le j \le n-1$. Notice that if $\xi = \varphi(\sigma)$ and $\sigma = (j^1 f_0(x_0), \ldots, j^1 f_s(x_s))$, $x = (x_1, \ldots, x_s)$ is a critical point of the map $F \circ (f_1 \times \cdots \times f_s)$ if and only if $c_{ij}(\xi) = 0$ for all $i = 1, \ldots, s$ and $j = 1, \ldots, n-1$. Introduce the map $K : G_p \to \mathbb{R}^{s(n-1)} \times \mathbb{R}^{n-1} \times \mathbb{R}$ by

$$K(\xi) = ((c_{ij}(\xi))_{\substack{1 \le i < s \\ 1 \le j \le n-1}}; (d^{(m)}(\xi))_{\substack{1 \le m \le n \\ m \neq p}}; L(\xi)),$$

where

$$d^{(m)}(\xi) = \frac{v_1^{(m)} - v_0^{(m)}}{\|v_1 - v_0\|} + \frac{v_2^{(m)} - v_0^{(m)}}{\|v_2 - v_0\|}$$

and $L(\xi)=\langle v_1-v_2, N_0(\xi)\rangle$. It is staightforward to see that $G_p\cap\varphi(\Omega\cap\Sigma)\subset K^{-1}(0)$. Consequently, to prove the assertion it remains to show K is submersion at any point of G_p.

Take $\xi\in G_p$ and suppose

$$\sum_{i=1}^{s}\sum_{j=1}^{n-1} C_{ij}\operatorname{grad} c_{ij}(\xi)+\sum_{\substack{m=1\\ m\neq p}}^{n} D_m \operatorname{grad} d^{(m)}(\xi)+E\operatorname{grad} L(\xi)=0$$

for some real constants C_{ij}, D_m and E. It is proved in [**9**] (and in fact is easy to see) that for every $v\in U_s$ and every $i=1,\ldots,s$ there exists $t=1,\ldots,n$ with $(\partial F/\partial y_i^{(t)})(v)\neq 0$. Now considering the derivatives with respect to $a_{ij}^{(t)}$ and taking into account the previous remark, we find $C_{ij}=0$ for all i and j. On the other hand, $(\partial L/\partial v_0^{(t)})(\xi)=0$ for $t=1,\ldots,n$, hence lemma 3.2 leads to $D_m=0$ for any $m\neq p$. Finally, taking $t=0,1,\ldots,n$ with $N_0^{(t)}(\xi)\neq 0$, from $E(\partial L/\partial v_1^{(t)})(\xi)=0$ we deduce $E=0$. This completes the proof of lemma 3.3.

Next, for $q=3,\ldots,s$ introduce the set $\mathscr{B}_q^{(s)}$ of those $f\in C^\infty_{\text{emb}}(X,\mathbb{R}^n)$ for which $F\circ f^s$ has no critical points $x\in X^{(s)}$ with $f^s(x)\in U_s$ and such that

$$(y'-y)/\|y'-y\|+(y''-y)/\|y''-y\|=0 \tag{3.5}$$

for $y'=f(x_1)$, $y''=f(x_2)$, $y=f(x_q)$. Clearly, if $f\in\mathscr{B}_q^{(s)}$, then for every simple non-symmetric periodic reflecting ray γ with successive reflection points $y_1,\ldots,y_s$ on $f(X)$, the segment $[f(x_1),f(x_2)]$ does not contain $f(x_q)$.

LEMMA 3.4. *Let $s>3$ and $3\leq q\leq s$. Then $\mathscr{B}_q^{(s)}$ contains a residual subset of $C^\infty_{\text{emb}}(x,\mathbb{R}^n)$.*

Proof. Let $M\subset J_s^1(X,\mathbb{R}^n)$ be given by (2.2) for $U=U_s$. Fix q, $3\leq q\leq s$, and denote by Σ the set of those $(j^1f_1(x_1),\ldots,j^1f_s(x_s))$ in M such that $x=(x_1,\ldots,x_s)$ is a critical point of the map $F\circ(f_1\times\cdots\times f_s)$ and (3.5) is fulfilled for $y'=f_1(x_1)$, $y''=f_2(x_2)$ and $y=f_q(x_q)$. Then we have

$$\mathscr{B}_q^{(s)}=\{f\in C^\infty_{\text{emb}}(X,\mathbb{R}^n): j_s^1f(X^{(s)})\cap\Sigma=\varnothing\}.$$

Now it is sufficient to show $\operatorname{Codim}\Sigma>\dim X^{(s)}$. This can be proved using a part of the arguments in the proof of lemma 3.3. We omit the details.

Combining lemmas 3.3 and 3.4 we get

COROLLARY 3.5. *$\mathscr{T}_1$ contains a residual subset of $C^\infty_{\text{emb}}(X,\mathbb{R}^n)$.*

Using arguments similar to those above, replacing U_s by U_s' and F by F_1, one can prove that $\mathscr{T}_2$ also contains a residual subset of $C^\infty_{\text{emb}}(X,\mathbb{R}^n)$. As we have mentioned above, this proves theorem 3.1.

4. *Proof of the main theorem*

Our aim in this section is to prove theorem 1.1.

Let $\mathscr{T}$ be the set defined in theorem 3.1, and let $\mathscr{A}$ and T_Q be as in subsection 2.2, where Q is the set of all roots of unity. According to 2.2 and theorem 3.1 the set $\mathscr{A}\cap\mathscr{T}\cap T_Q$ contains a residual subset of $C^\infty_{\text{emb}}(X,\mathbb{R}^n)$. Therefore to prove theorem 1.1 it is sufficient to establish the inclusion

$$\mathcal{A} \cap \mathcal{T} \cap T_Q \subset \mathcal{F}. \tag{4.1}$$

Fix $f \in \mathcal{A} \cap \mathcal{T} \cap T_Q$, $Y = f(X)$ and an integer $s \geq 2$. Denote by $\mathcal{L}(s)$ the set of those $(y_1, \ldots, y_s) \in Y^s$ such that $y_1, \ldots, y_s$ are all successive reflection points of some periodic reflecting ray on Y. It is sufficient to show $\mathcal{L}(s)$ is finite.

Suppose $\mathcal{L}(s)$ is infinite and fix a sequence $\{(y_{1m}, \ldots, y_{sm})\}_{m=1}^{\infty}$ of different elements of $\mathcal{L}(s)$ so that for every $i = 1, \ldots, s$ there exists $y_i = \lim_{m\to\infty} y_{im}$. Set for convenience $y_{s+1m} = y_{1m}$ and $y_{s+1} = y_1$.

LEMMA 4.1. *There exist $i \neq j$ with $y_i \neq y_j$.*

Proof. Set

$$e_{im} = (y_{i+1m} - y_{im})/\|y_{i+1m} - y_{im}\|,$$

$$a_{im} = \|y_{i+1m} - y_{im}\| \Big/ \sum_{j=1}^{s} \|y_{j+1m} - y_{jm}\|.$$

Then $\|e_{im}\| = 1$ and $\sum_{i=1}^{s} a_{im} = 1$. Without loss of generality we may assume that $e_{im} \to_{m\to\infty} e_i$ and $a_{im} \to_{m\to\infty} a_i$ for all $i = 1, \ldots, s$. Clearly, $\sum_{i=1}^{s} a_i = 1$ and $\|e_i\| = 1$ for every i.

Suppose $y_1 = y_2 = \cdots = y_s$, then by $\lim_m y_{1m} = \lim_m y_{2m} = \lim_m y_{3m}$, it is easy to see that $e_2 = e_1$. Similarly, we get $e_3 = e_2, \ldots, e_s = e_{s-1}$, so $e_1 = e_2 = \cdots = e_s$. Now by $\sum_{i=1}^{s} (y_{i+1m} - y_{im}) = 0$ we get $\sum_{i=1}^{s} a_{im} e_{im} = 0$ which implies $(\sum_{i=1}^{s} a_i) e = 0$ contradicting $e \neq 0$ and $\sum_{i=1}^{s} a_i = 1$.

Without loss of generality we may assume $y_1 \neq y_2$. Then there exists a unique sequence $i_1 = 1 < i_2 < \cdots < i_k \leq s$, $i_{k+1} = 1$ of indices such that for every $j = 2, \ldots, k$, i_j is the maximal index $i > i_{j-1}$ for which the points $y_{i_{j-1}}, y_{i_{j-1}+1}, \ldots, y_i$ lie on a common line. It is not difficult to see now that $y_{i_1}, y_{i_2}, \ldots, y_{i_k}$ are the successive reflection points of some periodic reflecting ray γ on Y.

LEMMA 4.2. *We have $k = s$ and $i_j = j$ for every $j = 1, \ldots, s$.*

Proof. Suppose $i_2 > 2$, then $i_2 \geq 3$. There are two cases.

Case 1. There exists i with $1 < i < i_2$ and $y_i \neq y_{i_2}$. In this case obviously the segment $[y_1, y_{i_2}]$ of the ray γ is tangent to Y at y_i - contradicting $Y = f(X)$ and $f \in \mathcal{T}$.

Case 2. $y_2 = y_3 = \cdots = y_{i_2}$. Denote by θ_m the angle between the vector $y_{3m} - y_{2m}$ and the tangent plane to Y at y_{2m}. Since $\lim_m y_{3m} = \lim_m y_{2m} = y_2$, we have $\theta_m \to_m 0$. On the other hand, θ_m equals the angle between $y_{1m} - y_{2m}$ and the tangent plane to Y at y_{2m}. That is why $y_1 - y_2 = \lim_m (y_{1m} - y_{2m})$ is tangent to Y at $y_2 = y_{i_2}$. This implies immediately that y_{i_2} lies on $[y_1, y_{i_3}]$ - contradicting the choice of i_2.

Therefore $i_2 = 2$. In a similar way we obtain $i_3 = 3, \ldots, i_s = s$ and $k = s$. This proves the assertion.

Remark. Some of the points $y_1, \ldots, y_s$ could coincide even if $(y_{1m}, \ldots, y_{sm}) \in Y^{(s)}$ for every m. For example, γ could be a symmetric periodic ray with $1 + s/2$ different reflection points.

Further, we need to use the billiard ball map on Y. Let Ω be the bounded domain with boundary $\partial\Omega = Y$. Denote by S^{n-1} the unit sphere in $\mathbb{R}^n$ and set

$$G = \{(y, \eta) \in Y \times S^{n-1}: \langle \eta, N_y \rangle > 0\},$$

where N_y is the inward (with respect to Ω) unit normal vector to Y at y. The billiard ball map $B = B_Y$ is defined on some open subset W of G as follows. For $(y, \eta) \in W$ let the ray passing through y with direction η hits transversally Y at some point $z \in Y$. Denote by ζ the reflected direction, that is $\|\zeta\| = 1$, $\langle \zeta, N_z \rangle = -\langle \eta, N_y \rangle$. Set $B(y, \eta) = (z, \zeta) \in G$. It should be mentioned that if $y_1, \dots, y_s$ are the successive reflection points of some ordinary periodic reflecting ray on Y, then we have $(y_1, \eta) \in W$, where $\eta = (y_2 - y_1)/\|y_2 - y_1\|$. It is known that B is differentiable on W.

Let η be as above and set $\eta_m = (y_{2m} - y_{1m})/\|y_{2m} - y_{1m}\|$. Then $B^j(y_1, \eta) \in W$ and $B^j(y_{1m}, \eta_m) \in W$ for all m and j. Moreover, for any m we have $B^s(y_{1m}, \eta_m) = (y_{1m}, \eta_m)$ and s is the smallest positive integer with this property (i.e. s is the smallest period of (y_{1m}, η_m)). Further, by lemma 4.2, $B^s(y_1, \eta) = (y_1, \eta)$. However, in general the smallest period k of (y_1, η) could be less than s. Nevertheless, $B^k(y_1, \eta) = (y_1, \eta)$ implies that k is a divisor of s. Mention that $y_1, \dots, y_k$ are the successive reflection points of some ordinary reflecting ray δ on Y (which coincides with γ as a subset of $\mathbb{R}^n$), and $d(B^k)(y_1, \eta)$ is the linear Poincaré map related to δ. Since $Y = f(X)$ and $f \in T_Q$, $d(B^k)(y_1, \eta)$ has no eigenvalues which are roots of unity. On the other hand, for $p = s/k$ we have $(dB^k)^p(y_1, \eta) = (dB^s)(y_1, \eta)$. Since (y_{1m}, η_m) are different elements of G, $(y_{1m}, \eta_m) \to_m (y_1, \eta)$ and $B^s(y_{1m}, \eta_m) = (y_{1m}, \eta_m)$ for any m, we get that every neighbourhood of (y_1, η) in $W \subset G$ contains fixed points of B^s different from (y_1, η). This implies that 1 is an eigenvalue of $(dB^s)(y_1, \eta)$, and therefore $(dB^k)(y_1, \eta)$ has an eigenvalue z with $z^p = 1$ which is a contradiction. We have shown in this way that $\mathscr{L}(s)$ is finite, and this proves theorem 1.1.

5. *Some remarks on scattering rays*

Throughout this section we assume that $\omega, \theta \in S^{n-1}$ are two fixed vectors and $\omega \neq \theta$. Let γ be a curve in $\mathbb{R}^n$ of the form $\gamma = \bigcup_{i=0}^k l_i$, where $l_i = [x_i, x_{i+1}]$ are finite segments for $i = 1, \dots, k-1$ ($k \geq 1$), $x_i \in X$ for all i, and l_0 (l_k) is the infinite segment starting at x_1 (resp. x_k) and having direction $-\omega$(resp. θ). The curve γ will be called a (ω, θ)-ray if the following conditions hold:

(i) the open segments $\mathring{l}_i$ do not intersect X transversally,

(ii) for every $i = 0, \dots, k-1$ the segments l_i and l_{i+1} satisfy the reflection law at x_{i+1}.

Again the points x_i will be called reflection points of γ. Some of them may coincide and some l_i could be tangent to X at some interior point of l_i.

The ray γ will be called symmetric if some l_i is orthogonal to X at x_i or x_{i+1}. Then we must have $\theta = -\omega$ and $l_0 = l_k$, moreover, if $k > 1$, then $k = 2m+1$, $l_{m-i} = l_{m+i-1}$ for $i = 0, \dots, m-1$ and $\gamma = \bigcup_{i=1}^m l_i$. If γ has no segments orthogonal to X, then it will be called a non-symmetric (ω, θ)-ray. As in §§ 2 and 3 we define tangent and ordinary (ω, θ)-rays.

Let $\mathscr{A}'$ be the set of those $f \in C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ such that for every non-symmetric (symmetric) (ω, θ)-ray γ on $f(X)$ there exist different points $y_1, \ldots, y_s$ such that $y_1, \ldots, y_s, y_1$ (resp. $y_1, \ldots, y_{s-1}, y_s, y_{s-1}, \ldots, y_1$) are all the successive reflection points of γ. It turns out (cf. [**14**]) that $\mathscr{A}'$ contains a residual subset of $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$.

THEOREM 5.1. *Let X be as in theorem* 1.1 *and let $\mathscr{T}'$ be the set of those $f \in C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ for which there are no (ω, θ)-rays on $f(X)$ tangent to $f(X)$. Then $\mathscr{T}'$ contains a residual subset of $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$.*

The proof of this theorem follows closely that of theorem 3.1 and we will discuss only the modifications which should be made.

As in § 3 we introduce the set $\mathscr{T}'_1$ $(\mathscr{T}'_2)$ related to ordinary non-symmetric (symmetric) rays. Notice that if $\{U_i\}_{i=1}^\infty$ are open subsets of $\mathbb{R}^n$ with $\bigcup_{i=1}^\infty U_i = \mathbb{R}^n$, then $C^\infty_{\text{emb}}(X, \mathbb{R}^n) = \bigcup_{i=1}^\infty C^\infty_{\text{emb}}(X, U_i)$ and every $C^\infty_{\text{emb}}(X, U_i)$ is open in $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$. A subset $T \subset C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ is residual in $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ if and only if $T \cap C^\infty_{\text{emb}}(X, U_i)$ is residual in $C^\infty_{\text{emb}}(X, U_i)$ for every i.

Fix a bounded open subset U of $\mathbb{R}^n$ with $X \subset U$. The above remark shows it is sufficient to prove that $\mathscr{T}' \cap C^\infty_{\text{emb}}(X, U)$ contains a residual subset of $C^\infty_{\text{emb}}(X, U)$. Let Z_1, Z_2 be two hyperplanes in $\mathbb{R}^n$ so that Z_1 (Z_2) is orthogonal to ω (resp. θ). Let H_i, $i = 1, 2$, be halfspaces determined by Z_i, we can choose Z_i and H_i in such a way that both H_1 and H_2 contain U. Denote by π_i the orthogonal projection from $\mathbb{R}^n$ onto Z_i. Fix a positive integer s and set

$$U''_s = \{(y_1, \ldots, y_s) \in U^{(s)}: y_i \notin [y_{i-1}, y_{i+1}] \text{ for } i = 2, \ldots, s-1 \text{ and } (y_2 - y_1)/\|y_2 - y_1\| \neq \omega,\ (y_{s-1} - y_s)/\|y_{s-1} - y_s\| \neq \theta\},$$

$$U'''_s = \{(y_1, \ldots, y_s) \in U^{(s)}: y_i \notin [y_{i-1}, y_{i+1}] \text{ for } i = 2, \ldots, s-1 \text{ and } (y_2 - y_1)/\|y_2 - y_1\| \neq \omega\}.$$

Introduce the maps $G: U''_s \times Z_1 \times Z_2 \to \mathbb{R}^n$, $G_1: U'''_s \times Z_1 \to \mathbb{R}^n$, given by

$$G(y; z_1, z_2) = \|z_1 - y_1\| + \sum_{i=1}^{s-1} \|y_i - y_{i+1}\| + \|y_s - z_2\|,$$

$$G_1(y; z_1) = \|z_1 - y_1\| + \sum_{i=1}^{s-1} \|y_i - y_{i+1}\|.$$

Notice that U''_s is open in $U^{(s)}$, G is smooth on $U''_s \times Z_1 \times Z_2$ and if γ is a non-symmetric (ω, θ)-ray with different successive reflection points $y_1, \ldots, y_s$, then $(y_1, \ldots, y_s) \in U''_s$ and $G(y; \pi_1(y_1), \pi_2(y_s))$ is the length of that part of γ which lies in $H_1 \cap H_2$. In a similar way U'''_s and G_1 are related to symmetric (ω, θ)-rays.

Now we apply the arguments from § 3, replacing U_s by U''_s and F by G, to prove that $\mathscr{T}'_1$ contains a residual subset of $C^\infty_{\text{emb}}(X, U)$. Similarly, we establish that $\mathscr{T}'_2$ (and therefore $\mathscr{T}'$) also contains a residual subset of $C^\infty_{\text{emb}}(X, U)$. This proves theorem 5.1.

Finally, we have the following:

THEOREM 5.2. *Let X be as in theorem* 1.1 *and let $\mathscr{F}'$ be the set of those $f \in C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ such that for every integer $s \geq 1$ there is at most a finite number of (ω, θ)-rays on $f(X)$ having exactly s reflection points. Then $\mathscr{F}'$ contains a residual subset of $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$.*

The proof of this result is similar to that of theorem 1.1, where instead of the properties of the Poincaré map we use the properties of the map dJ_γ (differential cross section related to γ) established in [**10**] (cf. also [**11**]). We leave the details to the reader.

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