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A Brunn-Minkowski Type Theorem on the Minkowski Spacetime

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Abstract. In this article, we derive a Brunn-Minkowski type theorem for sets bearing some relation to the causal structure on the Minkowski spacetime $\mathbb{L}^{n+1}$. We also present an isoperimetric inequality in the Minkowski spacetime $\mathbb{L}^{n+1}$ as a consequence of this Brunn-Minkowski type theorem.

1 Introduction

The Brunn-Minkowski theorem states that, for non-empty compact sets A, B in the Euclidean space $\mathbb{E}^n$,

$$V^{\frac{1}{n}}(A+B) \geq V^{\frac{1}{n}}(A) + V^{\frac{1}{n}}(B) \tag{1.1}$$

and equality holds for $A, B \subset \mathbb{E}^n$ with $V(A), V(B) \neq 0$ if and only if A and B are homothetic convex bodies (*i.e.*, convex and compact sets), where $A + B$ is the Minkowski sum or vector sum of A and B given by $A + B = \{a + b : a \in A, b \in B\}$; and $V(D)$ is the n-dimensional volume of $D \subset \mathbb{E}^n$ (See [BF], [BZ], [F], [W]). This is the core result of the Brunn-Minkowski theory in the theory of convex bodies and many prototypes of geometric inequalities originate from this, for example, the classical isoperimetric inequality and the Minkowski inequality of mixed volumes.

Our aim is to derive a Brunn-Minkowski type inequality in the Minkowski spacetime $\mathbb{L}^{n+1}$. Even though the Minkowski spacetime $\mathbb{L}^{n+1}$ is a vector space as is the Euclidean space, the nature of $\mathbb{L}^{n+1}$ is quite different from that of the Euclidean space; especially, the causality conditions of General Relativity are important in the differential geometry of the Minkowski spacetime. Thus, it is natural to try to find a Brunn-Minkowski type inequality in the Minkowski spacetime for sets bearing some relation to the causal structure. Our Brunn-Minkowski type theorem is not strongly related to convexity. In particular, the optimal situation does not occur for the convex case, *cf.* Remark 6.2. Let S be a compact, simply connected, achronal, piecewise smooth spacelike hypersurface of $\mathbb{L}^{n+1}$ contained in the chronological future $I^+(O)$ of the origin O of $\mathbb{L}^{n+1}$.

The (*upper*) *hyperbolic space* $\mathbb{H}(r)$ of radius $r > 0$ in $\mathbb{L}^{n+1}$ is

$$\mathbb{H}(r) = \{x = (x_0, x_1, \ldots, x_n) \in \mathbb{L}^{n+1} : d(O, x) = r\},$$

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where d is the Lorentzian distance on $\mathbb{L}^{n+1}$. Now $\mathbb{H}(r)$ is a smooth spacelike hypersurface of $\mathbb{L}^{n+1}$ with constant curvature $-1/r^2$. Let $\mu(S)$ be the subset of $\mathbb{H}(1)$ defined by

$$\mu(S) = \left\{ \frac{x}{\|x\|} \in \mathbb{H}(1) : x \in S \right\}.$$

Also let $t^* > 0$ be defined as $t^* = d(O, S) = \sup_{q \in S} d(O, q)$. The *infinite cone* Ω_S of S is defined by

$$\Omega_S = \{\lambda y \in \mathbb{L}^{n+1} : y \in \mu(S), \lambda \geq 0\}.$$

By the *cone* $K = C(S)$ of S (with respect to O) we mean the compact set enclosed by S and Ω_S in $\mathbb{L}^{n+1}$. Explicitly,

$$K = C(S) = \{\lambda q : q \in S, 0 \leq \lambda \leq 1\}.$$

The (*past*) *parallel* S_t of S with distance t is then given by

$$S_t = \{p \in C(S) : d(p, S) = t\}.$$

For the cone $K = C(S)$ of S, we let $K_t = C(S_t)$ and $\mathbb{B}_K(t) = C\big(t\mu(S)\big)$, where $t\mu(S) = \{tq : q \in \mu(S)\} \subset \mathbb{H}(t)$. Clearly, $S_0 = S$ and $K_0 = K$. Let $\tilde{\mathcal{B}}_p^-(t) = \mathcal{B}_p^-(t) \cap C(S)$ for $0 < t < t^*$ and $p \in S$, where $\mathcal{B}_p^-(t) = \{q \in J^-(p) : d(q, p) \geq t\}$, the *past outer ball* of radius $t > 0$ centered at p (*cf.* [BEE, p. 145]). Then from the definitions of S_t and K_t, we can see that

$$K_t = C(S_t) = \{x \in C(S) : d(x, S) \geq t\} = \bigcup_{p \in S} \tilde{\mathcal{B}}_p^-(t).$$

Throughout this paper, we will employ the following:

Convention 1.1 A *hypersurface* in $\mathbb{L}^{n+1}$ will always be a compact, simply connected, achronal, piecewise smooth spacelike hypersurface of $\mathbb{L}^{n+1}$ with piecewise smooth boundary contained in the chronological future $I^+(O)$ of the origin O of $\mathbb{L}^{n+1}$ unless explicitly mentioned.

We can now state our main result.

Brunn-Minkowski Type Theorem *Let S be a hypersurface in the Minkowski spacetime $\mathbb{L}^{n+1}$ with $t^* = d(O, S) > 0$ and let $K = C(S)$ be the cone of S. Let $V(B)$ be the $(n+1)$-dimensional Lorentzian volume of $B \subset \mathbb{L}^{n+1}$. Then, for $0 \leq t \leq t^*$,*

$$V^{\frac{1}{n+1}}(K_t) \geq V^{\frac{1}{n+1}}(K) - V^{\frac{1}{n+1}}\big(\mathbb{B}_K(t)\big). \tag{1.2}$$

Moreover, equality holds for some t $(0 < t \leq t^*)$ if and only if $S \subset \mathbb{H}(t^*)$.

As an application of this result, we will derive an isoperimetric inequality for cones in the Minkowski spacetime:

Isoperimetric Inequality *Let S be a hypersurface in the Minkowski spacetime $\mathbb{L}^{n+1}$ with $t^* = d(O, S)$ and let $K = C(S)$ be the cone of S. Let $A(S)$ be the n-dimensional Lorentzian volume of S and $\omega = V\big(\mathbb{B}_K(1)\big)$. Then,*

$$A^{n+1}(S) \leq (n+1)^{n+1} \omega V^n(K)$$

with equality only when $S \subset \mathbb{H}(t^*)$.

This last result is a Lorentzian version of the isoperimetric inequality for convex cones of P. L. Lions and F. Pacella [LP].

In Section 2, we recall some aspects of the Minkowski spacetime $\mathbb{L}^{n+1}$ needed in this paper. In Section 3, we present the k-dimensional Lorentzian volume systematically from the vector space structure of $\mathbb{L}^{n+1}$ and compare the k-dimensional Lorentzian volume with the k-dimensional Euclidean volume. In Section 4, we introduce an elementary cone of a PL-hypersurface in $\mathbb{L}^{n+1}$ and derive a Brunn-Minkowski type inequality for a PL-hypersurface by induction (Proposition 4.2). In Section 5, we present an approximation of a hypersurface by elementary cones (Proposition 5.2). In Section 6, we prove a Brunn-Minkowski type theorem for a hypersurface (Theorem 6.1). Finally, in Section 7, as an application of our Brunn-Minkowski type theorem, we prove an isoperimetric inequality for a hypersurface in $\mathbb{L}^{n+1}$ (Theorem 7.1).

2 Preliminaries

By the Minkowski $(n+1)$-spacetime $\mathbb{L}^{n+1}$ $(n \geq 1)$ we mean $\mathbb{R}^{n+1}$ with the scalar product g of index 1;

$$g(x, y) = x \cdot y = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i$$

for $x = (x_0, x_1, \dots, x_n)$, $y = (y_0, y_1, \dots, y_n) \in \mathbb{R}^{n+1}$. We shall consider $x, y, \dots$ not only as points but also vectors. The tangent vectors $v \in T\mathbb{L}^{n+1}$ are classified by the causal character; *timelike* if $g(v, v) < 0$, *spacelike* if $g(v, v) > 0$ or $v = 0$ and *null* if $g(v, v) = 0$ and $v \neq 0$. A smooth submanifold M of $\mathbb{L}^{n+1}$ is said to be *spacelike* provided all tangent vectors to M are spacelike. We assume that $\mathbb{L}^{n+1}$ is time-oriented by $e_0 = (1, 0, \dots, 0)$; thus, we say that a nonspacelike tangent vector v to $\mathbb{L}^{n+1}$ is *future-directed* if $g(e_0, v) < 0$. The norm of a vector v is defined by $\|v\| = \sqrt{|g(v, v)|}$. A curve $\gamma\colon [0, c] \to \mathbb{L}^{n+1}$ is said to be *timelike* (*spacelike, null, nonspacelike*, respectively) if $\gamma'(t)$ for all $0 \leq t \leq c$ is *timelike* (spacelike, null, nonspacelike, respectively). A nonspacelike curve γ is said to be *future-directed* if $\gamma'(t)$ for all $0 \leq t \leq c$ is future-directed. The arc length of γ is given by

$$L(\gamma) = \int_0^c \|\gamma'(t)\| \, dt.$$

For two points $p, q \in \mathbb{L}^{n+1}$, $p \ll q$ means that there is a smooth future-directed timelike curve from p to q, and $p \leq q$ means that either $p = q$ or there is a smooth future-directed nonspacelike curve from p to q. The chronological future (respectively, past) of p is the set $I^+(p) = \{q \in \mathbb{L}^{n+1} : p \ll q\}$ (respectively, $I^-(p) = \{q \in \mathbb{L}^{n+1} : q \ll p\}$). The causal future (respectively, past) of p is the set $J^+(p) = \{q \in \mathbb{L}^{n+1} : p \leq q\}$ (respectively, $J^-(p) = \{q \in \mathbb{L}^{n+1} : q \leq p\}$). For a point $p \in \mathbb{L}^{n+1}$, $I(p) = I^+(p) \cup I^-(p)$ and $J(p) = J^+(p) \cup J^-(p)$, and for a set $A \subset \mathbb{L}^{n+1}$, $I(A) = \bigcup_{p \in A} I(p)$ and $J(A) = \bigcup_{p \in A} J(p)$. A set $B \subset \mathbb{L}^{n+1}$ is said to be *achronal* if the relation $p \ll q$ never holds for $p, q \in B$. Given $p, q \in \mathbb{L}^{n+1}$ with $p \leq q$, let $\Omega_{p,q}$ denote the space of all future-directed piecewise smooth nonspacelike curves from p to q. The Lorentzian distance $d\colon \mathbb{L}^{n+1} \times \mathbb{L}^{n+1} \to \mathbb{R} \cup \{\infty\}$ is defined as follows; for any $p \in M$,

$$d(p,q) := \begin{cases} 0 & \text{for } q \notin J^+(p), \\ \sup_{\gamma \in \Omega_{p,q}} L(\gamma) & \text{for } q \in J^+(p). \end{cases}$$

For two sets A and B in $\mathbb{L}^{n+1}$, the Lorentzian distance $d(A,B)$ from A to B is defined by

$$d(A,B) := \sup_{p \in A, q \in B} d(p,q). \tag{2.1}$$

Sometimes, we will use the notations $d(x,B)$ and $d(A,y)$ instead of $d(\{x\},B)$ and $d(A,\{y\})$.

All geodesics in $\mathbb{L}^{n+1}$ are of the form $\alpha(s) = x + sy$ for $x, y \in \mathbb{L}^{n+1}$ and, if α is a future-directed timelike geodesic, α realizes the distance from $\alpha(0) = x$ to $\alpha(s) = x + sy$. Especially, let $\alpha\colon [0,1] \to \mathbb{L}^{n+1}$ be the curve defined by $\alpha(s) = sx$ for $x \in I^+(O)$. Then $d(O,x) = L(\alpha) = \|x\|$. The Lorentzian distance d is continuous on $\mathbb{L}^{n+1} \times \mathbb{L}^{n+1}$ and the reverse triangle inequality holds for the Lorentzian distance d (*cf.* [BEE, p. 140]): If $p \le r \le q$, then

$$d(p,q) \ge d(p,r) + d(r,q). \tag{2.2}$$

In the remaining part of this section, we present some properties of the cone and parallel of a hypersurface S needed later.

Lemma 2.1 *Let $K = C(S)$ be the cone of a hypersurface S in $\mathbb{L}^{n+1}$ with $d(O,S) = t^*$. Then $K \subset \mathbb{B}_K(t^*)$.*

Proof Suppose that $x \in K \subset \Omega_S$. Since $d(O,S) = t^*$, $\|x\| \le t^*$. By definition, $\mathbb{B}_K(t^*) = \{y \in \Omega_S : \|y\| \le t^*\}$. Thus, we have $K \subset \mathbb{B}_K(t^*)$. ■

Lemma 2.2 *Let $K = C(S)$ be the cone of a hypersurface S in $\mathbb{L}^{n+1}$ with $d(O,S) = t^*$. If $V(K) = V\big(\mathbb{B}_K(t^*)\big)$, then $K = \mathbb{B}_K(t^*)$, so $S \subset \mathbb{H}(t^*)$.*

Proof By Lemma 2.1, $K \subset \mathbb{B}_K(t^*)$. Since both K and $\mathbb{B}_K(t^*)$ are compact sets and $V(K) = V\big(\mathbb{B}_K(t^*)\big)$, $K = \mathbb{B}_K(t^*)$. ■

Lemma 2.3 *Suppose that S is a hypersurface in $\mathbb{L}^{n+1}$ with $d(O,S) = t^*$. Let $x \in S$ such that $\|x\| = t^*$ and let $\xi\colon [0,t^*] \to [0,1]$ be the function given by*

$$\xi(t) = \frac{t^* - t}{t^*}.$$

Then $\xi(t)x \in S_t$ for $0 \le t \le t^*$.

Proof Since $d\big(\xi(t)x, x\big) = t$, $d\big(\xi(t)x, S\big) \ge t$. Suppose that $d\big(\xi(t)x, S\big) > t$. Then since d is continuous and S is compact, there exists $y \in S$ so that $d\big(\xi(t)x, y\big) = d\big(\xi(t)x, S\big) > 0$. By the reverse triangle inequality (2.2) for the Lorentzian distance d,

$$d(O,y) \ge d\big(O, \xi(t)x\big) + d\big(\xi(t)x, y\big) > t^*.$$

This contradicts that $d(O,S) = t^*$. Therefore, we have $d\big(\xi(t)x, S\big) = t$ and $\xi(t)x \in S_t$. ■

Lemma 2.4 *Let S be a hypersurface in $\mathbb{L}^{n+1}$ such that $S = S^1 \cup S^2$, where S^i are hypersurfaces in $\mathbb{L}^{n+1}$ and S^i may have common boundary points but their interiors do not intersect. Let $K = C(S)$ and $K^i = C(S^i)$ for $i = 1, 2$. Then*

$$K_t^1 \cup K_t^2 \subset K_t.$$

Proof Suppose that $x \in K_t^1 \cup K_t^2$. Then x belongs to one of K_t^1 or K_t^2, say, K_t^1. So, $d(x, S^1) \geq t$. Since $S = S^1 \cup S^2$,

$$t \leq d(x, S^1) = \sup_{y \in S^1} d(x, y) \leq \sup_{z \in S} d(x, z) = d(x, S).$$

Thus, $x \in K_t$. ∎

Suppose that S_t is the past parallel of a hypersurface S of $\mathbb{L}^{n+1}$ with $t^* = d(O, S)$ and $0 < t < t^*$. Then by the reverse triangle inequality (2.2), S_t is achronal. Let $\tilde{\mathbb{S}}_p^-(t) = \mathbb{S}_p^-(t) \cap C(S)$ for $0 < t < t^*$ and $p \in S$, where $\mathbb{S}_p^-(t) = \{q \in I^-(p) : d(q, p) = t\}$, the *past sphere* of radius $t > 0$ centered at p. Then S_t can be regarded as the envelope of the family of $\tilde{\mathbb{S}}_p^-(t)$ for $p \in S$. Using the continuity of $f(m) = d(m, S)$, the basic geometric properties of the family S_t are readily established:

Proposition 2.5 *Let S be a hypersurface in $\mathbb{L}^{n+1}$ with $t^* = d(O, S)$. Then, for $0 < t < t^*$, S_t is also a hypersurface in $\mathbb{L}^{n+1}$ with $d(O, S_t) = t^* - t$ and $\mu(S_t) = \mu(S)$.* ∎

3 The Minkowski Spacetime as a Vector Space

In this section, we will define the Lorentzian volume in $\mathbb{L}^{n+1}$ systematically using elementary facts from linear algebra.

Let $e_0, e_1, \ldots, e_n$ be the standard orthonormal basis of $\mathbb{L}^{n+1}$ as a vector space; $e_0 = (1, 0, \ldots, 0)$, $e_1 = (0, 1, 0, \ldots, 0)$, $\ldots$, $e_n = (0, \ldots, 0, 1) \in \mathbb{L}^{n+1}$. Let $\Pi(a_1, \ldots, a_m)$ be the space spanned by the m linearly independent vectors $a_1, \ldots, a_m \in \mathbb{L}^{n+1}$, $1 \leq m \leq n+1$. We may consider $\Pi(a_1, \ldots, a_m)$ as an m-dimensional smooth submanifold of $\mathbb{L}^{n+1}$ with the embedding $X \colon \mathbb{R}^m \to \mathbb{L}^{n+1}$ defined by

$$X(\xi^1, \ldots, \xi^m) = \sum_{i=1}^{m} \xi^i a_i \in \mathbb{L}^{n+1}.$$

Then the induced metric on $\Pi(a_1, \ldots, a_m)$ from $\mathbb{L}^{n+1}$ is given by

$$g = \sum_{i,j=1}^{m} g_{ij}\, d\xi^i\, d\xi^j,$$

where

$$g_{ij} = X_{\xi^i} \cdot X_{\xi^j} = a_i \cdot a_j.$$

The volume element of $\Pi(a_1, \dots, a_m)$ is

$$dV = \sqrt{|g|}d\xi^1 \cdots d\xi^m,$$

where $|g|$ denotes the absolute value of the determinant of the matrix (g_{ij}). From this observation, we define the Lorentzian volume of the parallelotope:

Definition 3.1 The m-dimensional Lorentzian volume $V_m(\mathbf{P})$ of the m-dimensional parallelotope $\mathbf{P} = \mathbf{P}(a_1, \dots, a_m)$ whose edges are $a_1, \dots, a_m$ in $\mathbb{L}^{n+1}$ is given by

$$V_m(\mathbf{P}) = |\det(a_i \cdot a_j)|^{\frac{1}{2}}.$$

By a slight modification of a vector product of the $(n-1)$ vectors in the Euclidean n-space (*cf.* [Bl], [Hs]), we define a vector product of the n vectors in the Minkowski $(n+1)$-spacetime as follows:

Definition 3.2 Suppose that $a_1, \dots, a_n$ are linearly independent vectors in $\mathbb{L}^{n+1}$. Then the vector product $a_1 \times \cdots \times a_n$ of the n vectors $a_1, \dots, a_n$ is defined by

$$a_1 \times \cdots \times a_n = \det \begin{pmatrix} e_0 & e_1 & \cdots & e_n \\ -a_1^0 & a_1^1 & \cdots & a_1^n \\ \vdots & \vdots & & \vdots \\ -a_n^0 & a_n^1 & \cdots & a_n^n \end{pmatrix}, \tag{3.1}$$

where $a_i = (a_i^0, a_i^1, \dots, a_i^n)$ for $i = 1, \dots, n$.

We denote by $R(u_1, \dots, u_m)$ the matrix with row vectors $u_1, \dots, u_m$ and we let $\bar{u} = (-u^0, u^1, \dots, u^n)$ for $u = (u^0, u^1, \dots, u^n)$. For convenience, we denote $(e_0, \dots, e_n)$ by e as for a row vector. Then

$$a_1 \times \cdots \times a_n = \det R(e, \bar{a}_1, \dots, \bar{a}_n).$$

Lemma 3.3 *Let $a = a_1 \times \cdots \times a_n$ and $b \in \mathbb{L}^{n+1}$. Then*

$$\det R(\bar{b}, \bar{a}_1, \dots, \bar{a}_n) = b \cdot a.$$

Proof Let $A = R(e, \bar{a}_1, \dots, \bar{a}_n) = (a_{ij})_{0 \le i,j \le n}$. Then

$$a = A_{00}e_0 + A_{01}e_1 + \cdots + A_{0n}e_n,$$

where A_{ij} is the cofactor of a_{ij}. Let $B = R(\bar{b}, \bar{a}_1, \dots, \bar{a}_n) = (b_{ij})_{0 \le i,j \le n}$, where $b = (b^0, b^1, \dots, b^n)$. Then

$$\det B = -b^0 B_{00} + b^1 B_{01} + \cdots + b^n B_{0n},$$

where B_{ij} is the cofactor of b_{ij}. Since $A_{0j} = B_{0j}$ for $j = 0, 1, \dots, n$, we have

$$\det B = -b^0 A_{00} + b^1 A_{01} + \cdots + b^n A_{0n} = b \cdot a. \qquad \blacksquare$$

By Lemma 3.3, we have

Corollary 3.4 *$a_1 \times \cdots \times a_n$ is normal to the n-dimensional space spanned by the n linearly independent vectors $a_1, \ldots, a_n$ in $\mathbb{L}^{n+1}$.*

Let D be an $m \times (n+1)$ matrix over $\mathbb{R}$, $m, n \geq 1$, and let $u_j = (u_j^0, u_j^1, \ldots, u_j^n)$ be the j-th row vector of D. We set $D \cdot D^t = D\varepsilon D^t$, where ε is the diagonal $(n+1)$-matrix with diagonal entries $-1, 1, \ldots, 1$. Explicitly,

$$D \cdot D^t = \Big(-u_i^0 u_j^0 + \sum_{k=1}^{n} u_i^k u_j^k\Big) = (u_i \cdot u_j).$$

Lemma 3.5 *For any $m \times (n+1)$ matrix D over $\mathbb{R}$, $m, n \geq 1$,*

$$\det D \cdot D^t = -\sum_B (\det B)^2 + \sum_C (\det C)^2, \tag{3.2}$$

where the first sum is taken over all $m \times m$ submatrices B of D containing the first column of D and the second sum is taken over all $m \times m$ submatrices C of D not containing the first column of D.

Proof See the Appendix. ■

Lemma 3.6 *The norm of $a_1 \times \cdots \times a_n$ is equal to the n-dimensional Lorentzian volume of the n-dimensional parallelotope whose edges are the vectors $a_1, \ldots, a_n \in \mathbb{L}^{n+1}$.*

Proof Let $\mathbf{P} = \mathbf{P}(a_1, \ldots, a_n)$ be the n-dimensional parallelotope whose edges are the vectors $a_1, \ldots, a_n$. Then by Definition 3.1

$$V_n(\mathbf{P}) = |\det(a_i \cdot a_j)|^{\frac{1}{2}}.$$

Let $D = R(a_1, \ldots, a_n)$. Then, by Lemma 3.5,

$$\det(a_i \cdot a_j) = (\det B_0)^2 - \sum_{j=1}^{n} (\det B_j)^2, \tag{3.3}$$

where B_k is the $n \times n$ matrix obtained from D by deleting the $(k+1)$-th column. Note that the right hand side of (3.3) is

$$-(a_1 \times \cdots \times a_n \cdot a_1 \times \cdots \times a_n).$$

Thus

$$V_n(\mathbf{P}) = \|a_1 \times \cdots \times a_n\|.$$ ■

Remark 3.7 **(Lorentzian Volume vs. Euclidean Volume)** We denote by V_m^E the m-dimensional Euclidean volume, $1 \leq m \leq n+1$.

1. Let $\mathbf{P} = \mathbf{P}(u_1, \ldots, u_{n+1})$ be an $(n+1)$-dimensional parallelotope whose edges are the vectors $u_1, \ldots, u_{n+1}$ in $\mathbb{R}^{n+1}$, where $u_j = (u_j^0, u_j^1, \ldots, u_j^n)$. Then

$$V_{n+1}^E(\mathbf{P}) = |\det(\langle u_i, u_j \rangle)|^{\frac{1}{2}} = |\det R(u_1, \ldots, u_{n+1})|,$$

where $\langle u_i, u_j \rangle = \sum_{k=0}^n u_i^k u_j^k$ is the Euclidean metric. On the other hand, the $(n+1)$-dimensional Lorentzian volume of $\mathbf{P}$ is

$$V_{n+1}(\mathbf{P}) = |\det(u_i \cdot u_j)|^{\frac{1}{2}} = |\det R(\bar{u}_1, \ldots, \bar{u}_{n+1})|,$$

where $u_i \cdot u_j = -u_i^0 u_j^0 + \sum_{k=1}^n u_i^k u_j^k$ is the Minkowski metric. Notice that

$$\det R(\bar{u}_1, \ldots, \bar{u}_{n+1}) = -\det R(u_1, \ldots, u_{n+1}).$$

Thus $V_{n+1}(\mathbf{P})$ and $V_{n+1}^E(\mathbf{P})$ coincide for an $(n+1)$-dimensional parallelotope $\mathbf{P}$.

2. Let $\mathbf{P} = \mathbf{P}(u_1, \ldots, u_m)$ be an m-dimensional parallelotope in $\mathbb{R}^{n+1}$, $1 \leq m \leq n$. Let $D = R(u_1, \ldots, u_m)$. Then

$$V_m^E(\mathbf{P}) = \sqrt{\det DD^t} = \sqrt{\sum_A (\det A)^2},$$

where the sum is taken over all $m \times m$ submatrices A of D. On the other hand, by Definition 3.1 and Lemma 3.5, we have

$$V_m(\mathbf{P}) = \sqrt{|\det D \cdot D^t|} = \sqrt{\left| -\sum_B (\det B)^2 + \sum_C (\det C)^2 \right|},$$

where the first sum is taken over all $m \times m$ submatrices B of D containing the first column of D and the second sum is taken over all $m \times m$ submatrices C of D not containing the first column of D. Thus, the Lorentzian volume of the m-dimensional parallelotope $\mathbf{P}$ is quite different from the Euclidean volume of $\mathbf{P}$ for $1 \leq m \leq n$. For example, let $\mathbf{P} = \mathbf{P}\big((1,1,1),(0,1,1)\big)$ in $\mathbb{R}^3$; then $V_2^E(\mathbf{P}) = \sqrt{5} \neq V_2(\mathbf{P}) = \sqrt{2}$. Furthermore, one can find an example having a positive Euclidean volume, but a zero Lorentzian volume, say, $\mathbf{P}\big((0,1,0),(1,0,1)\big)$. Comparing these two formulas, we have

$$V_m(\mathbf{P}) \leq V_m^E(\mathbf{P})$$

for an m-dimensional parallelotope in $\mathbb{R}^{n+1}$, $1 \leq m \leq n$.

Let $\Pi = \Pi(a_1, \ldots, a_n)$ be the n-dimensional space spanned by the n linearly independent spacelike vectors $a_1, \ldots, a_n \in \mathbb{L}^{n+1}$, which will be called a *spacelike hyperplane* in $\mathbb{L}^{n+1}$. Then by Corollary 3.4 the vector $a_1 \times \cdots \times a_n$ is normal to Π, so it is timelike. Let σ be a permutation on the n numbers $1, \ldots, n$; then by definition,

$$a_{\sigma(1)} \times \cdots \times a_{\sigma(n)} = (\operatorname{sgn} \sigma) a_1 \times \cdots \times a_n,$$

where sgn σ is +1 or -1 according as the permutation σ is even or odd. So, we may assume that $a_1 \times \cdots \times a_n$ is future-directed by reordering a_i's if necessary. Let $\mathbf{P} = \mathbf{P}(c\mathbf{n}, a_1, \ldots, a_n)$ be an $(n+1)$-dimensional parallelotope whose edges are $c\mathbf{n}, a_1, \ldots, a_n$, where c is a positive constant and

$$\mathbf{n} = \frac{a_1 \times \cdots \times a_n}{\|a_1 \times \cdots \times a_n\|}.$$

Then by Lemma 3.3

$$V_{n+1}(\mathbf{P}) = c\|a_1 \times \cdots \times a_n\|. \tag{3.4}$$

4 Elementary Cones

In this section, we introduce elementary cones in $\mathbb{L}^{n+1}$ and prove a Brunn-Minkowski type inequality (1.2) for elementary cones by induction.

We first consider the simplest case of our hypersurfaces in $\mathbb{L}^{n+1}$. Let $P \subset I^+(O)$ be an n-dimensional convex polytope in the spacelike hyperplane $\Pi \subset \mathbb{L}^{n+1}$, which will be called a *spacelike convex polytope* in $\mathbb{L}^{n+1}$. Then P is clearly a hypersurface in $\mathbb{L}^{n+1}$ in the sense of Convention 1.1. (For convenience, we denote by $\Pi(P)$ the hyperplane Π containing P.) We define a *simple cone* K to be the cone $K = C(P)$ for a spacelike convex polytope P. Note that a simple cone $K = C(P)$ is convex. For convenience, we will denote by $V(K)$ the $(n+1)$-dimensional Lorentzian volume for a cone K and by $A(S)$ the n-dimensional Lorentzian volume for a hypersurface S in $\mathbb{L}^{n+1}$. By the *height* of P (or $K = C(P)$) we mean the Lorentzian distance $d\big(O, \Pi(P)\big)$ from O to $\Pi(P)$. Note that the $(n+1)$-dimensional Lorentzian and Euclidean volumes of an $(n+1)$-dimensional parallelotope are equivalent (Remark 3.7). Thus, from (3.4), we have

$$V(K) = V\big(C(P)\big) = \frac{1}{n+1} h\, A(P), \tag{4.1}$$

where h is the height of K. Recall that $\mathbb{B}_K(t) = C\big(t\mu(P)\big)$ for $K = C(P)$.

Lemma 4.1 *Let $K = C(P)$ be a simple cone with $d(O, P) = t^*$ and $K_t = C(P_t)$. Then*

$$\xi(t)K \subset K_t, \quad \big(1 - \xi(t)\big)K \subset \mathbb{B}_K(t)$$

for each $t \in [0, t^*]$.

Proof The second relation follows Lemma 2.1. The first relation is clear for $t = 0$, t^*, since $\xi(0)K = 1K = K = K_0$ and $\xi(t^*)K = O = K_{t^*}$. Since a spacelike convex polytope P is compact and the Lorentzian distance d is continuous, there exists $w^* \in P$ such that $d(O, w^*) = \|w^*\| = d(O, P)$. Then by Lemma 2.3, $\xi(t)w^* \in P_t \cap \xi(t)P$ for $t \in [0, t^*]$. For $z \in P_t$, let $\eta_t(z) = c \in [0, 1]$ if $z \in cP$. To prove the first relation, it suffices to show that for $z \in P_t$, $0 < t < t^*$, $\eta_t(\xi(t)w^*) = \xi(t) \leq \eta_t(z)$. Suppose that there exists $z \in P_t$ such that $\eta_t(z) < \xi(t) \leq 1$. Let $\gamma\colon [0, 1 - \eta_t(z)] \to \mathbb{L}^{n+1}$ be the curve given by

$$\gamma(s) = z + sw^*.$$

Then γ is a future-directed timelike geodesic. Clearly, $\gamma(0) = z \in \eta_t(z)P \subset \eta_t(z)K$ and $\bigl(1 - \eta_t(z)\bigr)w^* \in \bigl(1 - \eta_t(z)\bigr)P \subset \bigl(1 - \eta_t(z)\bigr)K$. From convexity, $K = \lambda K + (1 - \lambda)K$ for $0 \le \lambda \le 1$. So, $\gamma\bigl(1 - \eta_t(z)\bigr) = z + \bigl(1 - \eta_t(z)\bigr)w^* \in K$ and $\gamma\bigl([0, 1 - \eta_t(z)]\bigr) \subset K$. Notice that $\Omega_P + \Omega_P = \Omega_P$. Thus, we have

$$d(z, P) \ge d\Bigl(z, \gamma\bigl(1 - \eta_t(z)\bigr)\Bigr) \ge L(\gamma) = \bigl(1 - \eta_t(z)\bigr)t^* > \bigl(1 - \xi(t)\bigr)t^* = t.$$

This contradicts that $z \in P_t$. ■

Let Q be a hypersurface of $\mathbb{L}^{n+1}$ such that

$$Q = \bigcup_{j=1}^{m} P^j,$$

where each P^j is a spacelike convex polytope in $\mathbb{L}^{n+1}$, and P^j may have common boundary points but their interiors do not intersect; such a Q will be called a PL-*hypersurface* in $\mathbb{L}^{n+1}$. By an *elementary cone* we mean a cone $K = C(Q)$ for a PL-hypersurface Q. Note that

$$K = C(Q) = \bigcup_{j=1}^{m} K^j = \bigcup_{j=1}^{m} C(P^j).$$

Let h_j be the height of P^j. Then by (4.1) we have

$$V(K) = V\bigl(C(Q)\bigr) = \frac{1}{n+1} \sum_{j=1}^{n} h_j A(P^j). \tag{4.2}$$

Proposition 4.2 *Let Q be a* PL*-hypersurface in $\mathbb{L}^{n+1}$ with $t^* = d(O, Q)$ and let $K = C(Q)$ be the elementary cone of Q. Then*

$$V^{\frac{1}{n+1}}(K_t) \ge V^{\frac{1}{n+1}}(K) - V^{\frac{1}{n+1}}\bigl(\mathbb{B}_K(t)\bigr). \tag{4.3}$$

for $0 \le t \le t^*$.

Proof Notice that (4.3) is clear for $t = 0, t^*$ from the definitions K_t, $\mathbb{B}_K(t)$ and Lemma 2.1. We now prove (4.3) by induction on the number of constituent spacelike convex polytopes of a PL-hypersurface Q in $\mathbb{L}^{n+1}$. Suppose that Q consists of only one spacelike convex polytope, that is, $K = C(Q)$ is a simple cone. Then $K = C(Q)$ is a convex body, so $K = \lambda K + (1 - \lambda)K$ for $\lambda \in [0, 1]$, and $V(cK) = c^{n+1}V(K)$ for $c \ge 0$. From the original Brunn-Minkowski theorem (1.1) for convex bodies in the Euclidean space and Remark 3.7(1),

$$V^{\frac{1}{n+1}}(K) = V^{\frac{1}{n+1}}(\xi(t)K) + V^{\frac{1}{n+1}}\Bigl(\bigl(1 - \xi(t)\bigr)K\Bigr)$$

and by Lemma 4.1, for each $t \in [0, t^*]$,

$$\xi(t)K \subset K_t, \quad \bigl(1 - \xi(t)\bigr)K \subset \mathbb{B}_K(t).$$

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Thus, we have

$$V^{\frac{1}{n+1}}(K) \le V^{\frac{1}{n+1}}(K_t) + V^{\frac{1}{n+1}}\big(\mathbb{B}_K(t)\big).$$

Suppose that (4.3) is true when the number of constituent spacelike polytopes of a PL-hypersurface in $\mathbb{L}^{n+1}$ is $\le m-1$. Let $K = C(Q)$ be an elementary cone of a PL-hypersurface Q whose number of constituent spacelike polytopes is m. We can split $Q = \bigcup_{j=1}^{m} P^j$ into PL-hypersurfaces Q^1 and Q^2 so that the numbers of constituent spacelike convex polytopes of each Q^1 and Q^2 are $< m$. Let $K^i = C(Q^i)$ with $d(O, Q^i) = t_i^*$ for $i = 1, 2$. By induction hypothesis,

$$V^{\frac{1}{n+1}}(K_t^1) \ge V^{\frac{1}{n+1}}(K^1) - V^{\frac{1}{n+1}}\big(\mathbb{B}_{K^1}(t)\big) \tag{4.4}$$

for $0 \le t \le t_1^*$ and

$$V^{\frac{1}{n+1}}(K_t^2) \ge V^{\frac{1}{n+1}}(K^2) - V^{\frac{1}{n+1}}\big(\mathbb{B}_{K^2}(t)\big) \tag{4.5}$$

for $0 \le t \le t_2^*$. By the definitions of K_t, $\mathbb{B}_K(t)$ and Lemma 2.1, we see that $V(K^i) \le V\big(\mathbb{B}_{K^i}(t)\big)$ and $K_t^i = \varnothing$ for $t \ge t_i^*$. So, the inequalities (4.4) and (4.5) hold for $0 \le t \le t^*$. Thus, for $0 \le t \le t^*$, we have

$$\begin{aligned} V(K) &= V(K^1) + V(K^2) \\ &\le \Big[V^{\frac{1}{n+1}}(K_t^1) + V^{\frac{1}{n+1}}\big(\mathbb{B}_{K^1}(t)\big)\Big]^{n+1} + \Big[V^{\frac{1}{n+1}}(K_t^2) + V^{\frac{1}{n+1}}\big(\mathbb{B}_{K^2}(t)\big)\Big]^{n+1}. \end{aligned}$$

We recall an inequality of Minkowski (See [BB, Section 1.22]): If $a_i, b_i \ge 0$ and $0 < p < 1$, then

$$\Big(\sum_{i=1}^{m} a_i^p\Big)^{\frac{1}{p}} + \Big(\sum_{i=1}^{m} b_i^p\Big)^{\frac{1}{p}} \le \Big[\sum_{i=1}^{m} (a_i + b_i)^p\Big]^{\frac{1}{p}} \tag{4.6}$$

with equality only when $(a_1, \ldots, a_n) = \lambda(b_1, \ldots, b_n)$ for some constant λ. By the inequality of Minkowski (4.6), we have

$$V(K) \le \Big[\big(V(K_t^1) + V(K_t^2)\big)^{\frac{1}{n+1}} + \Big(V\big(\mathbb{B}_{K^1}(t)\big) + V\big(\mathbb{B}_{K^2}(t)\big)\Big)^{\frac{1}{n+1}}\Big]^{n+1}.$$

Since $\mathbb{B}_{K^1}(t) \cup \mathbb{B}_{K^2}(t) = \mathbb{B}_K(t)$ and $K_t^1 \cup K_t^2 \subset K_t$ by Lemma 2.4,

$$V(K) \le \Big[V^{\frac{1}{n+1}}(K_t) + V^{\frac{1}{n+1}}\big(\mathbb{B}_K(t)\big)\Big]^{n+1}.$$

Thus, the inequality (4.3) is proved. ■

5 Approximation

Let A, B be compact sets in $\mathbb{R}^n$ with the Euclidean distance $d_0 \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$d_0(x, y) = \sqrt{\sum_{i=1}^{n} (x^i - y^i)^2},$$

where $x = (x^1, \dots, x^n)$, $y = (y^1, \dots, y^n) \in \mathbb{R}^n$. The Euclidean distance $d_0(x, A)$ between a set A and a point x in $\mathbb{R}^n$ is defined by

$$d_0(x, A) = \min_{y \in A} d_0(x, y)$$

and the (Euclidean) diameter $\operatorname{diam}_0(A)$ of $A \subset \mathbb{R}^n$ is defined by

$$\operatorname{diam}_0(A) = \max_{x, y \in A} d_0(x, y).$$

The Hausdorff distance $d_0^H(A, B)$ of A and B in $\mathbb{R}^n$ is defined by

$$d_0^H(A, B) = \inf\{\rho : A \subset B^\rho, B \subset A^\rho\},$$

where $A^\rho = \{x \in \mathbb{R}^n : d_0(x, A) \leq \rho\}$. If $d_0^H(A, A_i) \to 0$, one says that the sequence of compact sets A_i converges to A in the Hausdorff distance. Note that $d_0^H(A, B) = 0$ if and only if $A = B$ (*cf.* [W, pp. 92–93]).

For sets A and B in the Minkowski spacetime $\mathbb{L}^{n+1}$, we let

$$d_J(A, B) = \sup_{x \in A, y \in B} d(x, y) + d(y, x), \tag{5.1}$$

where d denotes the Lorentzian distance in $\mathbb{L}^{n+1}$. Then clearly $d_J(A, B) = d_J(B, A)$ and $d_J(A, B) \geq 0$. Notice that $d_J(A, A) \neq 0$ in general. Moreover, $d_J(A, B)$ does not give any information about how close A is to B in general. For example, let $A = \{(0, x) \in \mathbb{L}^2 : 0 \leq x \leq 1\}$ and $B_y = \{(0, z) \in \mathbb{L}^2 : y \leq z \leq y + 1\}$ for $y \in \mathbb{R}$, then $d_J(A, B_y) = 0$ for all $y \in \mathbb{R}$. Inspired by this observation and restricting our attention to hypersurfaces A and B in $\mathbb{L}^{n+1}$ satisfying not only Convention 1.1, but also the further causality condition (5.2), we have the following properties for d_J, which are similar to those for d_0^H:

Lemma 5.1 *Suppose that A and B are hypersurfaces in $\mathbb{L}^{n+1}$ such that*

$$A \subset J(B), \quad B \subset J(A). \tag{5.2}$$

For a hypersurface D in $\mathbb{L}^{n+1}$ and $t \geq 0$, let

$$J(D, t) = \{x \in J(D) : d(x, D) + d(D, x) \leq t\}.$$

Then, if $d_J(A, B) \leq \varepsilon$ for $\varepsilon \geq 0$,

$$A \subset J(B, \varepsilon), \quad B \subset J(A, \varepsilon)$$

and $d_J(A, B) = 0$ if and only if $A = B$.

Proof Suppose that $d_J(A,B) \le \varepsilon$ and $x \in A$. Then $d(x,y) + d(y,x) \le \varepsilon$ for all $y \in B$. Thus, $d(x,B) + d(B,x) = \sup_{y\in B} d(x,y) + d(y,x) \le \varepsilon$. Hence, $x \in J(B,\varepsilon)$, so $A \subset J(B,\varepsilon)$. Similarly, we have $B \subset J(A,\varepsilon)$.

Suppose that $A = B$. Then $d(x,y) = 0$ for $x, y \in A$ since A is achronal. Thus, $d_J(A,B) = 0$.

Suppose that $d_J(A,B) = 0$. Then $A \subset J(B,0)$. Note that $J(B,0)$ can be written as

$$J(B,0) = B \,\dot\cup \Big(J(B) \setminus \big(I(B) \cup B\big)\Big),$$

where, as usual, $\dot\cup$ stands for a disjoint union. Using this, we may write

$$J(B) = J(B,0) \,\dot\cup\, B \,\dot\cup\, I(B).$$

Let $x \in A$ such that $x \in J(B) \setminus \big(I(B) \cup B\big)$. Then since A is simply connected and piecewise smooth spacelike, there exists $y \neq x \in A \subset J(B)$ such that $y \in I(B) \cup B$. If $y \in I(B)$, then $d_J(A,B) > 0$. If $y \in B$, then there must be $z \in A$ such that $z \in I(B)$ since A is simply connected and piecewise smooth spacelike, and $d_J(A,B) > 0$. Thus, $A \cap \Big(J(B) \setminus \big(I(B) \cup B\big)\Big) = \varnothing$, so $A \subset B$. Similarly, we have $B \subset A$. Therefore $A = B$. ■

We now approximate a hypersurface S by a sequence of PL-hypersurfaces Q^ε. The achronality of S implies that S may be considered as a graph of a function $f\colon D \subset \mathbb{R}^n \to \mathbb{R}$, and also plays a critical role for the estimation of the distance between S and Q^ε in our approximation.

Proposition 5.2 *Let S be a hypersurface in $\mathbb{L}^{n+1}$. Then there is a sequence of* PL*-hypersurfaces Q^ε in $\mathbb{L}^{n+1}$ converging to S.*

Proof Since S is a hypersurface in $\mathbb{L}^{n+1}$ (See Convention 1.1), there are a simply connected, compact set $D \in \mathbb{R}^n$ and a piecewise smooth function $f\colon D \to \mathbb{R}$ so that S may be parametrized as

$$S = \{\big(f(x), x\big) \in \mathbb{L}^{n+1} : x = (x^1, \dots, x^n) \in D\}.$$

Let $\bar f = \max_{x\in D} f(x)$, $\underline{f} = \min_{x\in D} f(x)$ and $c = 3(\bar f - \underline{f})$. Since $D \subset \mathbb{R}^n$ is compact, given $\varepsilon > 0$, we can approximate D from inside by a polytope $D^\varepsilon \subset \mathbb{R}^n$ with $d_0^H(D, D^\varepsilon) < \varepsilon$. Since f is continuous on D and $D^\varepsilon \subset D$ is compact, f is uniformly continuous on D^ε. Thus, for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ so that for $x, y \in D^\varepsilon$, if $d_0(x,y) < \delta$, then

$$|f(x) - f(y)| < \frac{\varepsilon^2}{c}. \tag{5.3}$$

Consider a triangulation $\Gamma^\varepsilon = \{\sigma_j^\varepsilon\}_{j=1}^{j_\varepsilon}$ of D^ε, where σ_j^ε are n-simplexes in $\mathbb{R}^n$ such that $\mathrm{diam}_0(\sigma_j^\varepsilon) < \delta$. Let $F(\sigma_j^\varepsilon)$ be the convex hull of the $(n+1)$-points $\big(f(y_0), y_0\big)$, $\big(f(y_1), y_1\big)$,

$\dots, \big(f(y_n), y_n\big)$ in $\mathbb{L}^{n+1}$, where y_k's are the vertices of σ_j^ε. Then $F(\sigma_j^\varepsilon)$ are spacelike convex polytopes in $\mathbb{L}^{n+1}$. Set

$$Q^\varepsilon = \bigcup_{j=1}^{j_\varepsilon} F(\sigma_j^\varepsilon), \quad S^\varepsilon = \{\big(f(x), x\big) \in S : x \in D^\varepsilon\}.$$

Then from construction, we see that Q^ε is a PL-hypersurface in $\mathbb{L}^{n+1}$ with $\mu(Q^\varepsilon) = \mu(S^\varepsilon) \subset \mu(S)$ and there exists a piecewise smooth function $\tilde{f}\colon D^\varepsilon \to \mathbb{R}$ such that

$$Q^\varepsilon = \{\big(\tilde{f}(x), x\big) \in \mathbb{L}^{n+1} : x \in D^\varepsilon\}.$$

Notice that for each $x \in D^\varepsilon$, there is a simplex $\sigma_j^\varepsilon \in \Gamma^\varepsilon$ such that $x \in \sigma_j^\varepsilon$, and since $\mathrm{diam}_0(\sigma_j^\varepsilon) < \delta$,

$$|\tilde{f}(x) - f(x)| \le \max_{x \in \sigma_j^\varepsilon} f(x) - \min_{y \in \sigma_j^\varepsilon} f(y) < \frac{\varepsilon^2}{c}. \tag{5.4}$$

Let $J^\varepsilon = \{(x, y) \in D^\varepsilon \times D^\varepsilon : d_0(x, y) \le |\tilde{f}(x) - f(y)|\}$. Then

$$d_J(Q^\varepsilon, S^\varepsilon)^2 = \sup_{(x,y) \in J^\varepsilon} |\tilde{f}(x) - f(y)|^2 - d_0(x, y)^2. \tag{5.5}$$

Note that for $(x, y) \in J^\varepsilon$,

$$\begin{aligned} |\tilde{f}(x) - f(y)|^2 - d_0(x, y)^2 &\le |\tilde{f}(x) - f(x)|^2 + 2|\tilde{f}(x) - f(x)|\,|f(x) - f(y)| \\ &\quad + |f(x) - f(y)|^2 - d_0(x, y)^2 \\ &\le |\tilde{f}(x) - f(x)|^2 + 2|\tilde{f}(x) - f(x)|\,|f(x) - f(y)|, \end{aligned}$$

where the last inequality comes from the achronality of S. Thus,

$$\begin{aligned} d_J(Q^\varepsilon, S^\varepsilon)^2 &\le \sup_{(x,y) \in J^\varepsilon} |\tilde{f}(x) - f(x)|^2 + 2|\tilde{f}(x) - f(x)|\,|f(x) - f(y)| \\ &\le \sup_{(x,y) \in J^\varepsilon} |\tilde{f}(x) - f(x)|\big(|\tilde{f}(x) - f(x)| + 2|f(x) - f(y)|\big) \\ &\le \sup_{x,y \in D^\varepsilon} 3(\bar{f} - \underline{f})|\tilde{f}(x) - f(x)| \\ &< c\frac{\varepsilon^2}{c} = \varepsilon^2. \end{aligned}$$

Thus, we have

$$d_J(Q^\varepsilon, S^\varepsilon) < \varepsilon. \tag{5.6}$$

Upon letting $\varepsilon \to 0$, we have $d_J(Q^\varepsilon, S^\varepsilon) \to 0$ and $S^\varepsilon \to S$. Note that $S^\varepsilon \subset \mathbb{R} \times D^\varepsilon \subset J(Q^\varepsilon)$ and $Q^\varepsilon \subset \mathbb{R} \times D^\varepsilon \subset J(S^\varepsilon)$. Thus, we have $S^\varepsilon \subset J(Q^\varepsilon)$ and $Q^\varepsilon \subset J(S^\varepsilon)$. By Lemma 5.1, we have $Q^\varepsilon \to S$ as $\varepsilon \to 0$. ■

By Proposition 5.2 and the continuity of the Lorentzian distance, we have the following:

Corollary 5.3 *Let Q^ε be a sequence of* PL*-hypersurfaces with $K^\varepsilon = C(Q^\varepsilon)$ in $\mathbb{L}^{n+1}$ converging to a hypersurface S in $\mathbb{L}^{n+1}$ with $K = C(S)$. Then $V(K^\varepsilon) \to V(K)$, $V(K_t^\varepsilon) \to V(K_t)$ and $V\big(\mathbb{B}_{K^\varepsilon}(t)\big) \to V\big(\mathbb{B}_K(t)\big)$.*

6 A Brunn-Minkowski Type Theorem

We now prove the main result.

***Theorem 6.1* (Brunn-Minkowski Type Theorem)** *Let S be a hypersurface in the Minkowski spacetime $\mathbb{L}^{n+1}$ with $t^* = d(O,S)$ and let $K = C(S)$ be the cone of S. Then, for $0 \le t \le t^*$,*

$$V^{\frac{1}{n+1}}(K_t) \ge V^{\frac{1}{n+1}}(K) - V^{\frac{1}{n+1}}\big(\mathbb{B}_K(t)\big). \tag{6.1}$$

Moreover, equality holds in (6.1) for some t $(0 < t \le t^*)$ if and only if $S \subset \mathbb{H}(t^*)$.

Proof The inequality (6.1) for a hypersurface S in $\mathbb{L}^{n+1}$ follows from Propositions 4.2, 5.2 and Corollary 5.3. Suppose that $S \subset \mathbb{H}(t^*)$. Then $S_t = (t^* - t)\mu(S)$ and $K_t = C(S_t) = C\big((t^* - t)\mu(S)\big) = \mathbb{B}_K(t^* - t) = (t^* - t)\mathbb{B}_K(1)$ for $0 \le t \le t^*$. Thus we have

$$V(K_t) = (t^* - t)^{n+1} V\big(\mathbb{B}_K(1)\big), \quad V\big(\mathbb{B}_K(t)\big) = t^{n+1} V\big(\mathbb{B}_K(1)\big)$$

for $0 \le t \le t^*$. Thus, equality holds in (6.1) for $0 \le t \le t^*$.

Suppose that equality holds in (6.1) for $t = t^*$. Then $V(K) = V\big(\mathbb{B}_K(t^*)\big)$ since $V(K_{t^*}) = 0$. Then by Lemma 2.2, $S \subset \mathbb{H}(t^*)$. So, we assume that

$$V^{\frac{1}{n+1}}(K_{t_0}) = V^{\frac{1}{n+1}}(K) - V^{\frac{1}{n+1}}\big(\mathbb{B}_K(t_0)\big) \tag{6.2}$$

for some $t_0 \in (0, t^*)$. Then, we have

$$V^{\frac{1}{n+1}}\big(\mathbb{B}_K(t^* - t_0)\big) - V^{\frac{1}{n+1}}(K_{t_0}) = V^{\frac{1}{n+1}}\big(\mathbb{B}_K(t^*)\big) - V^{\frac{1}{n+1}}(K). \tag{6.3}$$

Let us consider a decomposition $S = S^1 \cup S^2$ with $K^i = C(S^i)$, $V(K^i) > 0$ as in Lemma 2.4. Employing the arguments as in the proof of Proposition 4.2, we have

$$\begin{aligned} V(K) &= V(K^1) + V(K^2) \\ &\le \Big[V^{\frac{1}{n+1}}(K^1_{t_0}) + V^{\frac{1}{n+1}}\big(\mathbb{B}_{K^1}(t_0)\big)\Big]^{n+1} \\ &\qquad + \Big[V^{\frac{1}{n+1}}(K^2_{t_0}) + V^{\frac{1}{n+1}}\big(\mathbb{B}_{K^2}(t_0)\big)\Big]^{n+1} \\ &\le \Big[\big(V(K^1_{t_0}) + V(K^2_{t_0})\big)^{\frac{1}{n+1}} + V^{\frac{1}{n+1}}\big(\mathbb{B}_K(t_0)\big)\Big]^{n+1} \\ &\le \Big[V^{\frac{1}{n+1}}(K_{t_0}) + V^{\frac{1}{n+1}}\big(\mathbb{B}_K(t_0)\big)\Big]^{n+1}. \end{aligned} \tag{6.4}$$

By the assumption (6.2) and the condition for equality of the first inequality in (6.4), we have

$$V^{\frac{1}{n+1}}(K^i) = V^{\frac{1}{n+1}}(K^i_{t_0}) + V^{\frac{1}{n+1}}\big(\mathbb{B}_{K^i}(t_0)\big) \tag{6.5}$$

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and

$$V^{\frac{1}{n+1}}\big(\mathbb{B}_{K^i}(t^*-t_0)\big)-V^{\frac{1}{n+1}}(K^i_{t_0})=V^{\frac{1}{n+1}}\big(\mathbb{B}_{K^i}(t^*)\big)-V^{\frac{1}{n+1}}(K^i) \tag{6.6}$$

for $i=1,2$. By the condition for equality of the last inequality in (6.4), we have $V(K^1_{t_0})+V(K^2_{t_0})=V(K_{t_0})$ and so $K^1_{t_0}\cup K^2_{t_0}=K_{t_0}$ by Lemma 2.4. By the condition for equality of the second inequality in (6.4), we have

$$V(K^i_{t_0})=\lambda_i V(K_{t_0}),\quad V\big(\mathbb{B}_{K^i}(t_0)\big)=\lambda_i V\big(\mathbb{B}_K(t_0)\big) \tag{6.7}$$

where $\lambda_i\in(0,1)$ such that $\lambda_1+\lambda_2=1$. Finally, from (6.3), (6.6) and (6.7), we have

$$V^{\frac{1}{n+1}}\big(\mathbb{B}_{K^i}(t^*)\big)-V^{\frac{1}{n+1}}(K^i)=\lambda_i^{\frac{1}{n+1}}\Big(V^{\frac{1}{n+1}}\big(\mathbb{B}_K(t^*)\big)-V^{\frac{1}{n+1}}(K)\Big). \tag{6.8}$$

This means that for any two hypersurfaces $S',S''\subset S$ in $\mathbb{L}^{n+1}$ with $K'=C(S')$ and $K''=C(S'')$, if $V\big(\mathbb{B}_{K'}(t^*)\big)=V\big(\mathbb{B}_{K''}(t^*)\big)$, then $V(K')=V(K'')$. Suppose that S is not contained in $\mathbb{H}(t^*)$. Consider the function $f\colon S\to\mathbb{R}$ defined by $f(x)=\|x\|$. Since f is continuous and S is compact, there exist $w_*,w^*\in S$ so that $f(w_*)=t_*=\min_{x\in S}f(x)$, $f(w^*)=t^*=\max_{x\in S}f(x)$ and $t_*<t^*$. Since f is continuous on S, there exist open neighborhoods U of w^* and V of w_* in S such that $f(w^*)-f(x)<\frac{1}{3}(t^*-t_*)$ for $x\in U$ and $f(y)-f(w_*)<\frac{1}{3}(t^*-t_*)$ for $y\in V$. Then we can find hypersurfaces $S',S''\subset S$ in $\mathbb{L}^{n+1}$ such that $S'\subset U$, $S''\subset V$ and $V\big(\mathbb{B}_{K'}(t^*)\big)=V\big(\mathbb{B}_{K''}(t^*)\big)$, where $K'=C(S')$, $K''=C(S'')$. Let $\bar{t}=t_*+\frac{1}{2}(t^*-t_*)$. Then $\mathbb{B}_{K'}(\bar{t})\subsetneq K'$ and $K''\subsetneq\mathbb{B}_{K''}(\bar{t})$. So, $V(K'')<V(K')$. This is a contradiction. Thus, f is a constant function on S, that is, $f(x)=t^*$ for all $x\in S$. Hence, $S\subset\mathbb{H}(t^*)$. ■

Remark 6.2 Let A be a compact set and $\mathbb{B}^E(t)$ the Euclidean ball of radius $t\geq 0$ in $\mathbb{R}^{n+1}$. Let A^t be the parallel body of A with distance t given by

$$A^t=\{x+ty\in\mathbb{R}^{n+1}: x\in A,\, y\in\mathbb{B}^E(1)\}.$$

Then the original Brunn-Minkowski theorem says that

$$V_E^{\frac{1}{n+1}}(A^t)\geq V_E^{\frac{1}{n+1}}(A)+V_E^{\frac{1}{n+1}}\big(\mathbb{B}^E(t)\big), \tag{6.9}$$

where V_E denotes the $(n+1)$-dimensional Euclidean volume. On the other hand, our Brunn-Minkowski type theorem may be written as

$$V^{\frac{1}{n+1}}(K)\leq V^{\frac{1}{n+1}}(K_t)+V^{\frac{1}{n+1}}\big(\mathbb{B}_K(t)\big). \tag{6.10}$$

In view of inequalities (6.9) and (6.10) and recalling the inclusion that $K_t\subset K$ but $A\subset A^t$, we may say that the direction of the inequality of our Brunn-Minkowski type theorem is opposite to that of the original Brunn-Minkowski theorem. Non-trivial optimal cases of the original Brunn-Minkowski theorem occur only for convex bodies. However, our Brunn-Minkowski type theorem is not strongly related to convexity even though some properties of convexity are used in the proof. For example, our optimal case occurs only when $S\subset\mathbb{H}(t)$, but $K=C(S)$ is not convex.

7 An Isoperimetric Inequality

Let $Q = \bigcup_{j=1}^{m} P^j$ be a PL-hypersurface in $\mathbb{L}^{n+1}$ with $K = C(Q)$ and $\mathbf{n}_j$ the unit past-directed normal vector of the spacelike convex polytope P^j. Set

$$[t\mathbf{n}_j, P^j] = \{s\mathbf{n}_j + x : x \in P^j, 0 \le s \le t\}.$$

Then by (3.4),

$$V([t\mathbf{n}_j, P^j]) = tA(P^j). \tag{7.1}$$

Let R_1 be the set of points of $K \setminus K_t$ which are not contained in any $[t\mathbf{n}_j, P^j]$, R_2 the set of points of $K \setminus K_t$ which are contained in some $[t\mathbf{n}_j, P^j] \cap [t\mathbf{n}_k, P^k]$, $j \ne k$, and R_3 the set of points of $[t\mathbf{n}_j, P^j]$ which are not included in $K \setminus K_t$. Then, for a sufficiently small $t > 0$,

$$V(K) - V(K_t) = tA(Q) + \bigstar, \tag{7.2}$$

where the remainder $\bigstar$ is $V(R_1) - V(R_2) - V(R_3)$. For $\nu, r \ge 0$, let

$$\mathbb{B}^-(\nu, r) = \{x \in J^-(O) : \|x\| \le r, \varphi(x) \le \nu\},$$

where $\varphi(x)$ is the hyperbolic angle between e_0 and x (*cf.* [BH] or [On, p. 144]). Let $\mathbb{B}_p^-(\nu, r) = p + \mathbb{B}^-(\nu, r)$, which will be called the *past spherical cone* of radius r, center p and angle ν. Then every point of R_1, R_2 and R_3 is contained in the union of all past spherical cones $\mathbb{B}_p^-(\nu, t)$ of radius t, center p contained in the $(n-1)$-dimensional face of P^j and angle ν of some large, but finite ν. Thus, the remainder $\bigstar$ is $\le ct^2$, where c is a constant that is independent of t. For a fixed $K = C(Q)$, we may consider $V(K_t)$ as a function of t. Then by (7.2) we have

$$\lim_{t \to 0} \frac{V(K_t) - V(K)}{t} = -A(Q). \tag{7.3}$$

Let S be a hypersurface in $\mathbb{L}^{n+1}$ with $K = C(S)$. Then we may consider a decomposition

$$S = \bigcup_{k=1}^{m} S^k,$$

where each S^k is a smooth spacelike hypersurface in $\mathbb{L}^{n+1}$ and S^k may have common boundary points but their interiors do not intersect. Let dA_k be the volume element on S^k. Then the n-dimensional volume of S^k is defined by

$$A(S^k) = \int_{S^k} dA_k$$

and the n-dimensional volume of S is given by

$$A(S) = \sum_{k=1}^{m} A(S^k).$$

Let Q^ε be a sequence of PL-hypersurfaces constructed as in the proof of Proposition 5.2. Then $A(Q^\varepsilon) \to A(S)$ as $\varepsilon \to 0$. By Proposition 5.2 and Corollary 5.3, we have

$$\lim_{t\to 0} \frac{V(K_t) - V(K)}{t} = -A(S). \tag{7.4}$$

***Theorem 7.1* (Isoperimetric Inequality)** *Let S be a hypersurface in $\mathbb{L}^{n+1}$ with $t^* = d(O,S)$ and let $K = C(S)$ be the cone of S. Then,*

$$A^{n+1}(S) \le (n+1)^{n+1}\omega V^n(K) \tag{7.5}$$

with equality only when $S \subset \mathbb{H}(t^*)$, where $\omega = V\big(\mathbb{B}_K(1)\big)$.

Proof Let S be a hypersurface in $\mathbb{L}^{n+1}$ and $K = C(S)$, the cone of S. Then by Theorem 6.1,

$$\begin{aligned} V(K_t) &\ge [V^{\frac{1}{n+1}}(K) - (\omega t^{n+1})^{\frac{1}{n+1}}]^{n+1} \\ &= V(K) - (n+1)\omega^{\frac{1}{n+1}} V^{\frac{n}{n+1}}(K)t + o(t). \end{aligned}$$

So, we have

$$\lim_{t\to 0} \frac{V(K_t) - V(K)}{t} \ge -(n+1)\omega^{\frac{1}{n+1}} V^{\frac{n}{n+1}}(K).$$

By (7.4),

$$A(S) \le (n+1)\omega^{\frac{1}{n+1}} V^{\frac{n}{n+1}}(K),$$

which is equivalent to (7.5).

Suppose that $S \subset \mathbb{H}(t^*)$. Then we have

$$V(K) = {t^*}^{n+1}\omega, \quad A(S) = (n+1)\omega {t^*}^n.$$

Thus,

$$\begin{aligned} A^{n+1}(S) &= (n+1)^{n+1}\omega^{n+1}{t^*}^{n(n+1)} = (n+1)^{n+1}\omega(\omega {t^*}^{n+1})^n \\ &= (n+1)^{n+1}\omega V^n(K). \end{aligned}$$

Suppose that equality holds in (7.5). Consider a decomposition $S = S^1 \cup S^2$ with $K^i = C(S^i)$ and $V(K^i) > 0$ as in Lemma 2.4 and let $\omega_i = V\big(\mathbb{B}_{K^i}(1)\big)$ for $i = 1, 2$. Then we have

$$\begin{aligned} A(S) &= A(S^1) + A(S^2) \\ &\le (n+1)\Big[\big(\omega_1 V^n(K^1)\big)^{\frac{1}{n+1}} + \big(\omega_2 V^n(K^2)\big)^{\frac{1}{n+1}}\Big] \\ &\le (n+1)\big(\omega V^n(K)\big)^{\frac{1}{n+1}}, \end{aligned}$$

in view of another inequality of Minkowski (See [BB, Section 1.21]): for $a_i, b_i \ge 0$,

$$\Big(\prod_{i=1}^n a_i\Big)^{\frac{1}{n}} + \Big(\prod_{i=1}^n b_i\Big)^{\frac{1}{n}} \le \Big[\prod_{i=1}^n (a_i + b_i)\Big]^{\frac{1}{n}}$$

with equality only when $(a_1, \dots, a_n) = \lambda(b_1, \dots, b_n)$ for some constant λ. Since equality holds in (7.5), we have

$$\omega_i = \lambda_i \omega, \quad V(K^i) = \lambda_i V(K)$$

for $\lambda_i \in (0,1)$ such that $\lambda_1 + \lambda_2 = 1$. These imply (6.8). Therefore, we have $S \subset \mathbb{H}(t^*)$. ■

Remark 7.2 This isoperimetric inequality in the Minkowski spacetime corresponds to the isoperimetric inequality for convex cones of P. L. Lions and F. Pacella [LP] in the Euclidean space. Notice that the isoperimetric inequality of P. L. Lions and F. Pacella does not hold for nonconvex infinite cones (See Remark 1.3 in [LP]), and moreover its direction of inequality is opposite to that of our isoperimetric inequality in the Minkowski spacetime. The isoperimetric inequality (7.5) for the 2-dimensional case has been already established [BH], which corresponds to an isoperimetric inequality of C. Bandle for the Euclidean plane [Ba, Theorem 1.1]. The result in [BH] has been extended to a general Lorentzian surface in [B].

Appendix

Here, we give a proof of Lemma 3.5. One can find a similar proof of the Euclidean case in [Bl].

Lemma A.1 *If $\mathcal{R}$ is a row operation, then (3.2) in Lemma 3.5 holds for a given $m \times n$ matrix D if and only if it holds for the matrix $\mathcal{R}(D)$ in the place of D.*

Proof Let $\mathcal{C}$ be the column operation which does the same thing to columns that $\mathcal{R}$ does to rows. (Here, we use the right-hand notation for $\mathcal{C}$.) Then there is a constant $k \neq 0$ such that

$$\begin{aligned} k^2 \det D \cdot D^t &= k^2 \det(D\varepsilon D^t) = \det\bigl([\mathcal{R}(D\varepsilon D^t)]\mathcal{C}\bigr) \\ &= \det(\mathcal{R}D)\varepsilon(D^t\mathcal{C}) = \det(\mathcal{R}D)\cdot(D^t\mathcal{C}) \\ &= \det(\mathcal{R}D)\cdot(\mathcal{R}D)^t. \end{aligned}$$

Let B' be an $m \times m$ submatrix of $\mathcal{R}D$ containing the first column of $\mathcal{R}D$ and let C' be $m \times m$ submatrix of $\mathcal{R}D$ not containing the first column of $\mathcal{R}D$. Assume that (3.2) holds for $\mathcal{R}D$ in the place of D. Then we have

$$\begin{aligned} k^2 \det D \cdot D^t &= \det(\mathcal{R}D)\cdot(\mathcal{R}D)^t \\ &= -\sum_{B'}(\det B')^2 + \sum_{C'}(\det C')^2 \\ &= -\sum_{B}(k\det B)^2 + \sum_{C}(k \det C)^2 \\ &= k^2\Bigl(-\sum_{B}(\det B)^2 + \sum_{C}(\det C)^2\Bigr). \end{aligned}$$

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Since $k \neq 0$, (3.2) holds for D. Similarly, if (3.2) holds for D, then it holds for $\mathcal{R}D$. ■

For convenience, we will denote the $k \times k$ submatrices of a square matrix A containing the first column of A by $B(k, A)$ and the $k \times k$ submatrices of A not containing the first column of A by $C(k, A)$. Then, Lemma 3.5 may be written as follows:

Lemma A.2 *For any $m \times (n+1)$ matrix D over $\mathbb{R}$, $m, n \geq 1$,*

$$\det D \cdot D^t = -\sum \bigl(\det B(m, D)\bigr)^2 + \sum \bigl(\det C(m, D)\bigr)^2, \tag{A.1}$$

where the first sum is taken over all $B(m, D)$ of D and the second sum is taken over all $C(m, D)$ of D.

Proof We will prove (A.1) by induction on n. If $n = 1$ and $m = 1$, then (A.1) is just the standard flat Lorentzian metric on $\mathbb{L}^2$. If $n = 1$ and $m = 2$, then it is trivial since $B(2, D) = D$ is the only 2×2 submatrix of D and $\det(D \cdot D^t) = \det(D\varepsilon D^t) = -\det(DD^t) = -(\det D)^2$. If $n = 1$ and $m > 2$, the row vectors of D are linearly dependent, so are the rows of $D \cdot D^t$. Thus, $\det(D \cdot D^t) = 0$. The right hand side of (A.1) is also zero since there are no $m \times m$ submatrices of D. Hence, (A.1) holds for $n = 1$. Assume that (A.1) holds for $n = k$. Let D be an $m \times (k + 1)$ matrix. We may assume that some column $D^{(j)}$ of D, $1 \leq j \leq k + 1$, is non-zero. (Otherwise, the proof is trivial since D is the zero matrix.) Then, by Lemma A.1, we may assume without loss of generality that

$$D^{(j)} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Let D' be the matrix obtained from D by deleting the column $D^{(j)}$. Then, a straightforward computation from the definition of $D \cdot D^t$ gives

$$D \cdot D^t = D' \cdot (D')^t + \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Thus, we have

$$\det(D \cdot D^t) = \det\bigl(D' \cdot (D')^t\bigr) + \det E, \tag{A.2}$$

where E is the $(1, 1)$ minor of $D' \cdot (D')^t$. Now there are two types of $m \times m$ submatrices of D; (1) those of D' and (2) the consisting of the column $D^{(j)}$ together with an $m \times (m - 1)$ submatrix of D'. Let D^* be the matrix obtained from D' by deleting the first row. If we compute the determinant of an $m \times m$ matrix of type (2) using minors of the column $D^{(j)}$, we obtain ± 1 times the determinant of an $(m - 1) \times (m - 1)$ submatrix of D^*. Thus,

$$\sum \bigl(\det B(m, D)\bigr)^2 = \sum \bigl(\det B(m, D')\bigr)^2 + \sum \bigl(\det B(m - 1, D^*)\bigr)^2$$

and
$$\sum\big(\det C(m,D)\big)^2=\sum\big(\det C(m,D')\big)^2+\sum\big(\det C(m-1,D^*)\big)^2.$$
Notice that D' and D^* have k columns. So, we have

$$-\sum\big(\det B(m,D)\big)^2+\sum\big(\det C(m,D)\big)^2=\det\big(D'\cdot(D')^t\big)+\det\big(D^*\cdot(D^*)^t\big) \tag{A.3}$$

by induction hypothesis. By the definitions of D^* and E,

$$D^*\cdot(D^*)^t=E.$$

From (A.2) and (A.3) we see that (A.1) holds for an $m\times(k+1)$ matrix D. ■

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