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ON JOINT SPECTRA OF NON-COMMUTING NORMAL OPERATORS

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The purpose of the paper is to show that the Harte spectrum and the bicommutant spectrum of an arbitrary n-tuple of normal Hilbert space operators can be obtained from the spectral set γ introduced by McIntosh and Pryde. It is also proved that many commonly used joint spectra of an n-tuple of normal m by m matrices are equal. These results are non-commutative variants of some theorems proved by McIntosh, Pryde, and Ricker for commuting sets of operators.

Let H be a complex Hilbert space. All operators considered in the sequel are assumed to be bounded linear operators on H. In [**7, 8**] McIntosh and Pryde introduced a notion of a spectral set $\gamma(T)$ associated to each n-tuple $T = (T_1, \dots, T_n)$ of operators and defined as follows:

$$\gamma(T) = \Big\{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n : \sum_{j=1}^{n} (T_j - \lambda_j)^2 \text{ is not invertible in } \mathcal{B}(H)\Big\}.$$

(Here we write for simplicity $T_j - \lambda_j$ instead of $T_j - \lambda_j I$.) This set has proved useful not only in the spectral theory of self-adjoint operators but also in comparing various types of joint spectra of commuting n-tuples of operators (see [**9**]). One advantage of the set $\gamma(T)$ over other joint spectra is that it can be easily computed.

In this paper we show that the spectral set $\gamma(T)$ is also useful in the multiparameter spectral theory of normal (not necessarily commuting) operators. Moreover we prove that many known joint spectra coincide on the n-tuples of normal m by m matrices.

We recall some definitions of joint spectra. Let, as before, $T = (T_1, \dots, T_n)$ be an n-tuple of operators. A point $\lambda = (\lambda_1, \dots, \lambda_n)$ of $\mathbb{C}^n$ is not in the *left spectrum* of T if there exist operators $S_1, \dots, S_n \in \mathcal{B}(H)$ such that $\sum_{j=1}^{n} S_j(T_j - \lambda_j) = I$. The left spectrum of T will be denoted by $\sigma_l(T)$. The *right spectrum*, $\sigma_r(T)$, is defined analogously. The *Harte spectrum* of T $\big($in $\mathcal{B}(H)\big)$ is the union of the left and right joint spectra; in symbols

$$\sigma_H(T) = \sigma_l(T) \cup \sigma_r(T).$$

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It is well-known (see [**6**, Theorems 2.5 and 2.4]) that

$$\sigma_l(T) = \left\{\lambda \in \mathbb{C}^n : \inf_{\|x\|=1} \sum_{j=1}^{n} \|(T_j - \lambda_j)x\| = 0\right\}$$

(the approximate point spectrum) and

$$\sigma_r(T) = \left\{\lambda \in \mathbb{C}^n : \sum_{j=1}^{n} \big((T_j - \lambda_j)(H)\big) \neq H\right\}$$

(the defect spectrum).

Let $\mathcal{S}$ be a non-empty subset of $\mathcal{B}(H)$. The commutant $\mathcal{S}'$ of $\mathcal{S}$ is the set of all operators that commute with every element of $\mathcal{S}$. The bicommutant $\mathcal{S}''$ is the commutant of $\mathcal{S}'$. It is clear from the definition that $\mathcal{S}''$ is a closed, unital, and inverse-closed subalgebra of $\mathcal{B}(H)$ containing the set $\mathcal{S}$. The *bicommutant spectrum* of an n-tuple $T = (T_1, \dots, T_n)$, denoted by $\sigma''(T)$, is the Harte spectrum of T in its bicommutant $\{T_1, \dots, T_n\}''$. One can also consider the Harte spectrum of $T = (T_1, \dots, T_n)$ in the Banach algebra $[T_1, \dots, T_n]$ generated by $T_1, \dots, T_n$ and the identity. Let us denote this spectrum by $\widehat{\sigma}(T)$.

To define the last joint spectrum to be used in this paper we require "non-commutative polynomials" (see [**6**, pp. 98–99]). By $\mathcal{P}^{(n)}$ we denote the algebra of all polynomials over $\mathbb{C}$ in non-commutative indeterminates $X_1, \dots, X_n$. In other words, $\mathcal{P}^{(n)}$ is the free associative complex unital algebra generated by the symbols $X_1, \dots, X_n$. An n-tuple of operators $(T_1, \dots, T_n) \in \mathcal{B}(H)^n$ induces a homomorphism $f \mapsto f(T_1, \dots, T_n)$ from $\mathcal{P}^{(n)}$ to $\mathcal{B}(H)$ which preserves the identity and sends each X_j to the corresponding T_j $(j = 1, \dots, n)$. A system $(f_1, \dots, f_m) \in \big(\mathcal{P}^{(n)}\big)^m$ will be identified with a polynomial map $f : \mathcal{B}(H)^n \to \mathcal{B}(H)^m$ which sends $(T_1, \dots, T_n)$ to $\big(f_1(T_1, \dots, T_n), \dots, f_m(T_1, \dots, T_n)\big)$. The restriction of this mapping to the scalar multiples of the unit $\mathbb{C}^n \subset \mathcal{B}(H)^n$ takes its values in $\mathbb{C}^m \subset \mathcal{B}(H)^m$ and reduces to the system of "numerical" polynomials.

The *Waelbroeck spectrum* of an n-tuple $T = (T_1, \dots, T_n)$ (sometimes called the rational, polynomial or rationally convex joint spectrum, see [**1**, **9**], and [**10**] respectively) is defined to be the set:

$$\sigma_{\mathcal{R}}(T) = \left\{\lambda \in \mathbb{C}^n : f(\lambda) \in \sigma\big(f(T)\big) \text{ for every } f \in \mathcal{P}^{(n)}\right\}.$$

It is known (see [**10**]) that for an arbitrary n-tuple $T = (T_1, \dots, T_n)$ of operators the following inclusions are true:

$$\sigma_H(T) \subset \sigma''(T) \subset \sigma_{\mathcal{R}}(T) \subset \widehat{\sigma}(T). \tag{1}$$

Moreover all these spectra are compact (possibly empty) subsets of $\mathbb{C}^n$ (see [**6**, **10**], and [**11**]). Notice also that for a single operator T the sets $\sigma_H(T)$, $\sigma''(T)$, and $\sigma_{\mathcal{R}}(T)$ coincide and are equal to the usual spectrum of this element denoted by $\sigma(T)$.

For convenience of the reader we shall state the following result (see [**2**, Lemma 2.4]) which will be used repeatedly in the course of the paper.

DASH'S LEMMA. *Let $T = (T_1, \ldots, T_n)$ be an arbitrary n-tuple of operators. Then*

(a) $\lambda \in \sigma_l(T)$ *if and only if* $0 \in \sigma\Big(\sum_{j=1}^{n} (T_j - \lambda_j)^*(T_j - \lambda_j)\Big)$,

(b) $\lambda \in \sigma_r(T)$ *if and only if* $0 \in \sigma\Big(\sum_{j=1}^{n} (T_j - \lambda_j)(T_j - \lambda_j)^*\Big)$.

This lemma immediately implies

COROLLARY 1. *If $T = (T_1, \ldots, T_n)$ is an n-tuple of normal operators, then*

$$\sigma_l(T) = \sigma_r(T) = \sigma_H(T).$$

For n-tuples of self-adjoint operators we can prove more.

PROPOSITION 1. *Let $T = (T_1, \ldots, T_n)$ be an arbitrary n-tuple of self-adjoint operators. Then*

$$\sigma_l(T) = \sigma_r(T) = \sigma_H(T) = \sigma''(T) = \sigma_{\mathcal{R}}(T) = \widehat{\sigma}(T) = \gamma(T). \tag{2}$$

PROOF: From Dash's lemma we get

$$\sigma_l(T) = \sigma_r(T) = \sigma_H(T) = \gamma(T).$$

It is also clear that

$$\widehat{\sigma}(T) \subset \sigma(T_1) \times \cdots \times \sigma(T_n) \subset \mathbb{R}^n.$$

Hence, in view of (1), it is enough to show that $\widehat{\sigma}(T) \subset \gamma(T)$.

Suppose $\lambda \notin \gamma(T)$. This implies that the operator $S = \sum_{j=1}^{n} (T_j - \lambda_j)^2$ is invertible in $\mathcal{B}(H)$. As $[T_1, \ldots, T_n]$ is a C^*-algebra and $S \in [T_1, \ldots, T_n]$ there exists V in $[T_1, \ldots, T_n]$ such that $VS = I$. But this gives $\sum_{j=1}^{n} V(T_j - \lambda_j)(T_j - \lambda_j) = I$. So $\lambda \notin \widehat{\sigma}(T)$ and the proof is complete. ∎

REMARKS. 1. The equality $\sigma_l(T) = \gamma(T)$ for T an n-tuple of self-adjoint operators was proved in [**4**, Proposition 2].

2. For a commuting n-tuple $T = (T_1, \ldots, T_n)$ of Banach space operators, equalities (2) imply that the spectra $\sigma(T_j)$ $(j = 1, \ldots, n)$ are real (see [**9**, Theorem 1]). This

is no longer true if the operators T_j do not commute. To see this take the following 2 by 2 matrices: $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} i/2 & 1 \\ 1 & 0 \end{pmatrix}$. Then $\sigma(A_1) = \{0,1\}$, $\sigma(A_2) = \{-\sqrt{15}/4 + i/4, \sqrt{15}/4 + i/4\}$, but $\sigma_l(A_1,A_2) = \sigma_r(A_1,A_2) = \sigma_H(A_1,A_2) = \sigma''(A_1,A_2) = \sigma_{\mathcal{R}}(A_1,A_2) = \widehat{\sigma}(A_1,A_2) = \gamma(A_1,A_2) = \emptyset$.

Now we proceed to the case of normal operators. We start with

Lemma. *Let $T = (T_1, \dots, T_n)$ be an arbitrary n-tuple of normal operators. Then*

$$\sigma_H(T,T^*) = \{(\lambda, \overline{\lambda}) \in \mathbb{C}^{2n} : \lambda \in \sigma_H(T)\},$$

where $T^* = (T_1^*, \dots, T_n^*)$ and $\overline{\lambda} = (\overline{\lambda_1}, \dots, \overline{\lambda_n})$.

Proof: By Theorem 3.4(i) of [**6**] we have

$$\sigma_H(T,T^*) \subset \{(\lambda, \overline{\lambda}) \in \mathbb{C}^{2n} : \lambda \in \sigma_H(T)\}.$$

The reverse inclusion follows from the identity

$$\sum_{j=1}^{n} (T_j - \lambda_j)^*(T_j - \lambda_j) + \sum_{j=1}^{n} \left(T_j^* - \overline{\lambda_j}\right)^* \left(T_j^* - \overline{\lambda_j}\right) = 2\sum_{j=1}^{n} (T_j - \lambda_j)^*(T_j - \lambda_j)$$

and Dash's lemma. ∎

Corollary 2. *If $T = (T_1, \dots, T_n)$ is an arbitrary n-tuple of normal operators, then*

$$\{f(\lambda, \overline{\lambda}) : \lambda \in \sigma_H(T)\} \subset \sigma_H(f(T,T^*))$$

for every polynomial map $f \in \left(\mathcal{P}^{(2n)}\right)^m$.

Proof: This is an immediate consequence of the Lemma and Theorem 3.4(ii) of [**6**]. ∎

Now observe that if $T = (T_1, \dots, T_n)$ is an n-tuple of normal operators and $T_j = \mathrm{Re}T_j + i\,\mathrm{Im}T_j$, $j = 1, \dots, n$, then, in view of Corollary 2,

$$\lambda \in \sigma_H(T) \text{ implies } (\mathrm{Re}\lambda, \mathrm{Im}\lambda) \in \sigma_H(\mathrm{Re}T, \mathrm{Im}T),$$

where $\mathrm{Re}T = (\mathrm{Re}T_1, \dots, \mathrm{Re}T_n)$, $\mathrm{Im}T = (\mathrm{Im}T_1, \dots, \mathrm{Im}T_n)$, $\mathrm{Re}\lambda = (\mathrm{Re}\lambda_1, \dots, \mathrm{Re}\lambda_n)$, and $\mathrm{Im}\lambda = (\mathrm{Im}\lambda_1, \dots, \mathrm{Im}\lambda_n)$.

Let us introduce the following notation (see [**9**]):

$$\Pi(T) = (\mathrm{Re}T, \mathrm{Im}T) \text{ and } p(z_1, \dots, z_{2n}) = (z_1 + iz_{n+1}, \dots, z_n + iz_{2n}).$$

Theorem 1. *Let $T = (T_1, \dots, T_n)$ be an arbitrary n-tuple of normal operators. Then*

$$\sigma_H(T) = \sigma''(T) = p\Big(\gamma\big(\Pi(T)\big)\Big).$$

Proof: First we prove that $\sigma_H(T) = p\Big(\gamma\big(\Pi(T)\big)\Big)$. By the one-way spectral mapping property of σ_H (see [**6**, Theorem 3.2]) and Proposition 1 we get

$$p\Big(\gamma\big(\Pi(T)\big)\Big) = p\Big(\sigma_H\big(\Pi(T)\big)\Big) \subset \sigma_H\Big(p\big(\Pi(T)\big)\Big) = \sigma_H(T).$$

On the other hand, if $\lambda \in \sigma_H(T)$, then $(\mathrm{Re}\lambda, \mathrm{Im}\lambda) \in \sigma_H\big(\Pi(T)\big)$, and therefore

$$\lambda = p(\mathrm{Re}\lambda, \mathrm{Im}\lambda) \in p\Big(\sigma_H\big(\Pi(T)\big)\Big) = p\Big(\gamma\big(\Pi(T)\big)\Big)$$

as was to be proved.

To obtain the equality $\sigma_H(T) = \sigma''(T)$ it is enough to show that $\sigma''(T) \subset \sigma_H(T)$ as the reverse inclusion is obvious (see (1)). Suppose $\lambda \notin \sigma_H(T)$. By Dash's lemma the operator $S = \sum_{j=1}^{n} (T_j - \lambda_j)^*(T_j - \lambda_j)$ has an inverse V in $\mathcal{B}(H)$. Since the operators $T_1, \dots, T_n$ are normal, it follows from Fuglede's theorem that $S \in \{T_1, \dots, T_n\}''$. Hence $V \in \{T_1, \dots, T_n\}''$ and from $\sum_{j=1}^{n} V(T_j - \lambda_j)^*(T_j - \lambda_j) = I$ we conclude that $\lambda \notin \sigma''(T)$. This completes the proof. $\square$

Remarks. 1. Theorem 1 can be viewed as a generalisation of Theorem 2 and Lemma 2 of [**9**] to non-commuting n-tuples of normal operators.

2. Example 4.2, pp. 212-213 in [**1**], shows that even for a commuting pair of normal operators (T_1, T_2) the inclusion

$$\sigma''(T_1, T_2) \subset \sigma_{\mathcal{R}}(T_1, T_2)$$

can be proper.

3. One can easily modify Taylor's example [**12**, pp.189–191] to obtain two commuting Hilbert space operators T_1, T_2 (one of them not being normal) with

$$\sigma_H(T_1, T_2) \neq \sigma''(T_1, T_2).$$

So, Theorem 1 is not true when the operators $T_1, \dots, T_n$ are not normal.

4. Since Dash's lemma and Theorem 3.4 of [**6**] are valid for an arbitrary C^*-algebra the same is true for the results presented above.

In the finite dimensional case we can prove the following:

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Theorem 2. *Let $A = (A_1, \dots, A_n)$ be an arbitrary n-tuple of normal m by m matrices with complex entries. Then*

$$\begin{aligned}\sigma_l(A) &= \sigma_r(A) = \sigma_H(A) = \sigma''(A) = \sigma_{\mathcal{R}}(A) = \widehat{\sigma}(A) \\ &= p\Big(\gamma\big(\Pi(A)\big)\Big) = \{\textit{joint eigenvalues of } A_1, \dots, A_n\}.\end{aligned}$$

Proof: By Theorem 2.6 in [**6**]

$$\sigma_l(A) = \{\text{joint eigenvalues of } A_1, \dots, A_n\}.$$

In view of Theorem 1, Corollary 1, and (1) it is enough to show that $\widehat{\sigma}(A) \subset \sigma_H(A)$.

If $\widehat{\sigma}(A) = \emptyset$, then we are done. If, on the other hand, $\widehat{\sigma}(A) \neq \emptyset$, then by Theorem 1 of [**11**] (see also Proposition 2 of [**3**])

$$\widehat{\sigma}(A) = \Big\{\big(\varphi(A_1), \dots, \varphi(A_n)\big) : \varphi \text{ a multiplicative linear functional on } [A_1, \dots, A_n]\Big\}.$$

We always assume that multiplicative linear functionals are non-zero. Notice also that $A_j^* \in [A_1, \dots, A_n]$ for $j = 1, \dots, n$ as $A_j^* = f_j(A_j)$ for some polynomial f_j (see [**5**, p.172, Exercise 3(a)]). Therefore for an arbitrary multiplicative linear functional φ on the algebra $[A_1, \dots, A_n]$ we obtain

$$\sum_{j=1}^{n} \big(A_j - \varphi(A_j)\big)^* \big(A_j - \varphi(A_j)\big) \subset \text{ kernel } \varphi \neq [A_1, \dots, A_n].$$

Thus the element $\sum\limits_{j=1}^{n} \big(A_j - \varphi(A_j)\big)^* \big(A_j - \varphi(A_j)\big)$ is not invertible in $[A_1, \dots, A_n]$ and consequently not invertible in the algebra of all m by m matrices. An application of Dash's lemma concludes the proof. ∎

Remarks. 1. In [**4**, Proposition 5], it is shown that for self-adjoint matrices $A = (A_1, \dots, A_n)$, $\gamma(A) = \{\text{joint eigenvalues of } A_1, \dots, A_n\}$.

2. To show that Theorem 2 is not true when the matrices are not normal take the following two 5 by 5 matrices:

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then (see [**10**, Example 1])

$$p\Big(\gamma\big(\Pi(A_1,A_2)\big)\Big)=\sigma_l(A_1,A_2)=\sigma_r(A_1,A_2)=\sigma_H(A_1,A_2)=\sigma''(A_1,A_2)=\emptyset$$

while

$$\sigma_{\mathcal{R}}(A_1,A_2)=\widehat{\sigma}(A_1,A_2)=\big\{(0,0)\big\}.$$

3. To give another example showing that it can happen that $\sigma_H(A_1,A_2)\neq\sigma''(A_1,A_2)$, consider the following two 3 by 3 matrices:

$$A_1=\begin{pmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}\text{ and }A_2=\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & -1\\ 0 & 0 & 2\end{pmatrix}.$$

It is a matter of simple computation to show that

$$\begin{aligned}p\Big(\gamma\big(\Pi(A_1,A_2)\big)\Big)=\emptyset,\ \sigma_l(A_1,A_2)=\big\{(1,0)\big\},\ \sigma_r(A_1,A_2)=\big\{(0,2)\big\},\\ \sigma_H(A_1,A_2)=\big\{(1,0),(0,2)\big\}\neq\sigma''(A_1,A_2)=\sigma_{\mathcal{R}}(A_1,A_2)\\ =\widehat{\sigma}(A_1,A_2)=\big\{(1,0),(1,1),(0,2)\big\}.\end{aligned}$$

Finally we prove

Proposition 2. *For an arbitrary n-tuple $A=(A_1,\dots,A_n)$ of m by m complex matrices, $\sigma_{\mathcal{R}}(A)=\widehat{\sigma}(A)$.*

Proof: From the proof of Theorem 2 it is clear that it is enough to show

$$\Big\{\big(\varphi(A_1),\dots,\varphi(A_n)\big):\varphi\text{ a multiplicative linear functional on }[A_1,\dots,A_n]\Big\}\subset\sigma_{\mathcal{R}}(A).$$

Let $\varphi:[A_1,\dots,A_n]\to\mathbb{C}$ be an arbitrary multiplicative linear functional and f an arbitrary polynomial in $\mathcal{P}^{(n)}$. Then we get

$$f\big(\varphi(A_1),\dots,\varphi(A_n)\big)=\varphi\big(f(A_1,\dots,A_n)\big)\in\widehat{\sigma}\big(f(A_1,\dots,A_n)\big)=\sigma\big(f(A_1,\dots,A_n)\big),$$

and the result follows. ∎

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