Proceedings of the Edinburgh Mathematical Society (1989) **32**, 197–212 ©

APPROXIMATION AND INTERPOLATION BY COMPLEX SPLINES ON THE TORUS

by T. N. T. GOODMAN, S. L. LEE and A. SHARMA

(Received 17th August 1987)

1. Introduction

Let $T^2=\{(e^{ix_1},e^{ix_2}):0\leqq x_j<2\pi, j=1,2\}$ be a two dimensional torus and r, s, t and k be positive integers with $k>r+s+t-2$. Our main object is to study the approximation and interpolation properties of a class of smooth functions whose restrictions to each triangle of a three direction mesh lie in the linear span of $\{z_1^\mu z_2^\nu:0\leqq\mu\leqq r-1,\ 0\leqq\nu\leqq s-1$, or $0\leqq\mu\leqq r-1,\ r+s-1\leqq\mu+\nu\leqq r+s+t-2$, or $0\leqq\nu\leqq s-1,\ r+s-1\leqq\mu+\nu\leqq r+s+t-2\}$ where $(z_1,z_2)\in T^2$.

The one dimensional analogue is the class of uniform complex splines on the circle studied in [**1**], [**5**], [**8**], [**10**]. By analogy with [**5**], [**8**] and [**10**], the basic tool for our investigation will be a class of multivariate complex B-splines on the torus M_Γ which is a complex version of polynomial box splines, a subject which has received much interest recently (see the survey paper [**6**] and the reference therein).

The complex B-spline on the torus is a periodic case of a general class of compactly supported functions, known as the exponential box splines, introduced recently by Amos Ron [**9**].

In Section 2 we define the complex B-spline M_Γ on the d-dimensional torus and give a short proof of a basic relation for M_Γ. Section 3 deals with those properties of the linear combinations of translates of M_Γ which will be useful in the sequel. The proofs of the results in Section 3 are just slight modifications of those in [**6**] and [**9**] for polynomial and exponential box splines. In Section 4, we study the complex B-splines on a three direction mesh on the torus and state an interpolation problem. Section 5 deals with finite double Fourier series which is the tool for our solution of the interpolation problem. The solution is given in Section 6. In Section 7, we construct the Bernstein–Schoenberg type approximation operators on the torus.

2. Definition and elementary properties of complex B-splines on the torus

Let d be a positive integer and Γ a multiset consisting of a finite number of elements of the form $\gamma=(e,\lambda)$, where $e\in\mathbb{Z}^d\backslash\{0\}$ and $\lambda\in\mathbb{Z}$. We define the sets $V_\Gamma:\{e:(e,\lambda)\in\Gamma\}$, $\Lambda_\Gamma:=\{\lambda:(e,\lambda)\in\Gamma\}$, and assume that $\langle V_\Gamma\rangle=\mathbb{R}^d$. We shall also use the same notation V_Γ

to represent the $d\times|\Gamma|$ matrix whose columns are the vectors of Γ, and Λ_Γ to stand for the vectors in $\mathbb{Z}^{|\Gamma|}$ whose components are the corresponding λ's. We also define the set $J(\Gamma):=\{J\subset\Gamma: V_J$ is a basis of $\mathbb{R}^d\}$. To each $J\in J(\Gamma)$, there exists a unique $\theta_J\in\mathbb{R}^d$ such that $\theta_J V_J=\Lambda_J$.

Let $h=2\pi/k$, where k is a positive integer, such that $he\in[-\pi,\pi]^d$ for $\gamma=(e,\lambda)\in\Gamma$. We shall assume throughout that

$$|\det V_J|=1, \qquad J\in J(\Gamma) \tag{2.1}$$

and

$$0<|(\lambda-\theta_J e)h|<2\pi, \qquad \text{for } \gamma=(e,\lambda)\in\Gamma\backslash J. \tag{2.2}$$

Let M_Γ be a function on $\mathbb{R}^d$ defined by

$$M_\Gamma(x)=\left(\frac{1}{2\pi}\right)^d\sum_{v\in\mathbb{Z}^d}(M_\Gamma)_v^\wedge \exp ivx, \qquad x\in\mathbb{R}^d, \tag{2.3}$$

where

$$(M_\Gamma)_v^\wedge=\prod_{\gamma\in\Gamma}\left\{\frac{\exp i(\lambda-ve)h-1}{\lambda-ve}\right\}, \qquad v\in\mathbb{Z}^d, \tag{2.4}$$

where the factors in the product are taken to be ih if the denominators equal zero. We shall see later that the restrictions (2.1) and (2.2) will ensure that $M_\Gamma(x)$ is a piecewise polynomial in $z:=(\exp ix_1,\dots,\exp ix_d)\in T^d$. We shall call M_Γ the complex polynomial B-spline on the d-dimensional torus defined by Γ.

The Fourier coefficients (2.4) show that M_Γ is a convolution of periodic distributions $\mathcal{U}_\gamma$, $\gamma=(\lambda,e)\in\Gamma$, on $\mathbb{R}^d$ defined by

$$\mathcal{U}_\gamma(\phi)=i\int_0^h\exp(i\lambda t)\phi(et)\,dt, \qquad \phi\in\tilde{\mathcal{D}}(\mathbb{R}^d). \tag{2.5}$$

The distribution $\mathcal{U}_\gamma$ is supported on the line segment $\bigcup_{v\in\mathbb{Z}^d}(2\pi v+[0,he])$. Hence, the supp M_Γ is contained in $\bigcup_{v\in\mathbb{Z}^d}(2\pi v+\sum_{e\in V_\Gamma}[0,he])=\bigcup_{v\in\mathbb{Z}^d}(2\pi v+\{V_\Gamma t: t\in[0,h]^{|\Gamma|}\})$.

A straightforward computation of the Fourier coefficients shows that for each $J\in J(\Gamma)$, M_J is a periodic function given by

$$M_J(x)=\begin{cases}\exp i\theta_J x,\ x\in\bigcup_{v\in\mathbb{Z}^d}(2\pi v+\{V_J t: t\in[0,h)^d\}) \\ 0 \quad,\text{otherwise.}\end{cases} \tag{2.6}$$

Furthermore, if $\langle\Gamma\backslash\{\gamma\}\rangle=\mathbb{R}^d$, then

https://doi.org/10.1017/S0013091500028601 Published online by Cambridge University Press

$$M_\Gamma(x)=i\int_0^h M_{\Gamma\setminus\{\gamma\}}(x-te)\exp(i\lambda t)\,dt, \qquad x\in\mathbb{R}^d, \tag{2.7}$$

where $\gamma=(e,\lambda)\in\Gamma$.

Based on the Fourier coefficients (2.4) we define, for $\gamma=(e,\lambda)\in\Gamma$ two operators, the differential operator $D^\gamma f:=i(i\lambda-D_e)f$, and the difference operator $\nabla^\gamma f:=f-\exp(i\lambda h)f(\cdot-he)$, where D_e denotes the directional derivative along e. Then a straightforward calculation, using (2.4), gives

Proposition 2.1. *For* $\gamma\in\Gamma$,

$$D^\gamma M_\Gamma=\nabla^\lambda M_{\Gamma\setminus\{\gamma\}}, \tag{2.8}$$

and if $v=e/\|e\|^2$, *then*

$$D_e(\exp(-i\lambda(v\cdot))M_\Gamma)=i\exp(-i\lambda(v\cdot))\nabla^\lambda M_{\Gamma\setminus\{\gamma\}}. \tag{2.9}$$

The following result is similar to that of Ron [9], but we give a short and direct proof.

Proposition 2.2. *If conditions* (2.1) *and* (2.2) *are satisfied,*

$$M_\Gamma=\sum_{J\in J(\Gamma)} a_\Gamma(J)\nabla^{\Gamma\setminus J}M_J, \tag{2.10}$$

where

$$a_\Gamma(J)=\prod_{\gamma\in\Gamma\setminus J}(\theta_J e-\lambda)^{-1} \tag{2.11}$$

and

$$\nabla^{\Gamma\setminus J}=\prod_{\gamma\in\Gamma\setminus J}\nabla^\gamma.$$

Proof. To each $J\in J(\Gamma)$, $\lambda-\theta_J e=0\ \forall\gamma=(e,\lambda)\in J$, and by (2.2) $\lambda-\theta_J e\neq 0\ \forall\gamma=(e,\lambda)\in\Gamma\setminus J$. Using a partial fraction decomposition (see [3]) we have for $u\in\mathbb{R}^d$ for which $\lambda-ue\neq 0\ \forall\gamma=(e,\lambda)\in\Gamma$,

$$\frac{1}{\prod_{\gamma\in\Gamma}(\lambda-ue)}=\sum_{J\in J(\Gamma)}\frac{1}{\prod_{\gamma\in\Gamma\setminus J}(\lambda-\theta_J e)}\,\frac{1}{\prod_{\gamma\in J}(\lambda-ue)}, \tag{2.12}$$

where the products are over $\gamma=(e,\lambda)$. Multiplying equation (2.12) by $\prod_{\gamma\in\Gamma}\{\exp i(\lambda-ve)h-1\}$, and taking the limit as $u\to v$ gives

$$\prod_{\gamma\in\Gamma}\left\{\frac{\exp i(\lambda-ve)h-1}{\lambda-ve}\right\}=\sum_{J\in J(\Gamma)}a_\Gamma(J)\prod_{\gamma\in\Gamma\backslash J}\{1-\exp i(\lambda-ve)h\}\prod_{\gamma\in J}\left\{\frac{\exp i(\lambda-ve)h-1}{\lambda-ve}\right\},$$

where the quotient $(\exp i(\lambda-ve)h-1)/(\lambda-ve)$ is equal to ih if the denominator is zero.

Using (2.4) and the relation

$$(\nabla^{\Gamma\backslash J}M_\Gamma)_v^\wedge=\prod_{\gamma\in\Gamma\backslash J}\{1-\exp i(\lambda-ve)h\}(M_\Gamma)_v^\wedge$$

we obtain

$$(M_\Gamma)_v^\wedge=\sum_{J\in J(\Gamma)}a_\Gamma(J)(\nabla^{\Gamma\backslash J}M_\Gamma)_v^\wedge\quad\forall v\in\mathbb{Z}^d$$

from which (2.10) follows. □

Remark. From (2.6) and (2.10) we see that M_Γ is a linear combination of functions of the form $\exp(i\theta_J x)$, $J\in J(\Gamma)$, on each open set not crossed by the boundaries of the translates of supp $M_\Gamma, J\in J(\Gamma)$, along jh, $j\in\mathbb{Z}^d$. We shall call such a maximal open set a Γ-cell. Since $|(\det V_J|=1$, $M_\Gamma(x)$ equals a polynomial in $z:=(e^{ix_1},\dots,e^{ix_d})\in T^d$ in each Γ-cell.

3. Translates of complex box splines

Let Γ be as in Section 2 and assume that $\langle V_\Gamma\rangle=\mathbb{R}^d$. Let k be a positive integer, $h:=2\pi/k$ and $\mathbb{G}_h^d:=\{\alpha\in\mathbb{R}^d:\alpha_i=0,h,\dots,(k-1)h,\ i=1,2,\dots,d\}$. By (2.2) we have for each $J\in J(\Gamma)$, $|\lambda-\theta_J e|<k$ for all $\gamma=(e,\lambda)\in\Gamma\backslash J$.

Proposition 3.1. *Suppose conditions* (2.1.) *and* (2.2) *are satisfied. For each* $J\in J(\Gamma)$,

$$\sum_{\alpha\in\mathbb{G}_h^d}\exp(i\Lambda_J\alpha)M_\Gamma(x-V_J\alpha)=C_\Gamma(J)\exp(i\theta_J x)\tag{3.1}$$

holds for $x\in\mathbb{R}^d$, *where*

$$C_\Gamma(J)=\prod_{\gamma\in\Gamma\backslash J}\left\{\frac{\exp i(\lambda-\theta_J e)h-1}{\lambda-\theta_J e}\right\}\tag{3.2}$$

and the void product is taken to be 1.

Proof. The proof is by induction on $|\Gamma|$ and is a straightforward modification of Theorem 5.1 of [9]. If $|\Gamma|=d$, then $J(\Gamma)=\{\Gamma\}$. With $J=\Gamma$, $C_\Gamma(J)=1$ and $M_J(x)$ is given by (2.6). Hence

$$M_J(x-V_J\alpha)=\begin{cases}\exp(i\theta_J x)\exp(-i\Lambda_J\alpha), & x\in\bigcup_{v\in\mathbb{Z}^d}(2\pi v+\{V_J(t+\alpha):t\in[0,h)^d\})\\ 0 & , \text{ otherwise.}\end{cases}$$

Thus (3.1) holds for $|\Gamma|=d$.

Suppose $|\Gamma|>d$. Then for $J\in J(\Gamma)$ and $\gamma=(e,\lambda)\in\Gamma\backslash J$, by the convolution formula (2.7),

$$\sum_{\alpha\in G_h^d}\exp(i\Lambda_J\alpha)M_\Gamma(x-V_J\alpha)$$

$$=i\int_0^h\exp(i\lambda t)\sum_{\alpha\in G_h^d}\exp(i\Lambda_J\alpha)M_{\Gamma\backslash\{\gamma\}}(x-V_J\alpha-te)\,dt$$

$$=iC_{\Gamma\backslash\{\gamma\}}(J)\int_0^h\exp(i\lambda t)\exp(i\theta_J(x-te))\,dt$$

$$=C_\Gamma(J)\exp(i\theta_J x). \qquad \square$$

Remark. The constants $C_\Gamma(J)\neq 0$ for all $J\in J(\Gamma)$ because of the assumption that $|\lambda-\theta_J e|<k\ \forall\gamma=(e,\lambda)\in\Gamma\backslash J$.

Next, we shall prove

Proposition 3.2. *Let*

$$S:=\sum_{\alpha\in G_h^d}a_\alpha M_\Gamma(\cdot-\alpha).$$

Then for each $\gamma\in\Gamma$,

$$D^\gamma S=\sum_{\alpha\in G_h^d}\nabla^\gamma a_\alpha M_{\Gamma\backslash\{\gamma\}}(\cdot-\alpha) \tag{3.3}$$

where the equation is interpreted in the sense of distribution if $M_{\Gamma\backslash\{\gamma\}}$ *is supported on a set of measure zero.*

Proof. Suppose $\gamma=(e,\lambda)\in\Gamma$. By (2.8),

$$D^\gamma S=\sum_{\alpha\in G_h^d}a_\alpha\nabla^\gamma M_{\Gamma\backslash\{\gamma\}}(\cdot-\alpha)$$

$$=\sum_{\alpha\in G_h^d}a_\alpha M_{\Gamma\backslash\{\gamma\}}(\cdot-\alpha)-\exp(i\gamma h)\sum_{\alpha\in G_h^d}a_\alpha M_{\Gamma\backslash\{\gamma\}}(\cdot-\alpha-he)$$

$$=\sum_{\alpha\in G_h^d}\{a_\alpha-\exp(i\lambda h)a_{\alpha-he}\}M_{\Gamma\backslash\{\Gamma\}}(\cdot-\alpha)$$

from which (3.3.) follows. □

We shall now show that the conditions (2.1) and (2.2) imply that the translates $M_\Gamma(\cdot - \alpha), \alpha \in \mathbb{G}_h^d$ are locally linearly independent.

Proposition 3.3 *Suppose the conditions* (2.1) *and* (2.2) *hold. Then the translates* $M_\Gamma(\cdot - \alpha), \alpha \in \mathbb{G}_h^d$ *are locally linearly independent.*

Proof. The idea of the proof is the same as in [**9**]. If A is any non-empty set in a Γ-cell, then by Propositions 2.2 and 3.1, the span $\{M_\Gamma(\cdot - \alpha) | \alpha \in \mathbb{G}_h^d\}$ is precisely span $\{\exp i\theta_J x : J \in J(\Gamma)\}$ and so has dimension $|J(\Gamma)|$. By a result of Dahmen and Micchelli ([**4**, Theorem 3.1]), the number of α's for which $M_\Gamma(\cdot - \alpha)$ has support intersecting A is less than or equal to $\sum_{J \in J(\Gamma)} |\det V_J| = |J(\Gamma)|$. Hence the translates $M_\Gamma(x - \alpha) \neq 0$ for $x \in A$, $\alpha \in \mathbb{G}_h^d$ form a basis for the span of $\{M_\Gamma(\cdot - \alpha)|_A : \alpha \in \mathbb{G}_h^d\}$. □

Corollary. $\operatorname{supp} M_\Gamma = \bigcup_{v \in \mathbb{Z}^d} (2\pi v + \{V_\Gamma t : t \in [0, h]^{|\Gamma|}\})$.

Proof. This holds for $|\Gamma| = d$. By induction, using (2.9), $D_e(\exp(-i\lambda v x) M_\Gamma(x)) \neq 0$ on any Γ-cell in $\bigcup_{v \in \mathbb{Z}^d} (2\pi v + \{V_\Gamma t : t \in [0, h]^{|\Gamma|}\})$. □

4. Complex B-splines on a three direction mesh on the torus

Let k be a positive integer, $h := 2\pi/k$, $\omega = \exp ih$ and let $e^1 = (1, 0)$, $e^2 = (0, 1)$ and $e^3 = (1, 1)$. We consider the complex B-spline $M_{rst}(z) := M_\Gamma(x)$, $z = (z_1, z_2) = (e^{ix_1}, e^{ix_2}) \in T^2$, where M_Γ is defined by $\Gamma = \{(e^1, 0), \ldots, (e^1, r-1), (e^2, 0, \ldots, (e^2, s-1), (e^3, r+s-1), \ldots, (e^3, r+s+t-2)\}$, and r, s and t are positive integers. By (2.2), the Fourier coefficients of M_{rst} are given by

$$(M_{rst})_v^\wedge = a_{rv_1} b_{sv_2} c_{tv_1v_2}, \quad v = (v_1, v_2) \in \mathbb{Z}^2, \tag{4.1}$$

where

$$\begin{aligned} a_{rv_1} &= \prod_{j=0}^{r-1} \frac{\omega^{j-v_1} - 1}{j - v_1}, \quad b_{sv_2} = \prod_{j=0}^{s-1} \frac{\omega^{j-v_2} - 1}{j - v_2} \\ c_{tv_1v_2} &= \prod_{j=0}^{t-1} \frac{\omega^{r+s-1+j-v_1-v_2} - 1}{r+s-1+j-v_1-v_2}, \quad \omega = \exp ih, \end{aligned} \tag{4.2}$$

and the factors in the product (4.2) are taken to be equal ih when the denominators vanish.

Let $J(\Gamma) := J_1(\Gamma) \cup J_2(\Gamma) \cup J_3(\Gamma)$, where

$$J_1(\Gamma) := \{\{(e^1, \mu), (e^2, v)\} : \mu = 0, \ldots, r-1, v = 0, \ldots, s-1\},$$

https://doi.org/10.1017/S0013091500028601 Published online by Cambridge University Press

$$J_2(\Gamma):=\{\{(e^1,\mu),(e^3,\rho)\}:\mu=0,\ldots,r-1,\rho=r+s-1,\ldots,r+s+t-2\},$$

$$J_3(\Gamma):=\{\{(e^2,\nu),(e^3,\rho)\}:\nu=0,\ldots,s-1,\ \rho=r+s-1,\ldots,r+s+t-2\}.$$

For $J\in J_1(\Gamma)$, $\theta_J=(\mu,\nu)$, $\mu=0,\ldots,r-1$, $\nu=0,\ldots,s-1$, for

$J\in J_2(\Gamma)$, $\theta_J=(\mu,\rho-\mu)$, $\mu=0,\ldots,r-1$, $\rho=r+s-1,\ldots,r+s+t-2$, and for

$J\in J_3(\Gamma)$, $\theta_J=(\rho-\nu,\nu)$, $\nu=0,\ldots,s-1$, $\rho=r+s-1,\ldots,r+s+t-2$.

The $|\det V_J|=1$ for each $J\in J(\Gamma)$. Furthermore, for $J\in J(\Gamma)$ and $\gamma=(e,\lambda)\in\Gamma\backslash J, 0<|\lambda-\theta_J e|\leqq r+s+t-2$. Therefore, condition (2.2) is satisfied if we assume that $r+s+t-2<k$. This restriction and (4.2) implies that $M_\Gamma(x)$ is not a polynomial, but a proper spline function. The support,

$$\operatorname{supp} M_{rst}=\{(z,z_2)=(e^{ix_1},e^{ix_2}):0\leqq x_1\leqq(r+t)h,\ 0\leqq x_2\leqq(s+t)h,\ -sh\leqq x_1-x_2\leqq rh\},$$

and by (2.6) and (2.10) we see that the restriction of M_{rst} to each "triangle" bounded by the mesh lines corrresponding to $x_1=jh$, $x_2=jh$ and $x_1+x_2=jh$, $j\in\mathbb{Z}$, lies in the span of

$$\{z_1^\mu z_2^\nu:0\leqq\mu\leqq r-1,\ 0\leqq\nu\leqq s-1,\quad\text{or}\quad 0\leqq\mu\leqq r-1,$$

$$r+s-1\leqq\mu+\nu\leqq r+s+t-2,\quad\text{or}\quad 0\leqq\nu\leqq s-1,\quad r+s-1\leqq\mu+\nu\leqq r+s+t-2\}.$$

Since conditions (2.1) and (2.2) are satisfied, the translates

$$M_{rst}(z_1\omega^{-\alpha_1},z_2\omega^{-\alpha_2}),\ \alpha=(\alpha_1,\alpha_2)\in K:=\{(\nu_1,\nu_2)\in\mathbb{Z}^2:0\leqq\nu_i<k,i=1,2\},$$

are linearly independent. We shall consider the following interpolation problem.

Problem I: Given complex numbers y_β, $\beta\in K$, find $(c_\alpha)_{\alpha\in K}$ such that

$$\sum_{\alpha\in K}c_\alpha M_{rst}(\omega^{\beta_1-\alpha_1},\omega^{\beta_2-\alpha_2})=y_\beta,\qquad\forall\beta\in K.\tag{4.4}$$

The corresponding problem of interpolation by polynomial box spline has been solved by de Boor, Hollig and Riemenschneider **[2]**. For the one dimensional case of uniform complex splines on the circle, the corresponding problem has been solved by Ahlberg, Nilson and Walsh **[1]** and Schoenberg **[10]**. Following Schoenberg, we shall formulate the Problem I in terms of finite Fourier series.

5. Double finite Fourier series and the interpolation problem

Take $k \geqq 2$, and set $h = 2\pi/k$, $\omega = \exp ih$. Consider an array $\{a_\alpha : \alpha \in \mathbb{Z}^2\}$ satisfying $a_{\alpha+\beta k} = a_\alpha$ for any $\beta \in \mathbb{Z}^2$. We define its Fourier coefficients $\{\hat{a}_\nu : \nu \in \mathbb{Z}^2\}$ by

$$\hat{a}_\nu = \frac{1}{k^2} \sum_{\alpha \in K} a_\alpha \omega^{-\nu\alpha}. \tag{5.1}$$

Clearly, $\hat{a}_{\nu+\beta k} = \hat{a}_\nu$ for any $\beta \in \mathbb{Z}^2$, and

$$a_\alpha = \sum_{\nu \in K} \hat{a}_\nu \omega^{\alpha\nu}. \tag{5.2}$$

If we denote by Ω the $k^2 \times k^2$ matrix $(\omega^{\alpha\beta})_{\alpha,\beta \in K}$, then Ω^{-1} is the matrix $(1/k^2)(\omega^{-\alpha\beta})_{\alpha,\beta \in K}$, and the relations (5.1) and (5.2) can be written in matrix form

$$(a_\alpha)_{\alpha \in K} = \Omega(\hat{a}_\nu)_{\nu \in K}. \tag{5.3}$$

Furthermore, if A denotes the $k^2 \times k^2$ matrix $(a_{\alpha-\beta})_{\alpha,\beta \in K}$, then $\Omega^{-1} A \Omega = k^2 \operatorname{diag}(\hat{a}_\nu)_{\nu \in K}$. In particular

$$A \text{ is non singular if and only if } \hat{a}_\nu \neq 0 \; \forall \nu \in K. \tag{5.4}$$

By (2.3) the complex B-spline M_{rst} on the two dimensional torus can be written as

$$M_{rst}(z) = \frac{1}{4\pi^2} \sum_{\nu \in \mathbb{Z}^2} (M_{rst})^\wedge_\nu z_1^{\nu_1} z_2^{\nu_2}, \quad z = (z_1, z_2) \in U^2,$$

where $(M_{rst})^\wedge_\nu$ is given by equation (4.1). For $\alpha \in K$

$$M_{rst}(\omega^{\alpha_1}, \omega^{\alpha_2}) = \frac{1}{4\pi^2} \sum_{\nu \in \mathbb{Z}^2} (M_{rst})^\wedge_\nu \omega^{\alpha\nu} \tag{5.5}$$

which can be written as

$$M_{rst}(\omega^{\alpha_1}, \omega^{\alpha_2}) = \frac{1}{4\pi^2} \sum_{\nu \in K} A_\nu \omega^{\alpha\nu} \tag{5.6}$$

where

$$A_\nu = \sum_{\gamma \in \mathbb{Z}^2} (M_{rst})^\wedge_{\nu+k\gamma}. \tag{5.7}$$

The interpolation problem (4.4) is uniquely solvable if and only if the matrix

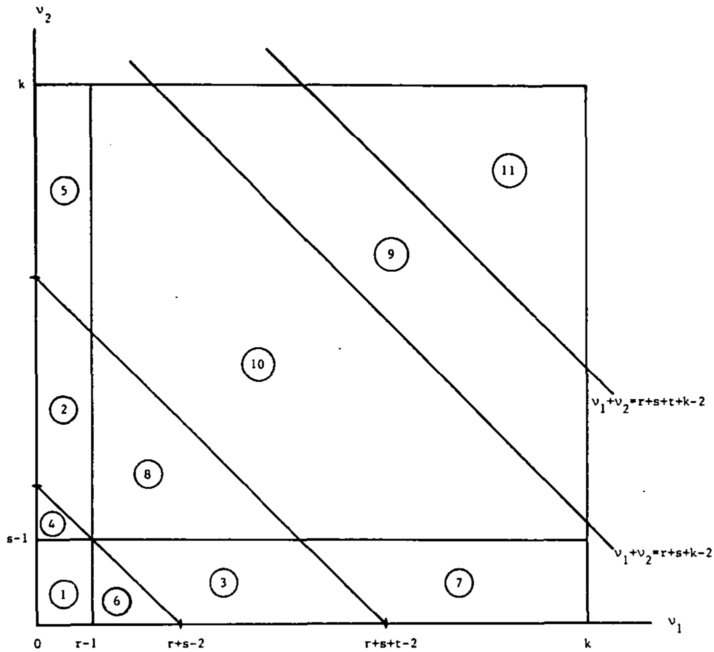

FIGURE 1 $K=\{(v_1,v_i):0\leqq v_2\leqq k-1,\ i=1,2\}$.

$(M_{rst}(\omega^{\beta_1-\alpha_1},\omega^{\beta_2-\alpha_2}))_{\alpha,\beta\in K}$ is non singular which is equivalent to $A_v\neq 0$ for all $v\in K$. By (4.1)

$$A_v=\sum_{\gamma\in\mathbf{Z}^2} a_{rv_1+k\gamma_1}b_{sv_2+k\gamma_2}c_{tv_1+k\gamma_1v_2+k\gamma_2},\quad v\in K, \tag{5.8}$$

where

$$\begin{aligned} a_{rv_1+k\gamma_1}&=\prod_{j=0}^{r-1}\frac{\omega^{j-v_1}-1}{j-v_1-k\gamma_1},\\ b_{sv_2+k\gamma_2}&=\prod_{j=0}^{s-1}\frac{\omega^{j-v_2}-1}{j-v_2-k\gamma_2},\\ c_{tv_1+k\gamma_1v_2+k\gamma_2}&=\prod_{j=0}^{t-1}\frac{\omega^{r+s-1+j-v_1-v_2}-1}{r+s-1+j-v_1-v_2-k\gamma_1-k\gamma_2}. \end{aligned} \tag{5.9}$$

6. Solution of the interpolation problem

In order to show that the finite Fourier coefficients A_v are non zero, we shall partition K into several regions (Figure 1) and look at the expressions for A_v in each case.

https://doi.org/10.1017/S0013091500028601 Published online by Cambridge University Press

Case 1. $0 \leqq v_1 \leqq r-1$ and $0 \leqq v_2 \leqq s-1$. Then $a_{rv_1} \neq 0$ and $a_{rv_1+k\gamma_1}=0$ for $\gamma_1 \neq 0$. Similarly $b_{sv_2} \neq 0$ and $b_{sv_2+k\gamma_2}=0$ for $\gamma_2 \neq 0$. Thus

$$A_v = \prod_{j=0}^{r-1} \left(\frac{\omega^{j-v_1}-1}{j-v_1} \right) \prod_{j=0}^{s-1} \left(\frac{\omega^{j-v_2}-1}{j-v_2} \right) \prod_{j=0}^{t-1} \left(\frac{\omega^{r+s-1+j-v_1-v_2}-1}{r+s-1+j-v_1-v_2} \right). \tag{6.1}$$

Case 2. $0 \leqq v_1 \leqq r-1$ and $r+s-1 \leqq v_1+v_2 \leqq r+s+t-2$. Then $a_{rv_1+k\gamma_1} \neq 0$ if and only if $\gamma_1=0$ and $c_{tv_1+k\gamma_1 v_2+k\gamma_2} \neq 0$ if and only if $\gamma_1+\gamma_2=0$. Then A_v reduces to the expression in (6.1).

Case 3. $0 \leqq v_2 \leqq s-1$ and $r+s-1 \leqq v_1+v_2 \leqq r+s+t-2$. Then, as in Case 2, A_v is as given in (6.1).

Case 4. $0 \leqq v_1 \leqq r-1$, $r>1$ and $s \leqq v_2 \leqq r+s-2-v_1$. Then $\prod_{j=0}^{s-1}(\omega^{j-v_2}-1) \neq 0$ and $\prod_{j=0}^{t-1}(\omega^{r+s-1+j-v_1-v_2}-1) \neq 0$, since $-r-s+2 \leqq j-v_2 \leqq -1$, for $0 \leqq j \leqq s-1$ and $1 \leqq r+s-1+j-v_1-v_2<k$, *for* $0 \leqq j \leqq t-1$. Hence

$$A_v = B_1 \sum_{\gamma=-\infty}^{\infty} \prod_{j=0}^{s-1} \frac{1}{j-v_2-k\gamma} \prod_{j=0}^{t-1} \frac{1}{r+s-1+j-v_1-v_2-k\gamma}, \tag{6.2}$$

where

$$B_1 = a_{rv_1} \prod_{j=0}^{s-1}(\omega^{j-v_2}-1) \prod_{j=0}^{t-1} (\omega^{r+s-1+j-v_1-v_2}-1) \neq 0.$$

Case 5. $0 \leqq v_1 \leqq r-1$ and $r+s+t-1 \leqq v_1+v_2 \leqq r+s+k-2$. Then $\prod_{j=0}^{s-1}(\omega^{j-v_2}-1) \neq 0$ and $\prod_{j=0}^{t-1}(\omega^{r+s-1+j-v_1-v_2}-1) \neq 0$, since $-k+1 \leqq j-v_2 \leqq -t-1$ for $0 \leqq j \leqq s-1$ and $-k+1 \leqq r+s-1+j-v_1-v_2 \leqq -1$, for $0 \leqq j \leqq t-1$. In this case A_v reduces to the same expression as in (6.2).

Case 6. $0 \leqq v_2 \leqq s-1$, $s>1$ and $r \leqq v_1 \leqq r+s-2-v_2$.

Case 7. $0 \leqq v_2 \leqq s-1$, and $r+s+t-1 \leqq v_1+v_2 \leqq r+s+k-2$.

As in Cases 4 and 5, Cases 6 and 7 gives

$$A_v = B_2 \sum_{\gamma=-\infty}^{\infty} \prod_{j=0}^{r-1} \frac{1}{j-v_1-k\gamma} \prod_{j=0}^{t-1} \frac{1}{r+s-1+j-v_1-v_2-k\gamma} \tag{6.3}$$

where

https://doi.org/10.1017/S0013091500028601 Published online by Cambridge University Press

$$B_2 = b_{sv_2} \prod_{j=0}^{r-1} (\omega^{j-v_1} - 1) \prod_{j=0}^{t-1} (\omega^{r+s-1+j-v_1-v_2} - 1) \neq 0.$$

Case 8. $r \leqq v_1 \leqq k-1$, $s \leqq v_2 \leqq k-1$ and $v_1 + v_2 \leqq r+s+t-2$. Then $\prod_{j=0}^{r-1}(\omega^{j-v_1}-1) \neq 0$, $\prod_{j=0}^{s-1}(\omega^{j-v_2}-1) \neq 0$ and $c_{tv_1+k\gamma_1 v_2+k\gamma_2} \neq 0$ if and only if $\gamma_1 + \gamma_2 = 0$. Hence

$$A_v = B_3 \sum_{\gamma=-\infty}^{\infty} \prod_{j=0}^{r-1} \frac{1}{j-v_1-k\gamma} \prod_{j=0}^{s-1} \frac{1}{j-v_2-k\gamma}, \tag{6.4}$$

where

$$B_3 = c_{tv_1v_2} \prod_{j=0}^{r-1} (\omega^{j-v_1} - 1) \prod_{j=0}^{s-1} (\omega^{j-v_2} - 1) \neq 0.$$

Case 9. $r+s+k-1 \leqq v_1 + v_2 \leqq r+s+t+k-2$. In this case $v_1 \geqq r$ and $v_2 \geqq s$. Furthermore $c_{tv_1+k\gamma_1 v_2+k\gamma_2} \neq 0$ if and only if $\gamma_1 + \gamma_2 = -1$. Hence

$$A_v = c_{tv_1v_2-k} \sum_{\gamma_1+\gamma_2=-1} \prod_{j=0}^{r-1} \left(\frac{\omega^{j-v_1}-1}{j-v_1-k\gamma_1} \right) \prod_{j=0}^{s-1} \left(\frac{\omega^{j-v_2}-1}{j-v_2-k\gamma_2} \right)$$

$$= B_4 \sum_{\gamma=-\infty}^{\infty} \prod_{j=0}^{r-1} \frac{1}{j-v_1-k\gamma} \prod_{j=0}^{s-1} \frac{1}{j-v_2+k\gamma+k}, \tag{6.5}$$

where

$$B_4 = c_{tv_1v_2-k} \prod_{j=0}^{r-1} (\omega^{j-v_1} - 1) \prod_{j=0}^{s-1} (\omega^{j-v_2} - 1) \neq 0.$$

Case 10. $r+s+t-1 \leqq v_1 + v_2 \leqq r+s+k-2$. In this case $v_1 \geqq r$ and $v_2 \geqq s$ which implies that $\prod_{j=0}^{r-1}(\omega^{j-v_1}-1) \neq 0$ and $\prod_{j=0}^{s-1}(\omega^{j-v_2}-1) \neq 0$. Furthermore, the above inequalities for $v_1 + v_2$ implies that $-k+1 \leqq r+s-1+j-v_1-v_2 \leqq -1$ for $j = 0, 2, \ldots, t$. Hence, we also have $\prod_{j=0}^{t-1}(\omega^{r+s-1+j-v_1-v_2}-1) \neq 0$, and therefore we can write

$$A_v = B_5 \sum_{\gamma \in Z^2} \prod_{j=0}^{r-1} \frac{1}{j-v_1-k\gamma_1} \prod_{j=0}^{s-1} \frac{1}{j-v_2-k\gamma_2} \prod_{j=0}^{t-1} \frac{1}{r+s-1+j-v_1-v_2-k\gamma_1-k\gamma_2}, \tag{6.6}$$

where

$$B_5 = \prod_{j=0}^{r-1} (\omega^{j-v_1} - 1) \prod_{j=0}^{s-1} (\omega^{j-v_2} - 1) \prod_{j=0}^{t-1} (\omega^{r+s-1+j-v_1-v_2} - 1) \neq 0.$$

Case II. $r+s+t+k-1 \leqq v_1+v_2 \leqq 2k-2$. In this case, a similar analysis shows that A_v is also given by (6.6).

From the above expressions for A_v, it is clear that if r, s and t are all even, then $A_v \neq 0$ for $v \in K$. Thus, in this case, the interpolation is uniquely solvable. We state the above result as

Theorem 6.1. *Given complex numbers* y_β, $\beta \in K$, *there exists a unique sequence* $(c_\alpha)_{\alpha \in K}$ *such that* (4.4) *holds if* r, s *and* t *are even.*

Remark. Theorem 6.1 is the solution of a particular case of a more general interpolation problem which can be stated as follows:

Problem II. Given complex numbers y_β, $\beta \in K$, find $(c_\alpha)_{\alpha \in K}$ such that

$$\sum_{\alpha \in K} c_\alpha M_{rst}(\omega^{\beta_1-\alpha_1+\varepsilon_1/2}, \omega^{\beta_2-\alpha_2+\varepsilon_2/2}) = y_\beta, \quad \beta \in K,$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2)$, $\varepsilon_i \in \{0, 1\}$, $i = 1, 2$. Theorem 6.1 gives a solution to the case $\varepsilon = 0$. We conjecture that Problem II is uniquely solvable if $(\varepsilon_1, \varepsilon_2) = (r+t, s+t) \bmod 2$.

7. Bernstein–Schoenberg type operator

We shall consider the complex B-splines $M_{rst}(z)$, $z \in T^2$. Since $|\det V_J| = 1$ for all $J \in J(\Gamma)$, by Proposition 3.1, we have for each J

$$C_\Gamma(J)^{-1} \sum_{\alpha \in K} \omega^{\theta_J \alpha} M_{rst}(z_1 \omega^{-\alpha_1}, \omega^{-\alpha_2}) = z^{\theta_J}, \quad z = (z_1, z_2) \in T^2, \tag{7.1}$$

where

$$C_\Gamma(J) = \prod_{\gamma \in \Gamma \backslash J} \left(\frac{\omega^{\lambda - \theta_J e} - 1}{\lambda - \theta_J e} \right), \tag{7.2}$$

and we have used the standard multivariate notation $z^{\theta_J} = z_1^\mu z_2^\nu$, $\theta_J = (\mu, \nu)$.

Let $J_0 = \{(e^1, 0), (e^2, 0)\}$. Then $\theta_{J_0} = 0$ and

$$C_\Gamma(J_0) = \binom{r+s-2}{r-1} \prod_{j=1}^{r-1} (\omega^j - 1) \prod_{j=1}^{s-1} (\omega^j - 1) \prod_{j=1}^{t-1} (\omega^{r+s+j+1} - 1)/(r+s+t-2)!$$

We now normalise the B-spline M_{rst} and set

$$N_{rst} = C_\Gamma(J_0)^{-1} M_{rst}. \tag{7.3}$$

It follows from (7.1) that

$$\sum_{\alpha\in K} N_{rst}(z_1\omega^{-\alpha_1}, z_2\omega^{-\alpha_2}) = 1\ \forall z=(z_1,z_2)\in T^2, \tag{7.4}$$

and for each J.

$$(C_\Gamma(J_0)/C_\Gamma(J)) \sum_{\alpha\in K} \omega^{\theta_J\alpha} N_{rst}(z_1\omega^{-\alpha_1}, z_2 w^{-\alpha_2}) = z^{\theta_J}, \quad z\in T^2. \tag{7.5}$$

We shall show that there is a unique linear operator

$$(Sf)(z) = \sum_{\alpha\in K} f(\tau_\alpha) N_{rst}(z_1\omega^{-\alpha_1}, z_2\omega^{-\alpha_2}), \quad z\in T^2, \tag{7.6}$$

where f is defined on a polyannulus, which reproduces $z^{\theta_{J_1}}$ and $z^{\theta_{J_2}}$ for any two distinct J_1, J_2 in $J(\Gamma)$. Let $\theta_{J_1}=(\mu_1,\mu_2)$ and $\theta_{J_2}=(\mu_2,\nu_2)$, where for $i=1,2$, $0\leqq\mu_i\leqq r-1$, $0\leqq\nu_i\leqq s-1$, or $0\leqq\mu_i\leqq r-1$, $r+s-1\leqq\mu_i+\nu_i\leqq r+s+t-2$, or $0\leqq\nu_i\leqq s-1$, $r+s-1\leqq\mu_i+\nu_i\leqq r+s+t-2$. In view of (7.5) the above requirement give

$$\tau_\alpha^{\theta_{J_1}} = \frac{C_\Gamma(J_0)}{C_\Gamma(J_1)}\omega^{\theta_{J_1}\alpha}, \quad \tau_\alpha^{\theta_{J_2}} = \frac{C_\Gamma(J_0)}{C_\Gamma(J_2)}\omega^{\theta_{J_2}\alpha},$$

which can be written as

$$\tau_{\alpha 1}^{\mu_1}\tau_{\alpha 2}^{\nu_1} = \frac{C_\Gamma(J_0)}{C_\Gamma(J_1)}\omega^{\mu_1\alpha_1+\nu_1\alpha_2} \tag{7.7}$$

$$\tau_{\alpha 1}^{\mu_2}\tau_{\alpha 2}^{\nu_2} = \frac{C_\Gamma(J_0)}{C_\Gamma(J_2)}\omega^{\mu_2\alpha_1+\nu_2\alpha_2} \tag{7.8}$$

where $\tau_\alpha=(\tau_{\alpha 1},\tau_{\alpha 2})$ and $\alpha=(\alpha_1,\alpha_2)\in K$. A straightforward computation gives

$$\begin{aligned}\tau_{\alpha 1} &= C_\Gamma(J_0)^{(\nu_2-\nu_1)\Delta^{-1}}(C_\Gamma(J_2)^{\nu_1}/C_\Gamma(J_1)^{\nu_2})\Delta^{-1}\omega^{\alpha_1}\\ \tau_{\alpha 2} &= C_\Gamma(J_0)^{(\mu_1-\mu_2)\Delta^{-1}}(C_\Gamma(J_1)^{\mu_2}/C_\Gamma(J_2)^{\mu_1})\Delta^{-1}\omega^{\alpha_2}\end{aligned} \tag{7.9}$$

where $\Delta=\mu_1\nu_2-\mu_2\nu_1$. Thus the operator (7.6) with $\tau_\alpha=(\tau_{\alpha 1},\tau_{\alpha 2})$ defined by (7.9) reproduces the constant function and the functions $z^{\theta_{J_1}}$ and $z^{\theta_{J_2}}$, where $\theta_{J_1}=(\mu_1,\nu_1)$, $\theta_{J_2}=(\mu_2,\nu_2)$.

For simplicity, we shall assume that $J_1=\{(e^1,1),(e^2,0)\}$ and $J_2=\{(e^1,0),(e^2,1)\}$. Then $\theta_{J_1}=(1,0)$, $\theta_{J_2}=(0,1)$, and $z^{\theta_{J_1}}=z_1$ and $z^{\theta_{J_2}}=z_2$. By (7.9).

$$\tau_\alpha = \left(\frac{C_\Gamma(J_0)}{C_\Gamma(J_1)}\omega^{\alpha_1}, \frac{C_\Gamma(J_0)}{C_\Gamma(J_2)}\omega^{\alpha_2}\right) \tag{7.10}$$

and it follows from (7.2) by elementary computation that

$$\tau_\alpha = (R_1\omega^{\alpha_1 + (r+t)/2}, R_2\omega^{\alpha_2 + (s+t)/2}), \tag{7.11}$$

where

$$\begin{aligned} R_1 &= \left(\frac{\sin(r-1)\pi/k}{(r-1)\pi/k}\right)\left(\frac{\sin(r+s+t-2)\pi/k}{(r+s+t-2)\pi/k}\right)\Big/\left(\frac{\sin\pi/k}{\pi/k}\right)\left(\frac{\sin(r+s-2)\pi/k}{(r+s-2)\pi/k}\right) \\ R_2 &= \left(\frac{\sin(s-1)\pi/k}{(s-1)\pi/k}\right)\left(\frac{\sin(r+s+t-2)\pi/k}{(r+s+t-2)\pi/k}\right)\Big/\left(\frac{\sin\pi/k}{\pi/k}\right)\left(\frac{\sin(r+s-2)\pi/k}{(r+s-2)\pi/k}\right). \end{aligned} \tag{7.12}$$

It is easy to see that

$$1 - R_i = 0(k^{-2}). \tag{7.13}$$

We shall prove the following.

Theorem 7.1 *Let $f(z)$ be defined and continuous on a polyannulus $A^2 := \{(z_1, z_2): \rho_i \leqq |z_i| \leqq 1,\ i = 1, 2\}$, where $0 < \rho_i < R_i$. Then the operator Sf defined by* (7.6), *where τ_α is given by* (7.11), *reproduces the functions* 1, $z_1 z_2$, *and*

$$|(Sf)(z) - f(z)| \leqq M\omega(f; k^{-1}),\ z \in T^2, \tag{7.14}$$

where M is independent of f and k, and $\omega(f; k^{-1})$ denotes the modulus of continuity.

We shall first establish a simple lemma.

Lemma 1. *For fixed integers r, s, t,*

$$|N_{rst}(z)| = 0(1), \qquad z \in T^2. \tag{7.15}$$

Proof. First, we observe that $|M_{110}(z)| = 0(1)$, and using (2.7) by induction on $n = r + s + t$, it is easy to see that $|M_{rst}(z)| = 0(k^{-n+2})$. The result then follows from (7.3), since

$$\begin{aligned} |C_\Gamma(J_0)| = \binom{r+s-2}{r-1} \prod_{j=1}^{r-1} (2\sin j\pi/k) \prod_{j=1}^{s-1} (2\sin j\pi/k). \\ \prod_{j=0}^{t-1} (2\sin(r+s+j-1)\pi/k)/(r+s+t-2)! \\ = 0(k^{-n+2}). \end{aligned}$$

□

Proof of Theorem 7.1. By (7.4) and (7.6), we have

https://doi.org/10.1017/S0013091500028601 Published online by Cambridge University Press

$$(Sf)(z)-f(z)=\sum_{\alpha\in K}(f(\tau_\alpha)-f(z))N_{rst}(z_1\omega^{-\alpha_1},z_2\omega^{-\alpha_2}),\qquad z\in T^2. \tag{7.16}$$

For a fixed $(z_1,z_2)=(e^{ix_1},e^{ix_2})\in T^2$, $N_{rst}(z_1\omega^{-\alpha_1},z_2\omega^{-\alpha_2})\neq 0$ if and only if $\alpha_1 h<x_1<(r+t+\alpha_1)h$, $\alpha_2 h<x_2<(s+t+\alpha_2)h$, and $(-s+\alpha_1-\alpha_2)h<x_1-x_2<(r+\alpha_1-\alpha_2)h$. Therefore from (7.11), in view of (7.13), a straightforward computation gives

$$|\tau_\alpha-z|:=(|\tau_{\alpha 1}-z_1|^2+|\tau_{\alpha 2}-z_2|^2)^{1/2}=0(k^{-1}).$$

The result now follows from (7.15) and (7.16). □

Remarks. 1. The order of approximation in (7.14) is best possible for the class of continuous functions on A^2. For if $f(z_1,z_2)=|z_1|^{1/2}$, then there exists a constant C such that $|(Sf)(z)-f(z)|\geqq Ck^{-1}$.

2. The Bernstein–Schoenberg operator (7.6), with τ_α given by (7.11), reproduces the constant function and the functions z_1 and z_2. In this case the function f is defined on the polyannulus A^2. However, for any $f\in C(T^2)$, we can define a linear operator

$$(Sf)(z)=\sum_{\alpha\in K}f(\omega^{\alpha_1+(r+t)/2},\omega^{\alpha_2+(s+t)/2})N_{rst}(z_1\omega^{-\alpha_1},z_2\omega^{-\alpha_2}),\quad z\in T^2. \tag{7.17}$$

This operator $\tilde{S}f$ does not reproduce z_1, z_2, and as in the proof of Theorem 7.1, we have

Theorem 7.2. *For* $f\in C(T^2)$,

$$|(\tilde{S}f)(z)-f(z)|\leqq M_1\omega(f,k^{-1}),\qquad z\in T^2, \tag{7.18}$$

where M_1 *is independent of f and k.*

REFERENCES

1. J. H. Ahlberg, E. N. Nilson and J. L. Walsh, Properties of analytic splines I: Complex polynomial splines, *J. Analysis and Appl.* **33** (1971), 234–257.

2. C. de Boor, K. Hollig and S. D. Riemenschneider, Bivariate cardinal interpolation by splines on a three direction mesh, *Illinois J. Math.* **29** (1985), 555–566.

3. W. Dahmen and C. A. Micchelli, On limits of multivariate B-splines, *J. Analyse Math.* **39** (1981), 256–278.

4. W. Dahmen and C. A. Micchelli, On the local linear independence of translates of a box spline, *Stud. Math. Appl.* **82** (1985), 243–263.

5. T. N. T. Goodman and S. L. Lee, B-splines on the circle and trigonometric B-splines, *Proc. Conference on Approx. Theory, St. John's, Newfoundland* (Reidel Pub. Co., 1983), 297–325.

6. K. Hollig, Box splines, *Approximation Theory V* (Academic Press, Boston, Mass., 1986), 71–95.

7. R. Q. Jia, On the linear independence of translates of box splines, *J. Approx. Theory* **40** (1984), 158–160.

8. C. A. Micchelli and A. Sharma, Spline functions on the circle: Cardinal L-splines revisited, *Canad. J. Math.* **32** (1980), 1459–1473.

9. Amos Ron, Exponential box splines, to appear in Constructive Approximation.

10. I. J. Schoenberg, On polynomial spline functions on the circle I and II, *Proc. Conference on Constructive Theory of Functions, Budapest* (1972), 403–433.

Department of Mathematical
and Computing Sciences
University of Dundee
Dundee DD1 4HN
Scotland

School of Mathematical
and Computer Sciences
Science University of Malaysia
Penang 11800
Malaysia and
Department of Mathematics
National University of Singapore
10 Kent Crescent
Singapore 0511

Department of Mathematics
University of Alberta
Edmonton
Alberta T6G 2G1
Canada