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Local VMO and Weak Convergence in h^1

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Abstract. A local version of VMO is defined, and the local Hardy space h^1 is shown to be its dual. An application to weak-$*$ convergence in h^1 is proved.

1 Introduction

This paper is concerned with the local case of the duality between the Hardy space H^1 and the spaces BMO and VMO. As proved by C. Fefferman [F], BMO is the dual of the real Hardy space H^1. On the other hand, H^1 itself is the dual of VMO. This space was originally defined by Sarason as the subspace of BMO whose elements have vanishing mean oscillation, but needs to be modified in the case of $\mathbf{R}^n$ in order for duality to hold. The paper begins with a survey of these results, together with complete proofs of the duality of H^1 and VMO in the cases of the circle and of $\mathbf{R}^n$.

In Section 3, a local version of VMO is defined as a subspace of the local BMO space bmo, defined by Goldberg [G], whose elements are characterized by two vanishing mean oscillation conditions. It is shown that this space is in fact the closure of $C_0(\mathbf{R}^n)$ in $\mathrm{bmo}(\mathbf{R}^n)$, and hence that the local Hardy space h^1 is the dual of this space. This allows the study of weak-$*$ convergence in h^1. In particular, in Section 4, a version of the Jones-Journé theorem [JJ] for h^1 is proved, which is useful in the application of Hardy spaces to compensated compactness (see [CLMS].)

2 Survey of Known Results

Recall that a (locally) integrable function on the circle $\mathbf{T}$ (respectively on the line $\mathbf{R}$) is said to have *bounded mean oscillation* if

$$\|f\|_* \stackrel{\text{def}}{=} \sup_I \frac{1}{|I|}\int_I |f(x) - f_I|\,dx < \infty.$$

Here and in the following I denotes a finite interval, $|I|$ its length and $f_I = \frac{1}{|I|}\int_I f(x)\,dx$ is the average of f over I. Modulo constants, $\|\cdot\|_*$ defines a norm and the Banach space of such functions is denoted by $\mathrm{BMO}(\mathbf{T})$ (resp. $\mathrm{BMO}(\mathbf{R})$).

***Definition 1* (Sarason [Sa])** A function f in $\mathrm{BMO}(\mathbf{T})$ (or $\mathrm{BMO}(\mathbf{R})$) is said to have vanishing mean oscillation if

(i)

$$\lim_{\delta\to 0} \sup_{|I|\le\delta} \frac{1}{|I|}\int_I |f(x) - f_I|\,dx = 0.$$

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The space of such functions, modulo constants, is denoted by VMO(**T**) (resp. VMO(**R**)), with the norm defined as the BMO norm $\|\cdot\|_*$.

Theorem 1 **(Sarason [Sa])** *VMO is the closure in the BMO norm of* UC $\cap$ BMO, *where* UC *is the space of uniformly continuous functions.*

Theorem 2 **(Sarason [Sa])** *$f \in$ VMO if and only if $f = u + \tilde{v}$, where u, v are bounded uniformly continuous functions and $\tilde{v}$ is the conjugate function (Hilbert transform) of v. Moreover, by varying f by a constant, we can take u and v with $\|u\|_\infty + \|v\|_\infty \le \|f\|_*$.*

Recall that the *complex* Hardy space $\mathcal{H}^1(\mathbf{T})$ is defined as the boundary values $F(t)$ of those holomorphic functions $F(re^{it})$ in the unit disk satisfying

$$\|F\|_{\mathcal{H}^1} \stackrel{\text{def}}{=} \sup_{r<1} \int_{\mathbf{T}} |F(re^{it})|\, dt < \infty.$$

Define

$$\operatorname{Re}\mathcal{H}^1(\mathbf{T}) := \{\operatorname{Re} F : F \in \mathcal{H}^1(\mathbf{T})\}$$

with norm

$$\|\operatorname{Re} F\|_{\operatorname{Re}\mathcal{H}^1} \stackrel{\text{def}}{=} \|F\|_{\mathcal{H}^1} \approx \|\operatorname{Re} F\|_{L^1} + \|\operatorname{Im} F\|_{L^1}$$

and the *real* Hardy space $\mathrm{H}^1(\mathbf{T})$ by

$$\mathrm{H}^1(\mathbf{T}) := \{f : \operatorname{Re} f, \operatorname{Im} f \in \operatorname{Re}\mathcal{H}^1(\mathbf{T})\}$$

with norm

$$\|f\|_{\mathrm{H}^1} \stackrel{\text{def}}{=} \|\operatorname{Re} f\|_{\operatorname{Re}\mathcal{H}^1} + \|\operatorname{Im} f\|_{\operatorname{Re}\mathcal{H}^1}.$$

Theorem 3 *The dual of* VMO(**T**) *is*

$$\mathrm{H}^1_0(\mathbf{T}) := \left\{ f : f \in \mathrm{H}^1(\mathbf{T}), \int_{\mathbf{T}} f(t)\, dt = 0 \right\}.$$

More precisely, every function $f \in \mathrm{H}^1_0(\mathbf{T})$ defines a bounded linear functional $\mathcal{L}_f$ on VMO(**T**) *by*

$$\mathcal{L}_f(g) = \int_{\mathbf{T}} fg \tag{1}$$

with $\|\mathcal{L}_f\| \le \|f\|_{\mathrm{H}^1}$. Conversely, every bounded linear functional $\mathcal{L}$ on VMO(**T**) *corresponds to a function $f \in \mathrm{H}^1_0(\mathbf{T})$ with $\mathcal{L} = \mathcal{L}_f$, $\|f\|_{\mathrm{H}^1} \le C\|\mathcal{L}\|$.*

Proof We follow the proof given by Garcia-Cuerva and Rubio de Francia [GCRdF]. Since VMO(**T**) $\subset$ BMO(**T**) which is the dual of $\mathrm{H}^1_0(\mathbf{T})$ (C. Fefferman, [F]), each $f \in \mathrm{H}^1_0(\mathbf{T})$ defines a bounded linear functional $\mathcal{L}_f$ on VMO(**T**) as in (1), of norm bounded by $\|f\|_{\mathrm{H}^1}$.

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Conversely, suppose $\mathcal{L}$ is a bounded linear functional on VMO. By taking real and imaginary parts, we may assume $\mathcal{L}$ is real-valued without changing its norm. Note that for $\phi \in C(\mathbf{T})$,

$$|\mathcal{L}(\phi)| \leq \|\mathcal{L}\| \, \|\phi\|_* \leq \|\mathcal{L}\| \, \|\phi\|_\infty,$$

so $\mathcal{L}$ acts on $C(\mathbf{T})$ by integration against a finite Borel measure μ, with $\|\mu\| \leq \|\mathcal{L}\|$.

By Theorem 2, the conjugate function of a continuous function is in VMO, so we can define the linear functional $\tilde{\mathcal{L}}$ acting on $C(\mathbf{T})$ by

$$\tilde{\mathcal{L}}(\phi) \stackrel{\text{def}}{=} -\mathcal{L}(\tilde{\phi}).$$

Then

$$|\tilde{\mathcal{L}}(\phi)| \leq \|\mathcal{L}\| \, \|\tilde{\phi}\|_* \leq C\|\mathcal{L}\| \, \|\phi\|_\infty$$

by the boundedness of the conjugate function operator from L^∞ to BMO. Thus $\tilde{\mathcal{L}}$ is given by integration against a finite Borel measure ν, with $\|\nu\| \leq C\|\mathcal{L}\|$.

Let $\phi = e^{int}$, $n > 0$. Then $\tilde{\phi} = -ie^{int}$ so $\tilde{\mathcal{L}}(\phi) = i\mathcal{L}(\phi)$ and hence

$$0 = \mathcal{L}(\phi) + i\tilde{\mathcal{L}}(\phi) = \int e^{int} \, (d\mu + i d\nu).$$

Since this holds for all $n > 0$, we can apply the F. and M. Riesz theorem to conclude that $\mu + i\nu$ is absolutely continuous with respect to Lebesgue measure and $\mu + i\nu = F dt$ for some $F \in \mathcal{H}^1(\mathbf{T})$. Note that since $\mathcal{L}$ vanishes on constants, $\int F = 0$. Writing $f = \operatorname{Re} F$, we have that $f \in \operatorname{Re} \mathrm{H}^1_0$, $\operatorname{Im} F = \tilde{f}$ and

$$\|f\|_{\mathrm{H}^1} = \|F\|_{L^1} \leq \|\mu\| + \|\nu\| \leq C\|\mathcal{L}\|.$$

Taking a real-valued $\phi \in C(\mathbf{T})$,

$$\mathcal{L}(\phi) + i\tilde{\mathcal{L}}(\phi) = \int \phi(f + i\tilde{f}) \, dt$$

and since $\mathcal{L}$ is real-valued,

$$\mathcal{L}(\phi) = \int \phi f \, dt \quad \text{and} \quad \tilde{\mathcal{L}}(\phi) = \int \phi \tilde{f} \, dt.$$

By linearity this extends to all $\phi \in C(\mathbf{T})$, and since $C(\mathbf{T})$ is dense in VMO, we see that $\mathcal{L}$ is given by integration against $f \in \operatorname{Re} \mathcal{H}^1$ with $\|f\|_{\operatorname{Re} \mathcal{H}^1} \leq C\|\mathcal{L}\|$. Dropping the assumption that $\mathcal{L}$ is real-valued, we have that a general complex-valued bounded linear functional on VMO($\mathbf{T}$) is given by a integration against a function $f = f_1 + if_2 \in \mathrm{H}^1_0(\mathbf{T})$, $f_1, f_2 \in \operatorname{Re} \mathcal{H}^1(\mathbf{T})$, with

$$\|f\|_{\mathrm{H}^1} = \|f_1\|_{\operatorname{Re} \mathcal{H}^1} + \|f_2\|_{\operatorname{Re} \mathcal{H}^1} \leq 2C\|\mathcal{L}\|. \qquad \blacksquare$$

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We now turn to the situation in $\mathbf{R}^n$. The space $\mathrm{BMO}(\mathbf{R}^n)$ is defined analogously to $\mathrm{BMO}(\mathbf{R})$, replacing intervals with cubes. This is the dual of the real Hardy space $\mathrm{H}^1(\mathbf{R}^n)$, which can be defined (Fefferman and Stein, [FS]) by requiring that the maximal function $M_\psi(f)$ be in L^1, with $\|f\|_{\mathrm{H}^1} \stackrel{\text{def}}{=} \|M_\psi(f)\|_{L^1}$, where

$$M_\psi(f)(x) = \sup_{t>0} |(f * \psi_t)(x)|,$$

ψ is some fixed function in $\mathcal{S}(\mathbf{R}^n)$, $\int \psi \neq 0$, and ψ_t denotes the dilation $\psi_t(x) = t^{-n}\psi(t^{-1}x)$.

In order to represent $\mathrm{H}^1(\mathbf{R}^n)$ as a dual space, we must define a slightly different version of VMO (see Coifman and Weiss [CW].)

Definition 2 The space $\mathrm{VMO}_0(\mathbf{R}^n)$ is the closure of $C_0(\mathbf{R}^n)$ in $\mathrm{BMO}(\mathbf{R}^n)$.

Clearly VMO_0 is a subspace of Sarason's VMO space, the closure of the uniformly continuous functions in BMO. However, it is easy to see that in addition to satisfying the vanishing mean oscillation condition (i), VMO_0 functions must also satisfy the following "vanishing mean oscillation" conditions:

(ii)

$$\lim_{N\to\infty} \sup_{\ell(Q)\geq N} \frac{1}{|Q|} \int_Q |f(x) - f_Q|\,dx = 0$$

(iii)

$$\lim_{R\to\infty} \sup_{Q\cap B(0,R)=\varnothing} \frac{1}{|Q|} \int_Q |f(x) - f_Q|\,dx = 0.$$

Here and in the following Q denotes a cube, $\ell(Q)$ its sidelength, $|Q|$ its measure, $f_Q = \frac{1}{|Q|}\int_Q f(x)\,dx$ the average of f over Q, and $B(0,R)$ the ball centered at the origin with radius R.

For example, when $n = 1$, the function $f(x) = \sin x$ is in VMO but not in VMO_0. Moreover, conditions (ii) and (iii) are both necessary. For example, the continuous function

$$f(x) = \sum_{k=1}^{\infty} (1 - |x - 2^k|)_+$$

satisfies conditions (i) and (ii) but not (iii), while the continuous function

$$f(x) = -\chi_{(-\infty,-1]} + x\chi_{[-1,1]} + \chi_{[1,\infty)}$$

satisfies conditions (i) and (iii) but not (ii).

Lemma 1 *The Riesz transforms* R_j *map* $C_0(\mathbf{R}^n)$ *into* $\mathrm{VMO}_0(\mathbf{R}^n)$.

Proof We use the fact that the Riesz transforms are bounded operators from L^∞ to BMO. Since C_0 is the closure of the Schwartz space $\mathcal{S}$ in L^∞, $\mathrm{R}_j(C_0)$ lies in the closure of $\mathrm{R}_j(\mathcal{S})$ in BMO. But $\mathrm{R}_j(\mathcal{S}) \subset C_0$ since for every $\phi \in \mathcal{S}$,

$$\widehat{\mathrm{R}_j(\phi)}(\xi) = -i\frac{\xi_j}{|\xi|}\hat{\phi}(\xi)$$

is in L^1. Thus $\mathrm{R}_j(C_0) \subset \overline{\mathrm{R}_j(\mathcal{S})}^{\mathrm{BMO}} \subset \overline{C_0}^{\mathrm{BMO}} = \mathrm{VMO}_0$. ■

Theorem 4 *The dual of* $\mathrm{VMO}_0(\mathbf{R}^n)$ *is the real Hardy space* $\mathrm{H}^1(\mathbf{R}^n)$.

Proof (see also Chang [Ch], Coifman and Weiss [CW]) The proof is similar to the case of the circle. Again, since $\mathrm{VMO}_0(\mathbf{R}^n)$ is a subspace of $\mathrm{BMO}(\mathbf{R}^n)$, which is the dual of H^1, every function f in H^1 determines a bounded linear functional on VMO_0 of norm bounded by $\|f\|_{\mathrm{H}^1}$.

Conversely, given a bounded linear functional $\mathcal{L}$ on VMO_0, we see as above that it acts on C_0 by means of a finite Borel measure μ. We can also define, thanks to Lemma 1, the "Riesz transforms" of $\mathcal{L}$ by

$$\mathcal{R}_j(\mathcal{L})(\phi) \overset{\mathrm{def}}{=} \mathcal{L}\big(\mathrm{R}_j(\phi)\big)$$

whenever $\phi \in C_0$, $j = 1, \dots, n$. Again the boundedness of the Riesz transforms R_j from L^∞ to BMO shows that each $\mathcal{R}_j(\mathcal{L})$ is given by a finite Borel measure ν_j. An n-dimensional version of the F. and M. Riesz theorem (see Stein [S, VII.3.2 Corollary 1]), applied to the n-tuple of measures $(\mu, \nu_1, \dots, \nu_n)$, gives that each is absolutely continuous with respect to Lebesgue measure on $\mathbf{R}^n$, with

$$d\mu = f dx$$

and

$$d\nu_j = \mathrm{R}_j(f) dx$$

for some $f \in \mathrm{H}^1(\mathbf{R}^n)$. Furthermore,

$$\|f\|_{\mathrm{H}^1} \approx \|f\|_{L^1} + \sum \|\mathrm{R}_j(f)\|_{L^1} \leq \|\mathcal{L}\| + \sum \|\mathcal{R}_j(\mathcal{L})\| \leq C\|\mathcal{L}\|.$$

3 Local Version of Duality

Fix a function $\psi \in \mathcal{S}(\mathbf{R}^n)$, $\int \psi \neq 0$. For $f \in L^1(\mathbf{R}^n)$, define the local maximal function

$$m_\psi(f)(x) = \sup_{0<t<1} |(f * \psi_t)(x)|,$$

where ψ_t denotes the dilation $\psi_t(x) = t^{-n}\psi(t^{-1}x)$.

Definition 3 **(Goldberg [G])** A function f belongs to the local Hardy space $h^1(\mathbf{R}^n)$ if and only if $m_\psi(f) \in L^1(\mathbf{R}^n)$, and

$$\|f\|_{h^1} \approx \|m_\psi(f)\|_{L^1}.$$

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Definition 4 **(Goldberg [G])** bmo($\mathbf{R}^n$) is the space of locally integrable functions f on $\mathbf{R}^n$ satisfying

$$\|f\|_{\mathrm{bmo}} \stackrel{\mathrm{def}}{=} \sup_{|Q|<1} \frac{1}{|Q|} \int_Q |f(x) - f_Q|\, dx + \sup_{|Q|\geq 1} \frac{1}{|Q|} \int_Q |f(x)|\, dx < \infty.$$

Note that $\|\cdot\|_* \leq 2\|\cdot\|_{\mathrm{bmo}}$ so bmo $\subset$ BMO, but in bmo constants are no longer identified to zero.

Theorem 5 **(Goldberg [G])** *The space* bmo($\mathbf{R}^n$) *is the dual of the local Hardy space* $h^1(\mathbf{R}^n)$.

Definition 5 The space vmo($\mathbf{R}^n$) is the subspace of bmo($\mathbf{R}^n$) consisting of those functions f on $\mathbf{R}^n$ satisfying

(i′)

$$\lim_{\delta\to 0} \sup_{\ell(Q)\leq\delta} \frac{1}{|Q|} \int_Q |f(x) - f_Q|\, dx = 0$$

and

(ii′)

$$\lim_{R\to\infty} \sup_{\ell(Q)\geq 1, Q\cap B(0,R)=\varnothing} \frac{1}{|Q|} \int_Q |f(x)|\, dx = 0.$$

Theorem 6 vmo($\mathbf{R}^n$) *is the closure of* $C_0(\mathbf{R}^n)$ *in* bmo($\mathbf{R}^n$).

Proof For $\phi \in C_0$, uniform continuity implies that

$$\begin{aligned}\lim_{\delta\to 0} \sup_{\ell(Q)\leq\delta} \frac{1}{|Q|} \int_Q |\phi(x) - \phi_Q|\, dx &= \lim_{\delta\to 0} \sup_{\ell(Q)\leq\delta} \frac{1}{|Q|} \int_Q |\phi(x) - \phi(x_Q)|\, dx \\ &\leq \lim_{\delta\to 0} \sup_{|x-y|<\delta} |\phi(x) - \phi(y)| \\ &= 0.\end{aligned}$$

The vanishing of ϕ at infinity implies that

$$\lim_{R\to\infty} \sup_{\ell(Q)\geq 1, Q\cap B(0,R)=\varnothing} \frac{1}{|Q|} \int_Q |\phi(x)|\, dx \leq \lim_{R\to\infty} \sup_{|x|\geq R} |\phi(x)| = 0.$$

We have shown $C_0 \subset$ vmo. Moreover, the space vmo is closed in bmo since for $f, g \in$ bmo, $\delta < 1$,

$$\sup_{\ell(Q)\leq\delta} \frac{1}{|Q|} \int_Q |f(x) - f_Q|\, dx \leq \sup_{\ell(Q)\leq\delta} \frac{1}{|Q|} \int_Q |g(x) - g_Q|\, dx + \|f - g\|_{\mathrm{bmo}}$$

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and

$$\sup_{\ell(Q)\geq 1, Q\cap B(0,R)=\varnothing} \frac{1}{|Q|}\int_Q |f(x)|\,dx \leq \sup_{\ell(Q)\geq 1, Q\cap B(0,R)=\varnothing} \frac{1}{|Q|}\int_Q |g(x)|\,dx + \|f-g\|_{\mathrm{bmo}}.$$

Conversely, suppose $f \in$ vmo. Take ϕ continuous with compact support in $B(0,1)$, $0 \leq \phi \leq 1$, and $\int \phi \neq 0$. For every $\epsilon > 0$, set $\phi_\epsilon(x) = \epsilon^{-n}\phi(\epsilon^{-1}x)$ and

$$f^\epsilon = f * \phi_\epsilon.$$

We first show $f^\epsilon \in C_0$ for every $\epsilon > 0$. In fact, f^ϵ is uniformly continuous since by the duality of H^1 and BMO,

$$\begin{aligned}
|f^\epsilon(x) - f^\epsilon(y)| &= \left| \int_{|t|<\epsilon} f(t)[\phi_\epsilon(x-t) - \phi_\epsilon(y-t)]\,dt \right| \\
&\leq C\|f\|_{\mathrm{BMO}}\,\|\phi_\epsilon(x-\cdot) - \phi_\epsilon(y-\cdot)\|_{\mathrm{H}^1} \\
&\leq C\|f\|_{\mathrm{bmo}}\,|B_{x,y,\epsilon}|\,\|\phi_\epsilon(x-\cdot) - \phi_\epsilon(y-\cdot)\|_\infty \\
&\leq C(\epsilon + |x-y|/2)^n\|f\|_{\mathrm{bmo}}\,\|\phi_\epsilon(x-\cdot) - \phi_\epsilon(y-\cdot)\|_\infty \\
&\to 0
\end{aligned}$$

as $|x-y| \to 0$. Here we've used the fact that $\phi_\epsilon(x-\cdot) - \phi_\epsilon(y-\cdot)$ is a multiple of an H^1-atom supported in $B_{x,y,\epsilon}$, the smallest ball containing $B(x,\epsilon)$ and $B(y,\epsilon)$.

Moreover, for a fixed $\epsilon > 0$,

$$\begin{aligned}
|f^\epsilon(x)| &\leq \int_{|t|<\epsilon} |f(x-t)|\phi_\epsilon(t)\,dt \\
&\leq \epsilon^{-n}\int_{B(x,\epsilon)} |f(t)|\,dt \\
&\leq \epsilon^{-n}\int_{Q_{x,\epsilon,1}} |f(t)|\,dt \\
&= C_\epsilon \frac{1}{|Q_{x,\epsilon,1}|}\int_{Q_{x,\epsilon,1}} |f(t)|\,dt \\
&\to 0
\end{aligned}$$

as $|x| \to \infty$. Here $Q_{x,\epsilon,1}$ is a cube centered at x with sidelength $\max\{1, 2\epsilon\}$. By condition (ii′) in the definition of vmo, the average of f over such a cube tends to zero as $|x| \to \infty$.

It remains to show that $f^\epsilon \to f$ in bmo. As a preliminary step we compute, for a cube Q,

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |f^\epsilon(x) - f^\epsilon_Q|\,dx \\
&\qquad = \frac{1}{|Q|}\int_Q \left| \int f(x-t)\phi_\epsilon(t)\,dt - \frac{1}{|Q|}\int_Q \int f(y-t)\phi_\epsilon(t)\,dt\,dy \right| dx
\end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{|Q|} \int_Q \left| \int \left[f(x-t) - \frac{1}{|Q|} \int_Q f(y-t)\,dy \right] \phi_\epsilon(t)\,dt \right| dx \\
&\le \int \frac{1}{|Q|} \int_Q \left| f(x-t) - \frac{1}{|Q|} \int_Q f(y-t)\,dy \right| dx\,\phi_\epsilon(t)\,dt \\
&= \int \frac{1}{|Q-t|} \int_{Q-t} |f(x) - f_{Q-t}|\,dx\,\phi_\epsilon(t)\,dt \\
&\le \sup_{|t|<\epsilon} \frac{1}{|Q-t|} \int_{Q-t} |f(x) - f_{Q-t}|\,dx
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{|Q|} \int_Q |f^\epsilon(x)|\,dx &= \frac{1}{|Q|} \int_Q \left| \int f(x-t)\phi_\epsilon(t)\,dt \right| dx \\
&\le \int \frac{1}{|Q|} \int_Q |f(x-t)|\,dx\,\phi_\epsilon(t)\,dt \\
&\le \sup_{|t|<\epsilon} \frac{1}{|Q-t|} \int_{Q-t} |f(x)|\,dx.
\end{aligned}
$$

Here $Q-t$ denotes the cube which is the translate of Q by $-t$. Note that we have used Minkowski's integral inequality in both estimates.

Let $\eta > 0$ be given. We want to show $\|f - f^\epsilon\|_{\mathrm{bmo}} < \eta$ for all sufficiently small ϵ. Take $\delta < 1$ such that

$$
\sup_{\ell(Q)\le\delta} \frac{1}{|Q|} \int_Q |f(x) - f_Q|\,dx < \eta/2.
$$

Then for cubes Q with $\ell(Q) \le \delta$,

$$
\begin{aligned}
&\frac{1}{|Q|} \int_Q |(f - f^\epsilon)(x) - (f - f^\epsilon)_Q|\,dx \\
&\quad \le \frac{1}{|Q|} \int_Q |f(x) - f_Q|\,dx + \frac{1}{|Q|} \int_Q |f^\epsilon(x) - f^\epsilon_Q|\,dx \\
&\quad \le \frac{1}{|Q|} \int_Q |f(x) - f_Q|\,dx + \sup_{|t|<\epsilon} \frac{1}{|Q-t|} \int_{Q-t} |f(x) - f_{Q-t}|\,dx \\
&\quad < \eta.
\end{aligned}
$$

Let $R > \sqrt{n}$ be such that

$$
\sup_{\ell(Q)\ge 1,\,Q\cap B(0,R)=\varnothing} \frac{1}{|Q|} \int_Q |f(x)|\,dx < \frac{\delta^n \eta}{4}.
$$

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Then for $\epsilon < R - \sqrt{n}$, if $\ell(Q) \geq \delta$ and $Q \cap B(0, 2R) = \varnothing$,

$$\begin{aligned}\frac{1}{|Q|}\int_Q |(f - f^\epsilon)(x)|\,dx &\leq \frac{1}{|Q|}\int_Q |f(x)|\,dx + \frac{1}{|Q|}\int_Q |f^\epsilon(x)|\,dx \\ &\leq \frac{1}{|Q|}\int_Q |f(x)|\,dx + \sup_{|t|<\epsilon}\frac{1}{|Q-t|}\int_{Q-t}|f(x)|\,dx \\ &\leq 2 \sup_{|Q'|=|Q|, Q'\cap B(0,R+\sqrt{n})=\varnothing} \frac{1}{|Q'|}\int_{Q'}|f(x)|\,dx \\ &\leq 2\delta^{-n} \sup_{\ell(Q^*)\geq 1, Q^*\cap B(0,R)=\varnothing} \frac{1}{|Q^*|}\int_{Q^*}|f(x)|\,dx \\ &< \frac{\eta}{2}.\end{aligned}$$

In particular, this also implies that for Q as above with $\ell(Q) < 1$,

$$\frac{1}{|Q|}\int_Q |(f - f^\epsilon)(x) - (f - f^\epsilon)_Q|\,dx \leq 2\frac{1}{|Q|}\int_Q |(f - f^\epsilon)(x)|\,dx < \eta.$$

Thus the only case remaining is that in which $\ell(Q) \geq \delta$ and $Q \cap B(0, 2R) \neq \varnothing$. If $\ell(Q) \geq 1$, write Q as a finite union of cubes Q_j with disjoint interiors and sidelengths < 1. Then

$$\begin{aligned}&\frac{1}{|Q|}\int_Q |(f - f^\epsilon)(x)|\,dx \\ &= \frac{1}{|Q|}\Big[\sum_{Q_j\subset B(0,2R+\sqrt{n})}\int_{Q_j}|(f - f^\epsilon)(x)|\,dx + \sum_{Q_j\setminus B(0,2R+\sqrt{n})\neq\varnothing}\int_{Q_j}|(f - f^\epsilon)(x)|\,dx\Big] \\ &\leq \delta^{-n}\|f - f^\epsilon\|_{L^1(B(0,2R+\sqrt{n}))} + \frac{1}{|Q|}\sum_{Q_j\cap B(0,2R)=\varnothing}\frac{\eta}{2}|Q_j| \\ &\leq \delta^{-n}\|f - f^\epsilon\|_{L^1(B(0,2R+\sqrt{n}))} + \frac{\eta}{2}.\end{aligned}$$

Since f is locally in L^1 and ϕ has compact support, we know that $f^\epsilon = f * \phi_\epsilon \to f$ in L^1 on any finite ball, hence we can make ϵ sufficiently small so that

$$\|f - f^\epsilon\|_{L^1(B(0,2R+\sqrt{n}))} < \delta^n\frac{\eta}{2}.$$

With this choice of ϵ we also have that if $\delta \leq \ell(Q) < 1$,

$$\begin{aligned}\frac{1}{|Q|}\int_Q |(f - f^\epsilon)(x) - (f - f^\epsilon)_Q|\,dx &\leq 2\frac{1}{|Q|}\int_Q |(f - f^\epsilon)(x)|\,dx \\ &< 2\delta^{-n}\|f - f^\epsilon\|_{L^1(B(0,2R+\sqrt{n}))} < \eta.\end{aligned}$$

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This finishes all the cases of the proof. ∎

To see that vmo $\neq C_0$, consider the following example on $\mathbf{R}^1$: for $\alpha > 0$ set

$$f_\alpha(x) = \min\bigl(1, (-\alpha \log |x|)_+\bigr).$$

Note that f_α is continuous, supported in $[-1, 1]$, $0 \leq f_\alpha \leq 1$, $f(0) = 1$ and for $x \neq 0$, $f_\alpha(x) \to 0$ as $\alpha \to 0$. Moreover,

$$\|f_\alpha\|_* \leq C\|(-\alpha \log |x|)_+\|_* \leq C\alpha\| \log |x| \,\|_* \leq C\alpha.$$

Let

$$f(x) = \sum_{n=1}^{\infty} f_{1/n}(x - 2n).$$

Since $0 \leq f \leq 1$, $f \in$ bmo. Condition (i') is satisfied because f is continuous, $\|f_{1/n}(\cdot - 2n)\|_* \to 0$ as $n \to \infty$ and $f \to 0$ on $[2n + 1/2, 2n + 3/2]$ uniformly as $n \to \infty$. Moreover, f satisfies condition (ii') since the average of f on any interval of length ≥ 1 tends to zero at infinity. However, it is not true that $f(x) \to 0$ as $x \to \infty$ since $f(2n) = 1$ for all n.

***Definition 6* (Goldberg [G])** The local Riesz transforms r_j can be defined on any tempered distribution f on $\mathbf{R}^n$ by

$$(\mathrm{r}_j\, f)^\wedge = -i(\xi_j/|\xi|)\tanh(\pi|\xi|)\hat{f}.$$

***Theorem 7* (Goldberg [G])** *The local Riesz transforms* r_j *are bounded operators from* $L^\infty(\mathbf{R}^n)$ *to* $\mathrm{bmo}(\mathbf{R}^n)$.

Lemma 2 *The local Riesz transforms* r_j *map* $C_0(\mathbf{R}^n)$ *into* $\mathrm{vmo}(\mathbf{R}^n)$.

The proof is the same as for Lemma 1.

Theorem 8 *A finite Borel measure* μ *on* $\mathbf{R}^n$ *whose local Riesz transforms* $\mathrm{r}_j\,\mu$, $1 \leq j \leq n$, *are also finite Borel measures on* $\mathbf{R}^n$ *is absolutely continuous with respect to Lebesgue measure on* $\mathbf{R}^n$. *Moreover,* $d\mu = f dx$, *and* $d(\mathrm{r}_j\,\mu) = f_j dx$ *with* $f \in h^1(\mathbf{R}^n)$, $f_j = \mathrm{r}_j\, f \in h^1(\mathbf{R}^n)$.

Proof The proof proceeds as in the case of the upper half-space. Let $\mu_0 = \mu$, $\mu_j = \mathrm{r}_j\mu$. As in Goldberg [G], convolution with the kernel P_y, given by its Fourier transform

$$\hat{P}_y(\xi) = \frac{\cosh(1 - 2y)\pi|\xi|}{\cosh \pi|\xi|},$$

converts the n-tuple of Borel measures $(\mu_0, \mu_1, \ldots, \mu_n)$ into a system of conjugate harmonic functions $(u_0, u_1, \ldots, u_n)$ in the strip $S = \{(x_1, \ldots, x_n, y) : 0 < y < 1\}$, even with respect to the line $y = 1/2$. Since

$$\|P_y * \mu_j\|_{L^1(\mathbf{R}^n)} \leq \|P_y\|_{L^1(\mathbf{R}^n)}\, \|\mu_j\| = \int_{\mathbf{R}^n} |d\mu_j| < \infty,$$

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this system satisfies Goldberg's h^1 condition

$$\sup_{0<y<1} \int_{\mathbf{R}^n} \sqrt{|u_0|^2 + \cdots + |u_n|^2}\, dx < \infty,$$

and therefore has boundary values $(f_0, f_1, \dots, f_n)$ with $f_0 \in L^1$, $f_j = \mathrm{r}_j f_0 \in L^1$, $1 \le j \le n$, and $u_j = P_y * f_j$. By uniqueness of the Poisson integral representation in the strip, $d\mu_j = f_j dx$. ■

Theorem 9 *The dual of* $\mathrm{vmo}(\mathbf{R}^n)$ *is the local Hardy space* $h^1(\mathbf{R}^n)$.

Proof The proof is similar to the case of VMO and H^1. Again, since $\mathrm{vmo}(\mathbf{R}^n)$ is a subspace of $\mathrm{bmo}(\mathbf{R}^n)$, which is the dual of h^1, every function f in h^1 determines a bounded linear functional on vmo of norm bounded by $\|f\|_{h^1}$.

Let $\mathcal{L}$ be a bounded linear functional on vmo. Since the bmo norm is bounded by the L^∞ norm, $\mathcal{L}$ acts on C_0 by means of a finite Borel measure μ. As above, Lemma 2 allows us to define the "local Riesz transforms" of $\mathcal{L}$ by

$$r_j(\mathcal{L})(\phi) \stackrel{\text{def}}{=} \mathcal{L}\big(\mathrm{r}_j(\phi)\big)$$

whenever $\phi \in C_0$, $j = 1, \dots, n$. Using the boundedness in Theorem 7, we get that each $r_j(\mathcal{L})$ is given by a finite Borel measure ν_j. Theorem 8 gives

$$d\mu = f dx$$

and

$$d\nu_j = \mathrm{r}_j f dx$$

for some $f \in h^1(\mathbf{R}^n)$. Furthermore,

$$\|f\|_{h^1} \approx \|f\|_{L^1} + \sum \|\mathrm{r}_j f\|_{L^1} \le \|\mathcal{L}\| + \sum \|r_j(\mathcal{L})\| \le C\|\mathcal{L}\|.$$

4 The Jones-Journé Theorem for h^1

We first quote the result for the Hardy space $\mathrm{H}^1(\mathbf{R}^n)$ and then give the version for $h^1(\mathbf{R}^n)$.

Theorem 10 **(Jones-Journé [JJ])** *If* $\{f_n\}$ *is a bounded sequence in* $\mathrm{H}^1(\mathbf{R}^n)$ *and* $f_n(x) \to f(x)$ *almost everywhere, then* $f \in \mathrm{H}^1(\mathbf{R}^n)$ *and* $f_n \stackrel{*}{\rightharpoonup} f$ *in* H^1, *i.e.,*

$$\int_{\mathbf{R}^n} f_n \phi \, dx \to \int_{\mathbf{R}^n} f \phi \, dx$$

for all $\phi \in \mathrm{VMO}_0(\mathbf{R}^n)$.

Theorem 11 *If $\{f_n\}$ is a bounded sequence in $h^1(\mathbf{R}^n)$ and $f_n(x) \to f(x)$ almost everywhere, then $f \in h^1(\mathbf{R}^n)$ and $f_n \overset{*}{\rightharpoonup} f$ in h^1, i.e.,*

$$\int_{\mathbf{R}^n} f_n \phi \, dx \to \int_{\mathbf{R}^n} f \phi \, dx \tag{2}$$

for all $\phi \in$ $\mathrm{vmo}(\mathbf{R}^n)$.

Proof We follow the proof of the Jones-Journé theorem given by Coifman, Lions, Meyer and Semmes in [CLMS] and make changes appropriate to the local case.

Since C_c^∞ is dense in vmo, it suffices to prove equation (2) for $\phi \in C^\infty(\mathbf{R}^n)$ with compact support in $B(0,R)$ for some $R > 0$. By Egorov's theorem applied to $\{f_n\}$ in $B(0,R)$, for every $\epsilon > 0$ there exists a measurable set $E \subset B(0,R)$ with $|E| < \epsilon$ and $f_n \to f$ uniformly on $B(0,R) \setminus E$. We want to write

$$\int (f_n - f)\phi \, dx = \int_{B(0,R)\setminus E} (f_n - f)\phi \, dx + \int_E f_n \phi \, dx - \int_E f \phi \, dx.$$

The first term on the right tends to zero as $n \to \infty$, and we can estimate the third term by $\|\phi\|_\infty \int_E f \, dx$, which tends to zero with ϵ (since by Fatou's lemma $f \in L^1$.) The duality of h^1 and bmo and the boundedness of $\{f_n\}$ in h^1 allows us to bound the second term by a constant multiple of $\|\chi_E \phi\|_{\mathrm{bmo}}$, but unfortunately this is not controlled by ϵ.

As in [CLMS], we fix $\lambda > 0$ and replace the characteristic function χ_E by the function

$$w_\lambda = \big(1 + \lambda \log M(\chi_E)\big)_+,$$

where $M(\cdot)$ denotes the Hardy-Littlewood maximal function. Since $\chi_E \le M(\chi_E) \le 1$ almost everywhere, we have also that $\chi_E \le w_\lambda \le 1$ almost everywhere, and hence $1 - w_\lambda$ vanishes on E. Writing

$$\int (f_n - f)\phi \, dx = \int (f_n - f)(1 - w_\lambda)\phi \, dx + \int f_n w_\lambda \phi \, dx - \int f w_\lambda \phi \, dx, \tag{3}$$

we can estimate the first term on the right by

$$\left| \int (f_n - f)(1 - w_\lambda)\phi \, dx \right| \le \|\phi\|_{L^1} \sup_{x \in B(0,R)\setminus E} |f_n(x) - f(x)|,$$

which tends to zero as $n \to \infty$, with ϵ fixed.

The third term on the right of equation (3) can be bounded as follows:

$$\left| \int f w_\lambda \phi \, dx \right| \le \|\phi\|_\infty \int_{\{w_\lambda > 0\}} |f| \, dx.$$

Note that

$$|\{w_\lambda > 0\}| = |\{M(\chi_E) > e^{-1/\lambda}\}| \le C\epsilon e^{1/\lambda}$$

by the weak L^1 boundedness of the maximal function. Thus the third term can be made arbitrarily small if we fix λ and let ϵ go to zero.

As for the second term on the right of equation (3), again by the duality of h^1 and bmo and the boundedness of $\{f_n\}$ in h^1,

$$\begin{aligned}\left|\int f_n w_\lambda \phi\, dx\right| &\le C\|w_\lambda \phi\|_{\mathrm{bmo}}\\ &= C \sup_{|Q|<1} \frac{1}{|Q|}\int_Q |\phi w_\lambda(x) - (\phi w_\lambda)_Q|\, dx + C \sup_{|Q|\ge 1} \frac{1}{|Q|}\int_Q |\phi w_\lambda(x)|\, dx\\ &\le C \sup_{|Q|<1} \frac{1}{|Q|}\int_Q |\phi(x) w_\lambda(x) - \phi(x)(w_\lambda)_Q|\, dx\\ &\quad + C \sup_{|Q|<1} \frac{1}{|Q|}\int_Q |\phi(x)(w_\lambda)_Q - \phi_Q(w_\lambda)_Q|\, dx + C\|\phi\|_\infty \|w_\lambda\|_{L^1}\\ &\le C[\|\phi\|_\infty \|w_\lambda\|_* + |(w_\lambda)_Q|\, \|\nabla\phi\|_\infty \operatorname{diam}(Q) + |\phi\|_\infty \|w_\lambda\|_{L^1}].\end{aligned}$$

As in [CLMS], we can estimate

$$\begin{aligned}\|w_\lambda\|_{\mathrm{BMO}} &= \left\| \left(1+\lambda \log M(\chi_E)\right)_+ \right\|_* \le \left\| \left(1+\lambda \log M(\chi_E)\right) \right\|_*\\ &= \lambda \|\log M(\chi_E)\|_* \le C_n \lambda\end{aligned}$$

by a result of Coifman and Rochberg [CR]. Moreover,

$$\|w_\lambda\|_{L^1} \le |\{w_\lambda > 0\}| \le C\epsilon e^{1/\lambda}$$

as above. Finally, fixing $\delta > 0$, we have

$$|(w_\lambda)_Q|\, \|\nabla\phi\|_\infty \operatorname{diam}(Q) \le \|\nabla\phi\|_\infty \delta$$

if $\operatorname{diam}(Q) \le \delta$ and

$$|(w_\lambda)_Q|\, \|\nabla\phi\|_\infty \operatorname{diam}(Q) \le C\|\nabla\phi\|_\infty \delta^{1-n} |\{w_\lambda > 0\}| \le C\|\nabla\phi\|_\infty \delta^{1-n}\epsilon e^{1/\lambda}$$

if $\operatorname{diam}(Q) > \delta$.

Thus all in all we can make the right-hand-side of equation (3) arbitrarily small by first letting $n \to \infty$, then $\epsilon \to 0$, then $\delta \to 0$ and $\lambda \to 0$. This proves (2) for all $\phi \in C_c^\infty(\mathbf{R}^n)$.

By weak-$*$ compactness of the ball in $h^1(\mathbf{R}^n)$, there exists a $g \in h^1(\mathbf{R}^n)$ such that $f_{n_k} \stackrel{*}{\rightharpoonup} g$ for some subsequence f_{n_k}. Thus $\int f\phi = \int g\phi$ for all $\phi \in C_c^\infty(\mathbf{R}^n)$, hence $f = g \in h^1(\mathbf{R}^n)$ and $f_n \stackrel{*}{\rightharpoonup} f$ in h^1.

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