ELEMENTS OF FINITE ORDER IN FREE PRODUCT SIXTH-GROUPS

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Introduction. A free product sixth-group (FPS-group) is, roughly speaking, a free product of groups with a number of additional defining relators, where, if two of these relators have a subword in common, then the length of this subword is less than one sixth of the lengths of either of the two relators.

Britton [1, 2] has proved a general algebraic result for FPS-groups and has used this result in a discussion of the word problem for such groups.

In this paper we use the results of [2] to obtain a characterization of the elements of finite order in any FPS-group, and also necessary and sufficient conditions for such a group to be torsion-free. Similar results have been obtained by Greendlinger [3, Theorem VIII] for free sixth-groups, and by Karass, Magnus and Solitar [4] for groups with one defining relation.

The main technical result of this paper (Lemma 5) was suggested by a corresponding result for free sixth-groups proved by Lipschutz [5, Lemma 4].

1. Notation. Statement of the main results.

1.1 Let Π be the free product of the set of groups $\{G_\gamma\colon \gamma\in\Gamma\}$. No restriction is placed on the *constituent groups* G_γ of Π and the index set Γ may be infinite. The non-identity elements of the groups G_γ are called *components* of Π and are denoted by small letters. We write $x \sim y$ or $x \sim' y$ according to whether x and y belong to the same constituent group or not. The identity element of Π is denoted by I and general elements of Π are denoted by capital letters. We write $X\,.\,Y$ for the product of the elements X and Y of Π. Every element X of Π except I has a unique normal form expression

$$X = x_1 . x_2 . \;\ldots\; . x_n, \tag{1.11}$$

where $n \geqq 1$, $I \neq x_i \in G_{\gamma(i)}$ $(i = 1, 2, \ldots, n)$ and $\gamma(i) \neq \gamma(i+1)$ $(i = 1, 2, \ldots, n-1)$. The elements $x_1, x_2, \ldots, x_n$ are called the components of X. We write $l(X) = n$, $\mathrm{In}\,(X) = x_1$ and $\mathrm{Fin}\,(X) = x_n$. We call $l(X)$ the *length* of X and define the length $l(I)$ of I to be zero.

If $X = X_1 . X_2 . \;\ldots\; . X_r$ and $l(X) = \sum_{i=1}^{r} l(X_i)$, we write $X = X_1 X_2 \ldots X_r$; thus we write (1.11) as

$$X = x_1 x_2 \ldots x_n. \tag{1.12}$$

Let X have form (1.12). A *double segment* of X is defined to be any element with form

$$x'_j x_{j+1} \ldots x_{s-1} x'_s, \tag{1.13}$$

where $1 \leqq j \leqq s \leqq n$, $x'_j \sim x_j$ and $x'_s \sim x_s$.

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A *subword* of X is any element with form (1.13), where $x'_j = x_j$ and $x'_s = x_s$. The subword $x_j x_{j+1} \dots x_{s-1} x_s$ of X is said to *cover* the double segment $x'_j x_{j+1} \dots x_{s-1} x'_s$ of X. We note that, if Y is a double segment of X, then $Y \neq I$. If $l(X) \geqq 2$ and $\mathrm{In}(X) \sim' \mathrm{Fin}(X)$, we define a *cyclic arrangement* of X to be any element with form $x_r x_{r+1} \dots x_n x_1 x_2 \dots x_{r-1}$ $(1 \leqq r \leqq n)$.

We define the number $\beta(X, Y)$ of cancellations and the number $\varepsilon(X, Y)$ of amalgamations in the product $X.Y$ as follows:

Suppose first that $X \neq I$ and $Y \neq I$ and let $X = x_1 x_2 \dots x_m$, $Y = y_1 y_2 \dots y_n$. Then

$$\beta(X, Y) = \begin{cases} 0 & \text{if} \quad x_m . y_1 \neq I, \\ \beta & \text{if} \quad x_m . y_1 = I, \end{cases}$$

where β is the largest integer for which $x_{m-\beta+1} \dots x_m . y_1 \dots y_\beta = I$, and

$$\varepsilon(X, Y) = \begin{cases} 1 & \text{if} \quad \beta < \mathrm{Min}(m, n) \quad \text{and} \quad x_{m-\beta} \sim y_{\beta+1}, \\ 0 & \text{otherwise.} \end{cases}$$

If either $X = I$ or $Y = I$, we define $\beta(X, Y)$ and $\varepsilon(X, Y)$ to be zero. Note that

$$l(X.Y) = l(X) + l(Y) - 2\beta(X, Y) - \varepsilon(X, Y).$$

We put

$$\alpha(X, Y) = \beta(X, Y) + \varepsilon(X, Y).$$

If $X = A.B.C$ and $\alpha\{(A.B), C\} = 0$ we write $X = A.BC$; similarly if $\alpha\{A, (B.C)\} = 0$ we write $X = AB.C$.

1.2. Let Ω be a subset of Π such that, for all elements R, R' of Ω, the following conditions are satisfied:

1) $l(R) \geqq 7$.

(2) $\mathrm{In}(R) \sim' \mathrm{Fin}(R)$.

(3) Every cyclic arrangement of R and R^{-1} belongs to Ω.

(4) Either $R' = R^{-1}$ or

$$6\alpha(R', R) < \mathrm{Min}(l(R'), l(R)).$$

(5) $\mathrm{Max}\, \alpha(R_1, V) \neq 0$, where R_1 is a cyclic arrangement of R or R^{-1} and V is an element of Ω such that $R_1 . V \neq I$.

It is easy to show that, if Ω satisfies conditions (2) to (5), then condition (1) is also satisfied. [Let $R \in \Omega$. By (5) there exist R_1 (a cyclic arrangement of R or R^{-1}) and $V \in \Omega$ such that $R_1 . V \neq I$ and $\alpha(R_1, V) \neq 0$. By (4) we have $6 \leqq 6\alpha(R_1, V) < l(R_1) = l(R)$.] Thus the conditions we have imposed on the set Ω are the same as those of [**2**].

Let $[\Omega]$ be the normal subgroup of Π generated by the set Ω and let ϕ be the natural homomorphism of Π onto $\Pi/[\Omega]$. We denote by $|X|$ the image under ϕ of the element X of Π. If X and Y are elements of Π such that $|X| = |Y|$, then we write $X \approx Y$; it is clear that $\approx$ is a congruence relation on Π. We note that, if W is an element of Π such that $W \in [\Omega]$, then $|W| = |I|$ and so $W \approx I$.

DEFINITION. An FPS-group is defined to be any group isomorphic to a quotient group $\Pi/[\Omega]$ of a free product Π of groups, where Ω is a subset of Π such that the five conditions stated above are satisfied.

We now state the main theorem of [2].

THEOREM 1. *Let $\Pi/[\Omega]$ be an FPS-group. If W is an element of Π such that $W \approx I$ but $W \neq I$, then*

(i) *there exist elements X, Y, Z, T of Π such that $W = XYZ$, $YT^{-1} \in \Omega$ and*

$$l(X.T.Z) < l(W),$$

(ii) $l(W) \geqq l_0 = \mathrm{Min}\{l(R)\colon\ R \in \Omega\}$, *and $l(W) = l_0$ implies $W \in \Omega$.*

From (ii) it follows that the intersection of $[\Omega]$ with any constituent group consists only of the identity element. Thus, if U is an element of Π of length one, we have $U \not\approx I$, and if $U^\lambda \approx I$ for any integer λ, then $U^\lambda = I$.

DEFINITION. An element V of a group G is said to be a *proper power* if V is not the identity and there exist an element J of G and an integer λ greater than one such that $V = J^\lambda$.

We shall prove the following theorems.

THEOREM 2. *Let $\Pi/[\Omega]$ be an FPS-group. If Ω contains a proper power R, $R = J^\lambda$ $(\lambda > 1)$, then $|J|$ has order λ in $\Pi/[\Omega]$.*

THEOREM 3. *Let $\Pi/[\Omega]$ be an FPS-group. Then $\Pi/[\Omega]$ is torsion-free if and only if*

(i) *each constituent group of Π is torsion-free, and*

(ii) *no element of Ω is a proper power.*

Arising from these results we have the following characterization of the elements of finite order in any FPS-group.

COROLLARY 1. *Let $\Pi/[\Omega]$ be an FPS-group. If A is an element of Π such that $A \not\approx I$ and $|A|$ has finite order in $\Pi/[\Omega]$, then either*

(i) *$|A|$ is the image under ϕ of a conjugate of an element of finite order in a constituent group of Π, and has the same order as this element, or*

(ii) *there exist $J \in \Pi$, $R \in \Omega$ and positive integers m, n such that $R = J^n$, $|A|$ is conjugate to $|J^m|$, and $|A|$ has order n/t, where t is the highest common factor of m and n.*

These results may be summarised as stating that the only elements of finite order in an FPS-group are the obvious ones.

2. Preliminary definitions and results.

2.1. Let $X_1, X_2, \ldots, X_r$ be a sequence of elements of Π. The sequence of components of Π obtained by writing down successively the components of the X_i is called the *component sequence* of $X_1, X_2, \ldots, X_r$. Thus, if $X_1 = abc$ and $X_2 = c^{-1}b$, the component sequence of X_1, X_2 is a, b, c, c^{-1}, b.

DEFINITION.† A *partition* is a sequence $X_1, X_2, \ldots, X_r$ ($r \geqq 1$) of elements of Π such that the following two conditions are satisfied.

(a) At least one X_i is different from I.

(b) If the component sequence of $X_1, X_2, \ldots, X_r$ is bracketed so that the terms inside a bracket belong to the same constituent group and terms in adjacent brackets belong to different constituent groups, then the product of the terms in each bracket is not the identity.

For example, if $X_1 = abc$, $X_2 = c^{-1}$, $X_3 = cd$, $X_4 = I$, then the bracketing of the component sequence of X_1, X_2, X_3, X_4 is $\{a\}$, $\{b\}$, $\{c, c^{-1}, c\}$, $\{d\}$ and so X_1, X_2, X_3, X_4 is a partition. Here X_1, X_2 is not a partition, since the bracketing is $\{a\}$, $\{b\}$, $\{c, c^{-1}\}$.

If $X_1, X_2, \ldots, X_r$ is a partition and $X_1 . X_2 . \ldots . X_r = X$, we say that $X_1, X_2, \ldots, X_r$ is a partition of X, or just that $X_1 . X_2 . \ldots . X_r$ is a partition of X. We note that in this case $X \neq I$, for the length of X is clearly the number of brackets occurring in the bracketing described in (b) above, and this is non-zero since at least one of the X_i has non-zero length.

Remark. In Lemma 1 we give two general ways in which we can alter a partition of X to obtain a different partition of X. These arise from the following ways in which we can alter the component sequence $d_1, d_2, \ldots, d_n$ of a partition of X so that the component sequence obtained still satisfies condition (b) above. Thus we can

(i) replace two adjacent terms d_p, d_{p+1} in the same bracket by their product, provided it is not the identity,

(ii) delete the pair d_p, d_{p+1} if their product is the identity,

(iii) replace the term c by the pair of terms c_1, c_2 if $c = c_1 . c_2$ (Note that the notation implies that c_1 and c_2 are non-identity elements of the same constituent group as c.),

(iv) insert a pair a, a^{-1} before or after the term d, provided that $a \sim d$.

We note that, if $X \neq I$, $X = a_1 a_2 \ldots a_n$, say, then the sequence consisting of X alone is a partition of X with component sequence $a_1, a_2, \ldots, a_n$. It is clear that the component sequence of any partition of X can be reduced to that of X by a finite number of changes of types (i) and (ii) in the remark above. Conversely, the component sequence of any partition of X can be obtained from that of X by a finite number of changes of types (iii) and (iv) above.

LEMMA 1. *Let $X_1, X_2, \ldots, X_r$ be a partition of X. We have the following results.*

(i) *If $X_i . X_{i+1} = Y$, then $X_1, \ldots, X_{i-1}, Y, X_{i+2}, \ldots, X_r$ is a partition of X.*

(ii) *If $Y_1, Y_2, \ldots, Y_n$ is a partition of X_i, then $X_1, \ldots, X_{i-1}, Y_1, \ldots, Y_n, X_{i+1}, \ldots, X_r$ is a partition of X.*

(iii) *If $X \neq I$, then X_i is a double segment of X.*

† This definition is based on a similar one used by J. L. Britton in some unpublished work.

Proof. We prove the results under the assumption that none of the X_i is the identity. It is easy to see that the results continue to hold when this restriction is removed.

1^0. To prove (i), we let $X_i = a_1 \dots a_m$, $X_{i+1} = b_1 \dots b_n$. We consider separately the cases $a_m \sim' b_1$; $a_m \sim b_1$ but $a_m . b_1 \neq I$; $a_m . b_1 = I$.

If $a_m \sim' b_1$ then the component sequence of $X_1, \dots, X_{i-1}, Y, X_{i+2}, \dots, X_r$ is just the component sequence of $X_1, \dots, X_r$, so that (i) holds in this case.

If $a_m \sim b_1$ but $a_m . b_1 \neq I$, then the component sequence of $X_1, \dots, X_{i-1}, Y, X_{i+2}, \dots, X_r$ is obtained from that of $X_1, \dots, X_r$ by replacing the pair of terms a_m, b_1 by their product. Hence (i) holds in this case.

Finally, if $a_m . b_1 = I$, then at least one of X_i, X_{i+1} has length one (for if they both had length greater than one the bracket $\{a_m, b_1\}$ would occur in the bracketing of the component sequence of $X_1, \dots, X_r$ and so we could not have $a_m . b_1 = I$). We suppose firstly that not both $l(X_i) = 1$ and $l(X_{i+1}) = 1$ hold. We take $l(X_i) = 1, l(X_{i+1}) > 1$; the case $l(X_i) > 1, l(X_{i+1}) = 1$ is similar. Thus $X_i = a_1$, $X_{i+1} = a_1^{-1} b_2 \dots b_n$ and so $Y = b_2 \dots b_n$. Hence the component sequence of $X_1, \dots, X_{i-1}, Y, X_{i+2}, \dots, X_r$ is obtained from that of $X_1, \dots, X_r$ by deleting the pair of terms a_1, a_1^{-1}, so that (i) holds in this case. Now suppose that $l(X_i) = l(X_{i+1}) = 1$, so that $X_i = a_1$, $X_{i+1} = a_1^{-1}$. Then by a similar argument we see that (i) also holds in this case. This completes the proof of (i).

2^0. For (ii) we note that, if the terms due to X_i in the component sequence of $X_1, \dots, X_r$ are replaced by the terms due to $Y_1, \dots, Y_n$, then the component sequence obtained satisfies condition (b) in the definition of a partition, since the new sequence can be obtained from the original one by a finite number of changes of types (iii) and (iv) in the remark above. This proves (ii).

3^0. We shall indicate the proof of (iii) for the case when $r \geqq 3$ and $i \neq 1, r$. The proofs of the other cases are similar.

From (i) of the lemma it follows that, if $Y = X_1 . \dots . X_{i-1}$ and $Z = X_{i+1} . \dots . X_r$, then $Y . X_i . Z$ is a partition of X. Let Y, X_i, Z have normal forms

$$Y = a_1 \dots a_l, \quad X_i = b_1 \dots b_m, \quad Z = c_1 \dots c_n.$$

If $a_l \sim' b_1$ and $b_m \sim' c_1$, then X_i is a subword of X, so that (iii) holds in this case. If $a_l \sim b_1$ and $b_m \sim' c_1$, then the normal form of X is

$$a_1 \dots a_{l-1}(a_l . b_1) b_2 \dots b_m c_1 \dots c_n,$$

for $a_l . b_1 \neq I$ since Y, X_i, Z is a partition. Clearly X_i is a double segment of X in this case. A similar argument shows that (iii) holds if $a_l \sim' b_1$ and $b_m \sim c_1$. Thus to prove (iii) we need only consider the case when $a_l \sim b_1$ and $b_m \sim c_1$. Then, if $m \neq 1$, the normal form of X is

$$a_1 \dots a_{l-1}(a_l . b_1) b_2 \dots b_{m-1}(b_m . c_1) c_2 \dots c_n,$$

where $a_l . b_1 \neq I$ and $b_m . c_1 \neq I$, since Y, X_i, Z is a partition, while, if $m = 1$, then X has normal form

$$a_1 \dots a_{l-1}(a_l . b_1 . c_1) c_2 \dots c_n$$

and $a_l . b_1 . c_1 \neq I$, since Y, X_i, Z is a partition. In both these cases it is easily seen that X_i is a double segment of X. This completes the proof of (iii).

DEFINITION. Let $Y_1, \ldots, Y_{i-1}, Y_i, Y_{i+1}, \ldots, Y_n$ be a partition of $R \in \Omega$. The element $(Y_1 . \ldots . Y_{i-1})^{-1}(Y_{i+1} . \ldots . Y_n)^{-1}$ of Π is called the *complement of Y_i with respect to the partition $Y_1, \ldots, Y_n$ of R*. We write

$$(Y_1 . \ldots . Y_{i-1})^{-1}(Y_{i+1} . \ldots . Y_n)^{-1} = \mathscr{C}(Y_i; Y_1, \ldots, Y_n).$$

Note that $\alpha\{(Y_1 . \ldots . Y_{i-1})^{-1}, (Y_{i+1} . \ldots . Y_n)^{-1}\} = 0$, since $\mathrm{In}(R) \sim' \mathrm{Fin}(R)$ and

$$(Y_1 . \ldots . Y_{i-1}), Y_i, (Y_{i+1} . \ldots . Y_n)$$

is a partition of R. If no ambiguity arises concerning the partition being referred to, we may speak of $(Y_1 . \ldots . Y_{i-1})^{-1}(Y_{i+1} . \ldots . Y_n)^{-1}$ as the complement of Y_i with respect to R and denote it by $\mathscr{C}(Y_i; R)$. We note that $\mathscr{C}(Y_i; R) \approx Y_i$, since $R \approx I$.

LEMMA 2. *Let A, Y, B be a partition of $R \in \Omega$ and C, Y, D a partition of $R' \in \Omega$. Suppose that one of the following three conditions is satisfied.*

(i) $R = A . YB$, $R' = C . YD$ *and* $l(Y) \geqq \frac{1}{6}\mathrm{Min}(l(R), l(R'))$.

(ii) $R = AY . B$, $R' = CY . D$ *and* $l(Y) \geqq \frac{1}{6}\mathrm{Min}(l(R), l(R'))$.

(iii) $l(Y) \geqq \frac{1}{6}\mathrm{Min}(l(R), l(R')) + 1$.

Then $\mathscr{C}(Y; A, Y, B) = \mathscr{C}(Y; C, Y, D)$.

Proof. 1^0. Suppose that (i) holds. Then $\beta(A, Y) = 0$, since A, Y, B is a partition and $\alpha\{(A . Y), B\} = 0$. Similarly $\beta(C, Y) = 0$. Let $R_1 = BA . Y$, $R_1' = DC . Y$. Then R_1 and R_1' belong to Ω and

$$\alpha(R_1', R_1^{-1}) \geqq l(Y) \geqq \tfrac{1}{6}\mathrm{Min}(l(R'), l(R_1^{-1})).$$

Hence $R_1' = R_1$ and so $BA = DC$, which proves the lemma in this case. A similar argument proves the lemma if condition (ii) holds.

2^0. Suppose that condition (iii) holds. Then $Y \neq I$ and Y is a double segment of R and R', since A, Y, B and C, Y, D are partitions. Let $Y = Y_1 Y_2$, where $Y_2 = \mathrm{Fin}(Y)$. Then $R = A . Y_1(Y_2 . B)$, $R' = C . Y_1(Y_2 . D)$, $l(Y_1) \geqq \frac{1}{6}\mathrm{Min}(l(R), l(R'))$ and so (i) of the lemma is satisfied with Y_1 in place of Y and $Y_2 . B$, $Y_2 . D$ in place of B and D respectively. Hence $Y_2 . BA = Y_2 . DC$ and so $BA = DC$. This completes the proof of the lemma.

2.2. Let S be a subword of $R \in \Omega$ such that $l(S) \geqq [\frac{1}{2}l(R)]$ (where the square brackets denote " the integral part of "). We define the subset Ω_1 of Π to be the set of all such S, where R ranges over all the elements of Ω.

We note that, if $S \in \Omega_1$, where S is a subword of $R \in \Omega$ say, then

$$l(S) \geqq \tfrac{2}{6}\, l(R) \tag{2.21}$$

and

$$l(S) \geqq \tfrac{1}{6}\, l(R) + 1. \tag{2.22}$$

For we can write $l(R) = 6\mu + \lambda$, where $\mu \geqq 1$ and $0 \leqq \lambda \leqq 5$. Then, since $l(S) \geqq [\frac{1}{2}l(R)]$, we have

$$l(S) \geqq 3\mu + [\tfrac{1}{2}\lambda] \geqq 2\mu + 1 + [\tfrac{1}{2}\lambda] \geqq 2\mu + \tfrac{1}{3}\lambda = \tfrac{2}{6}l(R),$$

so that (2.21) holds. Now (2.22) follows from (2.21) since $l(R) \geqq 7$.

From property (3) of Ω it follows that, if $S \in \Omega_1$, we can find $R \in \Omega$, $T \in \Pi$ such that $R = ST$; moreover R and T are unique. For if $S \in \Omega_1$, there is an element R of Ω such that $R = ASB$ for some A and B, and $l(S) \geqq [\frac{1}{2}l(R)]$. Thus $R_1 = SBA \in \Omega$, and, if $R_2 = ST \in \Omega$, then

$$\alpha(R_1^{-1}, R_2) \geqq l(S) \geqq \tfrac{2}{6}\, l(R) = \tfrac{2}{6}\, l(R_1^{-1}),$$

by (2.21), and it follows that $R_1 = R_2$. We note also that, if $S \in \Omega_1$, then we can speak of *the complement* of S, for by (2.21) and Lemma 2 any two complements of S are equal.

Let X, Y, Z be a partition of $W \in \Pi$ and A, Y, B a partition of $R \in \Omega$, where $Y \neq I$. Put $W' = X.(A^{-1}B^{-1}).Z$. We say that Y is *Ω-replaceable* in W and that W' is the *result* of replacing Y by $\mathscr{C}(Y; A, Y, B)$ in the partition $X.Y.Z$ of W. If no ambiguity arises concerning the partitions being referred to, we abbreviate this by saying that W' is the result of replacing Y in W by $\mathscr{C}(Y; R)$. We note that, if W' is the result of replacing Y in W by $\mathscr{C}(Y; R)$, then $W' \approx W$, since $Y \approx \mathscr{C}(Y; R)$.

Two replacements in W are said to be *equivalent* if they yield the same result. Thus if $L.M.N$ is a partition of Y above, so that $X.L.M.N.Z$ is a partition of W and $A.L.M.N.B$ is a partition of R, then replacing Y in $X.Y.Z$ by $A^{-1}B^{-1}$ is equivalent to replacing M in $X.L.M.N.Z$ by $(A.L)^{-1}(N.B)^{-1}$.

Let Y be Ω-replaceable in W with result W'. If Y is a subword of W, $Y \in \Omega_1$ and

$$l(W') < l(W),$$

then we say that W is *Ω-reducible* by Y. If $W \neq I$ and W contains no subword Y by which it is Ω-reducible, then we say that W is *Ω-reduced*. We note that, if W is Ω-reduced, then so are W^{-1} and any subword of W or W^{-1}; for if $W = X.Y.Z$ and Y is Ω-replaceable in W with result W_1, then $W^{-1} = Z^{-1}.Y^{-1}.X^{-1}$ and Y^{-1} is Ω-replaceable in W^{-1} with result W_1^{-1}, so that, if W is Ω-reducible by Y, then W^{-1} is Ω-reducible by Y^{-1}, which proves the first part of the statement. If $W = ABC$ and B is Ω-reducible by some subword Y, $B = XYZ$ say, then $W = AXYZC$ and clearly W is Ω-reducible by Y, which proves the second part of the statement.

The following lemma is a restatement of (i) of Theorem 1 together with some consequences of it.

LEMMA 3. *Let W be any non-identity element of Π. We have the following results:*

(i) *If $W \approx I$, then W contains a subword Y by which it is Ω-reducible.*

(ii) *If W is Ω-reduced, then $W \not\approx I$.*

(iii) *If $W \not\approx I$, then there exists an element W' of Π such that W' is Ω-reduced, $W' \approx W$ and $l(W') \leqq l(W)$.*

Proof. (i) is a restatement of (i) of Theorem 1, and (ii) follows immediately from (i) and the definition of an Ω-reduced element.

To prove (iii) we note that $W' = W$ will do if W is Ω-reduced. If W is not Ω-reduced, then W contains a subword Y by which it is Ω-reducible. Let W_1 be the result of replacing Y in W. Then $W_1 \approx W$ and $l(W_1) < l(W)$. If W_1 is Ω-reduced, then we can take $W' = W_1$;

if not, then we carry out the above process with W_1 in place of W to obtain an element W_2 such that $W_2 \approx W_1$ and $l(W_2) < l(W_1)$. Repeating the process if necessary, we must eventually obtain an element W' satisfying (iii) of the lemma, since $I \not\approx W \approx W_1 \approx W_2 \approx \dots$ and $l(W) > l(W_1) > l(W_2) > \dots$.

Let W be Ω-reducible by Y, where $W = XYZ$, $YT^{-1} = R \in \Omega$ say. Let W' be the result of replacing Y in W, so that $W' = X.T.Z$. Then, if $\beta(X, T) = \beta(T, Z) = 0$, we say that Y is *maximal in* W. We note that, if W is Ω-reducible by Y, then we can find a subword S of W such that W is Ω-reducible by S, Y is a subword S, S is maximal in W and replacing S in W is equivalent to replacing Y in W. {One way of choosing such an S is as follows. We have

$$W = XYZ, \quad Y \in \Omega_1, \quad YT^{-1} \in \Omega, \quad W' = X.T.Z.$$

We can find elements X_1, X_2 of Π such that

$$W = X_1 X_2 YZ \quad \text{and} \quad \beta(X, T) = l(X_2).$$

Then $T = X_2^{-1}T_1$ for some $T_1 \in \Pi$, $X_2 YT_1^{-1} \in \Omega$, since $YT_1^{-1}X_2 \in \Omega$, $X_2 Y \in \Omega_1$, W is Ω-reducible by $X_2 Y$ and $\beta(X_1, T_1) = 0$. The result of replacing $X_2 Y$ in W is $X_1.T_1.Z$. We can find Z_1, Z_2 such that

$$W = X_1 X_2 YZ_1 Z_2 \qquad \text{and} \qquad \beta(T_1, Z) = l(Z_1).$$

Then $T_1 = T_2 Z_1^{-1}$ for some $T_2 \in \Pi$, $X_2 YZ_1 T_2^{-1} \in \Omega$, $X_2 YZ_1 \in \Omega_1$, W is Ω-reducible by $X_2 YZ_1$ and $\beta(X_1, T_2) = \beta(T_2, Z_2) = 0$ ($\beta(X_1, T_2) = 0$ since $\beta(X_1, T_1) = 0$ and either $T_2 = I$ or $\text{In}(T_1) = \text{In}(T_2)$). Clearly $X_2 YZ_1$ has the properties required of S.}

In view of this, if W is Ω-reducible by Y, we shall usually choose the Ω-replaceable subword Y maximally. We note further that, if Y is maximal in W, then, with the above notation, $X.T.Z$ will be a partition unless either

$$T = I \quad \text{and} \quad \beta(X, Z) > 0 \tag{2.23}$$

or

$$l(T) = 1, \quad \varepsilon(X, T) = \varepsilon(T, Z) = 1 \quad \text{and} \quad \text{Fin}(X).T.\text{In}(Z) = I. \tag{2.24}$$

We now introduce the concepts of *cyclic length* and *cyclic Ω-reduction*. Let W be any element of Π. We define the *cyclic length* of W, written $l^0(W)$, by

$$l^0(W) = \text{Min}\,\{l(X)\colon\ X \text{ is a conjugate of } W \text{ in } \Pi\}.$$

It follows easily that any two conjugate elements of Π have the same cyclic length and that, if $l(W) \geqq 2$ and $\text{In}(W) \sim' \text{Fin}(W)$, then $l^0(W) = l(W)$.

DEFINITION. An element W of Π is said to be *cyclically Ω-reduced* if either $l(W) = 1$, or $l(W) > 1$ and W satisfies the following conditions:

(i) $\text{In}(W) \sim' \text{Fin}(W)$.

(ii) If $Y \in \Omega_1$ is a subword of W and W' is the result of replacing Y in W, then

$$l^0(W') \geqq l(W).$$

(iii) Every cyclic arrangement of W satisfies (ii).

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We note that if W is cyclically Ω-reduced, then W is Ω-reduced and every cyclic arrangement of W and W^{-1} is cyclically Ω-reduced.

3. The main result.

3.1. Before proving the main technical result of this paper we need a lemma concerning commuting elements of Π.

LEMMA 4. *Let A, B be elements of Π such that $\alpha(A, B) = \alpha(B, A) = 0$. If $AB = BA$, then there exist $J \in \Pi$ and integers λ, μ such that $A = J^\lambda$, $B = J^\mu$ and $\alpha(J, J) = 0$.*

Proof. Put $m = l(AB)$. We prove the lemma by induction on m. The result holds for $m = 0$, since then $A = B = I$ and we can take $J = I$, $\lambda = \mu = 1$. Assume that $m > 0$ and that the result holds for all pairs A_1, B_1 satisfying the requirements of the lemma and for which $l(A_1 B_1) < m$. We suppose also, without loss of generality, that $l(A) \leqq l(B)$. Then, since $AB = BA$, we have, from the uniqueness of the normal form of the elements of a free product, that there exists an element C of Π such that $B = AC$. Hence $AAC = ACA$ and so $AC = CA$. If $A \neq I$, then $l(AC) < m$ and so by the induction hypothesis there exist $J \in \Pi$ and integers λ, μ such that $A = J^\lambda$, $C = J^\mu$ and $\alpha(J, J) = 0$. Thus $B = J^{\lambda+\mu}$ and the result holds for the pair A, B in this case. If $A = I$, then we can take $J = B$, $\lambda = 0$ and $\mu = 1$. Hence the result holds for the pair A, B. This proves the lemma.

Now we come to the main result.

LEMMA 5. *Let W be cyclically Ω-reduced and $l(W) > 1$.*

(A). *If W^2 is cyclically Ω-reduced, then either*

(i) *W^n is Ω-reduced for all integers n, or*

(ii) *there exist $J \in \Pi$, $R \in \Omega$ and non-zero integers σ, τ such that $W = J^\sigma$, $R = J^\tau$.*

(B). *If W^2 is not cyclically Ω-reduced, then either*

(iii) *there exist $R \in \Omega$, $T \in \Pi$ and a cyclic arrangement $W_1 W_2$ of W such that*

$$R = W_1 W_2 W_1 T^{-1},$$

$(T.W_2)^n$ is Ω-reduced for all integers n and $(T.W_2)$, $(T.W_2)^2, \ldots$ are increasing in length, or

(iv) *part* (ii) *of* (A) *above holds.*

Note. We note that, if (iii) holds, then $T.W_2$ is the result of replacing $W_1 W_2 W_1$ by T in $(W_1 W_2 W_1)W_2$, so that $T.W_2 \approx (W_1 W_2)^2$. Now $(W_1 W_2)^2$ is conjugate to W^2, since $W_1 W_2$ is a cyclic arrangement of W, and so $|T.W_2|$ is conjugate to $|W|^2$ in $\Pi/[\Omega]$.

It follows that, if either (i) or (iii) holds, then $|W|$ has infinite order in $\Pi/[\Omega]$, while, if (ii) holds, then $|W|$ has finite order.

Proof of (A). 1^0. We have W^2 cyclically Ω-reduced. If W^n is Ω-reduced for all positive integers n, then the result follows; so we assume that W^m is not Ω-reduced for some positive integer m.

Thus we have $W^m = XSY$ say, where $S \in \Omega_1$ and is chosen to be maximal, $ST^{-1} = R \in \Omega$, $XSY \approx X.T.Y$,

$$\beta(X, T) = \beta(T, Y) = 0 \tag{3.11}$$

and

$$l(XSY) > l(X.T.Y). \tag{3.12}$$

2^0. We show $l(S) \geqq l(W^2)$.

For suppose that this is not the case. Then there exists a cyclic arrangement W' of W such that $W'^2 = SP$ for some $P \neq I$. We note that neither $X = I$ nor $Y = I$ can hold, since otherwise, as $l(S) < l(W^2)$, we could take $m = 2$ and then (3.11) and (3.12) would contradict the fact that W^2 is cyclically Ω-reduced. Thus $\operatorname{Fin}(X) = \operatorname{Fin}(P) = a$, $\operatorname{In}(Y) = \operatorname{In}(P) = b$, say. From (3.11) and (3.12) we have either

(i) $l(S) > l(T)$, or

(ii) $l(S) = l(T)$ and one of $\varepsilon(a, T)$, $\varepsilon(T, b)$ is non-zero, or

(iii) $l(S) = l(T) - 1$ and $\varepsilon(a, T) = \varepsilon(T, b) = 1$.

But each of (i), (ii) and (iii) above contradicts the fact that W^2 is cyclically Ω-reduced, as is easily seen by replacing S by T in $W'^2 = SP$. Hence $l(S) \geqq l(W^2)$.

3^0. It follows that there exist a cyclic arrangement AB of W and an integer $r \geqq 2$ such that

$$S = (AB)^r A,$$

where without loss of generality we may take $B \neq I$. Thus $R = (AB)^r AT^{-1}$, and since $S \in \Omega_1$ we have

$$l\{(AB)^r A\} \geqq [\tfrac{1}{2} l(R)]. \tag{3.13}$$

We show that

$$l\{(AB)^{r-1}\} \geqq \tfrac{1}{6} l(R). \tag{3.14}$$

Suppose that $l\{(AB)^{r-1}\} < \frac{1}{6} l(R)$. Then $l\{(AB)^{r-1}\} \leqq [\frac{1}{6} l(R)]$. From (3.13) we have

$$[\tfrac{1}{2} l(R)] \leqq l\{(AB)^r A\} = l\{(AB)^{r-1}\} + l(AB) + l(A) < 3[\tfrac{1}{6} l(R)]$$

(the last inequality on the right hand side is strict, since $l(A) < l(AB)$ as $B \neq I$). This is a contradiction, hence (3.14) must hold.

Now $(AB)^{r-1}AT^{-1}AB$ and $(AB)^{r-1}ABAT^{-1}$ are elements of Ω, and from (3.14) and property (4) of Ω it follows that

$$(AB)^{r-1}AT^{-1}AB = (AB)^{r-1}ABAT^{-1}$$

and so

$$AT^{-1}AB = ABAT^{-1}.$$

It now follows from Lemma 4 that there exist $J \in \Pi$ and integers λ, μ such that

$$\alpha(J, J) = 0, \quad AB = J^\lambda, \quad AT^{-1} = J^\mu$$

and so

$$R = J^{r\lambda + \mu}.$$

Since $AB \neq I$ and $R \neq I$, it is clear that $J \neq I$ and that the integers λ and $r\lambda+\mu$ are non-zero. Now W is a cyclic arrangement of AB and it is easy to see that there exist elements J_1, J_2 of Π such that $J = J_1 J_2$ and $W = (J_2 J_1)^{\lambda}$. Now $R_1 = (J_2 J_1)^{r\lambda+\mu}$ is an element of Ω, and so we see that (ii) of the lemma holds. This concludes the proof of (A).

Proof of (B). 1^0. We have $l(W) > 1$, W cyclically Ω-reduced and W^2 not cyclically Ω-reduced. It follows that there exist a cyclic arrangement U of W, an element V of Π and an element S of Ω_1 such that

$$U^2 = SV, \quad R = ST^{-1} \in \Omega$$

and

$$l^0(T.V) < l(U^2). \tag{3.15}$$

It is clear that by choosing U and S suitably we can ensure in addition that

$$\beta(T, V) = \beta(V, T) = 0. \tag{3.16}$$

(Suppose, for example, that $\beta(V, T) = r$, so that $V = V_1 V_2$, $T = V_2^{-1} T_1$, where $l(V_2) = r$ and $\beta(V_1, T_1) = 0$. Then $U^2 = SV_1 V_2$ and $R = ST_1^{-1} V_2$. It is easy to see that we can find a cyclic arrangement U_1 of W such that $U_1^2 = V_2 S V_1$. Now $R_1 = V_2 S T_1^{-1} \in \Omega$, $V_2 S \in \Omega_1$,

$$l^0(T_1 . V_1) = l^0(T.V) < l(U^2) = l(U_1^2) \quad \text{and} \quad \beta(V_1, T_1) = 0.)$$

2^0. We now show that

$$\text{if } V = I, \text{ then (ii) of the lemma holds,} \tag{3.17}$$

and

$$\text{if } l(V) = 1, \text{ then } \varepsilon(T, V) = 0. \tag{3.18}$$

We suppose that either $V = I$ or $l(V) = 1$ and $\varepsilon(T, V) = 1$. Then we can write

$$R = U^2 . T_1^{-1}, \tag{3.19}$$

where $\beta(U, T_1^{-1}) = 0$ and

$$l^0(T_1) < l(U^2). \tag{3.20}$$

{For, if $V = I$, then we can take $T_1 = T$, while, if $l(V) = 1$ and $\varepsilon(T, V) = 1$, then

$$R = ST^{-1} = (U^2 . V^{-1})T^{-1} = U^2 .(V^{-1} . T^{-1}) = U^2 . T_1^{-1},$$

where $T_1 = T.V$, $\mathrm{Fin}(U).\mathrm{In}(T_1^{-1}) = V.V^{-1}.\mathrm{In}(T^{-1})$, so that $\beta(U, T_1^{-1}) = 0$, and

$$l^0(T_1) = l^0(T.V) < l(U^2).\}$$

Now, if $T_1 = I$, then $R = U^2$ by (3.19), and, since U is a cyclic arrangement of W, this is clearly equivalent to (ii) of the lemma. (Note also that, if $T_1 = I$, then $\varepsilon(T, V) = 0$.) Hence we can assume that $T_1 \neq I$. Then we can write $T_1^{-1} = T_2^{-1} A T_2$ say, where either $l(A) = 1$ or $l(A) > 1$ and $\beta(A, A) = 0$. We show that

$$l(T_2) < \tfrac{1}{6} l(R). \tag{3.21}$$

For suppose that $l(T_2) \geqq \frac{1}{6} l(R)$. Now $R = (U^2 . T_2^{-1}) A T_2$ and so both $AT_2(U^2 . T_2^{-1})$ and $A^{-1}(T_2 . U^{-2}) T_2^{-1}$ are elements of Ω. Hence, by property (4) of Ω, we have

$$AT_2(U^2 . T_2^{-1}) = A^{-1}(T_2 . U^{-2}) T_2^{-1},$$

so that

$$AT_2 U^2 = A^{-1} T_2 . U^{-2}$$

and in particular $\mathrm{Fin}(U) = \mathrm{Fin}(U^{-1}) = \{\mathrm{In}(U)\}^{-1}$. This is a contradiction, since $l(U) > 1$ and U is cyclically Ω-reduced. Hence (3.21) must hold.

Now from (3.19) we have

$$l(R) = 2l(U)+l(T_1)-\varepsilon(U, T_1^{-1}) = 2l(U)+l(A)+2l(T_2)-\varepsilon(U, T_1^{-1}),$$

so that

$$2l(U)+l(A) \geqq l(R)-2l(T_2),$$

and, by (3.21),

$$2l(U)+l(A) > \tfrac{4}{6}l(R). \tag{3.22}$$

Also, from (3.20), we have

$$2l(U) > l^0(T_1) = l^0(T_2^{-1}AT_2) = l^0(A) \geqq l(A)-1;$$

hence

$$2l(U) \geqq l(A)$$

and from (3.22) it follows that

$$l(U) > \tfrac{1}{6}l(R).$$

Now $UU.T_1^{-1}$ and $U.T_1^{-1}U$ are elements of Ω of length $l(R)$; hence

$$UU.T_1^{-1} = U.T_1^{-1}U$$

and so

$$U.T_1^{-1} = T_1^{-1}U. \tag{3.23}$$

In particular $l(U.T_1^{-1}) = l(T_1^{-1}U)$, so that $\varepsilon(U, T_1^{-1}) = 0$ and therefore

$$R = U^2T_1^{-1}. \tag{3.24}$$

Now, if $l(V) = 1$ and $\varepsilon(T, V) = 1$, then $R = U^2.T_1^{-1}$ and $\varepsilon(U, T_1^{-1}) = 1$, which contradicts (3.24). Hence $T_1 = T$ and we have proved (3.18). Also, from (3.23) we have $UT^{-1} = T^{-1}U$, and by Lemma 4 there exist $J \in \Pi$ and integers λ, μ such that

$$U = J^{\mu}, \qquad T^{-1} = J^{\lambda} \qquad \text{and so} \qquad R = J^{2\mu+\lambda}.$$

Clearly μ and $2\mu+\lambda$ are non-zero. Since U is a cyclic arrangement of W the above result is equivalent to (ii) of the lemma, so that we have proved (3.17).

3^0. In view of (3.17) we can assume for the remainder of the proof that $V \neq I$. Under this assumption we shall show that (iii) of the lemma holds. We note firstly that

$$T \neq I. \tag{3.25}$$

For, if $T = I$, then $U^2 = RV$, so that

$$2l(U) = l(R)+l(V) > l(R).$$

It follows that we can write $U = S_1P_1$, where S_1 is a subword of R such that $l(S_1) > \tfrac{1}{2}l(R)$. Now replacing S_1 in U by $\mathscr{C}(S_1; R)$ yields an element whose length is less than $l(U)$, contradicting the fact that U is cyclically Ω-reduced. Hence (3.25) must hold.

We now show that

$$\text{if } l(T) = 1, \text{ then } \varepsilon(T, V) = 0. \tag{3.26}$$

Suppose that $l(T) = 1$ and $\varepsilon(T, V) = 1$. Then $l(V) \geqq 2$, by (3.18), and

$$2l(U) = l(U^2) = l(S) + l(V) = l(R) - 1 + l(V) > l(R).$$

Now $l(S) > \frac{1}{2}l(R)$, since $l(T) = 1$, and, since $U^2 = SV$ and $l(U) > \frac{1}{2}l(R)$, it follows that we can write $U = S_1 P_1$, where S_1 is a subword of R and $l(S_1) > \frac{1}{2}l(R)$. This yields a contradiction as in the proof of (3.25). Hence (3.26) must hold.

Now $V \neq I$ and $T \neq I$ and in view of (3.16), (3.18) and (3.26) it follows that

$$l^0(T.V) = l(T.V) - \varepsilon(V, T);$$

for, if $l(V) \geqq 2$ and $l(T) \geqq 2$, then this is so by (3.16), while, if $l(V) = 1$, then it is so by (3.16) and (3.18), and if $l(T) = 1$, then it is so by (3.16) and (3.26). Hence instead of (3.15) we can write either

$$l(T.V) < l(U^2) \tag{3.27}$$

or

$$l(T.V) = l(U^2) \quad \text{and} \quad \mathrm{In}(T.V) \sim \mathrm{Fin}(T.V). \tag{3.28}$$

4^0. We show that $l(S) \geqq l(U)$.

Suppose that $l(S) < l(U)$. Then $U = SP$ for some $P \neq I$. Hence $V = PSP$ and so $\mathrm{In}(P) = \mathrm{In}(V)$, $\mathrm{Fin}(P) = \mathrm{Fin}(V)$. From (3.16), (3.27) and (3.28) it follows that either

$$l(T.P) < l(U)$$

or

$$l(T.P) = l(U) \quad \text{and} \quad \mathrm{In}(T.P) \sim \mathrm{Fin}(T.P).$$

Each of these contradicts the fact that U is cyclically Ω-reduced. Hence we must have $l(S) \geqq l(U)$.

Thus S has form $W_1 W_2 W_1$, say, where $U = W_1 W_2$, so that $V = W_2 \neq I$ and

$$R = W_1 W_2 W_1 T^{-1}.$$

Put $l(R) = \lambda$. Then

$$l(W_1 W_2 W_1) \geqq [\tfrac{1}{2}\lambda], \tag{3.29}$$

since $S \in \Omega_1$, and

$$l(W_1 W_2) \leqq \tfrac{1}{2}\lambda \tag{3.30}$$

since $l(W_1 W_2) > \frac{1}{2}\lambda$ would imply that U was not cyclically Ω-reduced.

Rewriting (3.16), (3.27) and (3.28), we have

$$\beta(T, W_2) = \beta(W_2, T) = 0 \tag{3.31}$$

and either

$$l(T.W_2) < l(U^2) \tag{3.32}$$

or

$$l(T.W_2) = l(U^2) \quad \text{and} \quad \mathrm{In}(T.W_2) \sim \mathrm{Fin}(T.W_2). \tag{3.33}$$

5^0. We now obtain upper bounds for $l(W_1)$ and lower bounds for $l(W_2)$ and $l(T)$.

Suppose that $l(W_1) \geqq \frac{1}{6}\lambda$. Then, using property (4) of Ω, we have

$$W_1 W_2 W_1 T^{-1} = W_1 T^{-1} W_1 W_2$$

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and so $\mathrm{In}(W_2) = \mathrm{In}(T^{-1}) = \{\mathrm{Fin}(T)\}^{-1}$, which contradicts (3.31). Hence

$$l(W_1) < \tfrac{1}{6}\lambda. \tag{3.34}$$

Moreover, if either $\mathrm{In}(W_2) \sim \mathrm{In}(T^{-1})$ or $\mathrm{Fin}(W_2) \sim \mathrm{Fin}(T^{-1})$, a similar argument shows that in these cases we must have

$$l(W_1) < \tfrac{1}{6}\lambda - 1. \tag{3.35}$$

We now prove that

$$l(W_2) > \tfrac{1}{6}\lambda. \tag{3.36}$$

Suppose firstly that $\varepsilon(W_2, T) = \varepsilon(T, W_2) = 0$. Then (3.32) must hold, that is,

$$l(T) + l(W_2) < l(W_1 W_2 W_1 W_2),$$

and so $l(W_1 W_2 W_1) > \frac{1}{2}\lambda$. From (3.34) it follows that in this case

$$l(W_2) > \tfrac{1}{6}\lambda. \tag{3.37}$$

Suppose now that at least one of $\varepsilon(W_2, T)$, $\varepsilon(T, W_2)$ is non-zero. Let $\lambda = 6\sigma + \tau$, where $\sigma \geqq 1$ and $0 \leqq \tau \leqq 5$. From (3.29) we have $l(W_1 W_2 W_1) \geqq 3\sigma + [\frac{1}{2}\tau]$ and from (3.35) we have $l(W_1) < \sigma + \frac{1}{6}\tau - 1$. Hence

$$\begin{aligned} l(W_2) = l(W_1 W_2 W_1) - 2l(W_1) &> 3\sigma + [\tfrac{1}{2}\tau] - 2(\sigma + \tfrac{1}{6}\tau - 1) \\ &= \sigma + 1 + \tfrac{1}{6}\tau + (1 + [\tfrac{1}{2}\tau] - \tfrac{1}{3}\tau) \\ &\geqq \sigma + 1 + \tfrac{1}{6}\tau = \tfrac{1}{6}\lambda + 1. \end{aligned} \tag{3.38}$$

Combining (3.37) and (3.38) we see that (3.36) holds.

Now we look at $l(T)$. We have

$$l(T) = \lambda - l(W_1 W_2 W_1) = \lambda - l(W_1 W_2) - l(W_1) > \lambda - \tfrac{1}{2}\lambda - \tfrac{1}{6}\lambda = \tfrac{1}{3}\lambda,$$

and since $\lambda \geqq 7$ we have

$$l(T) > \tfrac{1}{6}\lambda + 1. \tag{3.39}$$

We note that from (3.36) and (3.39) we have $l(W_2) \geqq 2$ and $l(T) \geqq 2$. From (3.31) it follows that $(T.W_2), (T.W_2)^2, \ldots$ are increasing in length, and that, for any positive integer n, $(T.W_2)^n$ is a partition.

6^0. We complete the proof by showing that $(T.W_2)^n$ is Ω-reduced for all positive integers n. We suppose that, for some positive integer n, $(T.W_2)^n$ contains a subword Q by which it is Ω-reducible, $QY^{-1} = R_1 \in \Omega$ say. We show that either

replacing Q in $(T.W_2)^n$ is equivalent to replacing some T in $(T.W_2)^n$ by $\mathscr{C}(T; R^{-1})$ (3.40)

or

replacing Q in $(T.W_2)^n$ is equivalent to replacing some W_2 in $(T.W_2)^n$ by $\mathscr{C}(W_2; R)$. (3.41)

We consider the various forms Q can take.

(a) Q "contains" T, that is there is a partition $K.L.T.M.N$ of $(T.W_2)^n$ such that $Q = L.T.M$, $(T.W_2)^n = KQN$ and, for some $\gamma \geqq 1$, $K.L = (T.W_2)^{n-\gamma}$ and

$$M.N = W_2.(T.W_2)^{\gamma-1}.$$

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Then we have $R_1 = L.T.MY^{-1}$ and $R^{-1} = T(W_1 W_2 W_1)^{-1}$. Now clearly replacing Q in KQN by $\mathscr{C}(Q; R_1)$ is equivalent to replacing T in $K.L.T.M.N$ by $\mathscr{C}(T; R_1)$; this in turn is equivalent to replacing T in $K.L.T.M.N$ by $\mathscr{C}(T; R^{-1})$, for by (3.39) and Lemma 2 we have $\mathscr{C}(T; R_1) = \mathscr{C}(T; R^{-1})$. Hence in this case replacing Q in KQN is equivalent to replacing T in $K.L.T.M.N$ by $\mathscr{C}(T; R^{-1})$, that is, (3.40) holds.

Since the proofs of the following cases are similar to that of case (a) we shall only give outlines of them.

(b) Q " contains " W_2, that is, there is a partition $K.L.W_2.M.N$ of $(T.W_2)^n$ such that $Q = L.W_2.M$, $(T.W_2)^n = KQN$ and, for some $\gamma \geqq 1$, $K.L = (T.W_2)^{n-\gamma}.T$ and

$$M.N = (T.W_2)^{\gamma-1}.$$

Then (3.41) follows from (3.36), (3.39) and Lemma 2.

(c) T " contains " Q. Then (3.40) holds.

(d) W_2 " contains " Q. Then (3.41) holds.

(e) $Q = T_2.W_{21}$, where $T = T_1 T_2$, $W_2 = W_{21} W_{22}$ and none of T_1, T_2, W_{21}, W_{22} is the identity. Then, since $l(Q) \geqq [\frac{1}{2}l(R_1)]$, it follows easily that either $l(T_2) \geqq \frac{1}{6}l(R_1)$ or $l(W_{21}) \geqq \frac{1}{6}l(R_1)$, so that either (3.40) or (3.41) must hold.

(f) $Q = W_{22}.T_1$, where $W_2 = W_{21} W_{22}$, $T = T_1 T_2$ and none of W_{21}, W_{22}, T_1, T_2 is the identity. Then, as in (e) above, either (3.40) or (3.41) must hold.

Now Q must take one of the forms considered in (a) to (f) above; hence we have shown that either (3.40) or (3.41) must hold.

7^0. We now suppose that (3.40) holds and obtain a contradiction. We have

$$\mathscr{C}(T; R^{-1}) = W_1 W_2 W_1,$$

and replacing the $(n-\gamma+1)$th T in $(T.W)^n$ we obtain

$$l\{(T.W_2)^n\} > l\{(T.W_2)^{n-\gamma}.(W_1 W_2 W_1).W_2.(T.W_2)^{\gamma-1}\}. \tag{3.42}$$

Now $\alpha(W_2, W_1) = \alpha(W_1, W_2) = 0$, and, if $W_1 = I$, then $\alpha(W_2, W_2) = 0$. Hence (3.42) can be written

$$l\{(T.W_2)^n\} > l\{(T.W_2)^{n-\gamma}(W_1 W_2 W_1 W_2).(T.W_2)^{\gamma-1}\}. \tag{3.43}$$

If $n = 1$, (3.43) reduces to

$$l(T.W_2) > l\{(W_1 W_2)^2\} = l(U^2), \tag{3.44}$$

while if $n > 1$ we have

$$l\{(T.W_2)^n\} = l\{(T.W_2)^{n-\gamma}.(T.W_2).(T.W_2)^{\gamma-1}\} \leqq l\{(T.W_2)^{n-\gamma}\} + l(T.W_2) + l\{(T.W_2)^{\gamma-1}\} - \varepsilon(W_2, T)$$

and

$$l\{(T.W_2)^{n-\gamma}(W_1 W_2 W_1 W_2).(T.W_2)^{\gamma-1}\} \geqq l\{(T.W_2)^{n-\gamma}\} + l\{(W_1 W_2)^2\} + l\{(T.W_2)^{\gamma-1}\} - \varepsilon(W_2, T),$$

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so that (3.44) remains true. Now (3.44) contradicts (3.32) and (3.33) and so we conclude that (3.40) cannot hold.

8^0. We now suppose that (3.41) holds. We have $\mathscr{C}(W_2; R) = W_1^{-1}TW_1^{-1}$, and replacing the $(n-\gamma+1)$th W_2 in $(T.W_2)^n$, we obtain

$$l\{(T.W_2)^n\} > l\{(T.W_2)^{n-\gamma}.T.(W_1^{-1}TW_1^{-1}).(T.W_2)^{\gamma-1}\}. \tag{3.45}$$

We consider separately the cases $W_1 \neq I$; $W_1 = I$.

Suppose firstly that $W_1 \neq I$. Then, since $\alpha(T, W_1^{-1}) = \alpha(W_1^{-1}, T) = 0$, we can write (3.45) as

$$l\{(T.W_2)\}^n > l\{(T.W_2)^{n-\gamma}.(TW_1^{-1}TW_1^{-1})(T.W_2)^{\gamma-1}\}. \tag{3.46}$$

If $n = 1$, (3.46) reduces to

$$l(T.W_2) > l(TW_1^{-1}TW_1^{-1}), \tag{3.47}$$

while if $n > 1$ we have

$$l\{(T.W_2)^n\} \leqq l\{(T.W_2)^{n-\gamma}\} + l(T.W_2) + l\{(T.W_2)^{\gamma-1}\} - \varepsilon(W_2, T)$$

and

$$l\{(T.W_2)^{n-\gamma}.(TW_1^{-1}TW_1^{-1})(T.W_2)^{\gamma-1}\} \geqq l\{(T.W_2)^{n-\gamma}\} + l\{(TW_1^{-1}TW_1^{-1}) + l\{(T.W_2)^{\gamma-1}\} - \varepsilon(W_2, T),$$

so that (3.47) remains true. From (3.47) we have

$$l(T) + l(W_2) > 2l(T) + 2l(W_1),$$

that is,

$$l(W_2) > l(T) + 2l(W_1).$$

Since $\lambda = l(T) + 2l(W_1) + l(W_2)$, it follows that $l(W_2) > \frac{1}{2}\lambda$, which contradicts (3.30). Hence (3.41) cannot hold if $W_1 \neq I$.

Now suppose that $W_1 = I$. Then $U = W_2$, $R = W_2 T^{-1}$ and so

$$\alpha(W_2, W_2) = \alpha(W_2, T^{-1}) = \alpha(T^{-1}, W_2) = 0. \tag{3.48}$$

Rewriting (3.32) and (3.33), we have either

$$l(T.W_2) < (W_2^2) \tag{3.49}$$

or

$$l(T.W_2) = (W_2^2) \qquad \text{and} \qquad \operatorname{In}(T) \sim \operatorname{Fin}(W_2). \tag{3.50}$$

Suppose firstly that (3.49) holds. Now $l(W_2) \leqq l(T)$ since W_2 is cyclically Ω-reduced; hence we must have $\varepsilon(T, W_2) = 1$. It follows that $\alpha(T, T) = 0$, for, if $\alpha(T, T)$ were non-zero, then, since $\varepsilon(T, W_2) = 1$, we would have $\alpha(T^{-1}, W_2) > 0$, which contradicts (3.48). Hence we can write (3.45) as

$$l\{(T.W_2)^n\} > l\{(T.W_2)^{n-\gamma}.T^2(T.W_2)^{\gamma-1}\},$$

from which it follows that

$$l(T.W_2) > l(T^2) = 2l(T).$$

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Hence $l(W_2) > l(T)$, which contradicts the fact that W_2 is cyclically Ω-reduced. Thus (3.41) cannot hold if $W_1 = I$ and (3.49) holds. Now suppose that (3.50) holds. Then again we must have $\alpha(T, T) = 0$, since otherwise, as $\varepsilon(W_2, T) = 1$, we would have $\alpha(W_2, T^{-1}) > 0$, contradicting (3.48). Now we obtain a contradiction from (3.45) as in the previous case. Hence (3.41) cannot hold if $W_1 = I$ and (3.50) holds. Since either (3.49) or (3.50) must hold we have shown that (3.41) cannot hold if $W_1 = I$.

Combining our results we have shown that neither (3.40) nor (3.41) can hold. Hence the assumption made in 6^0, that $(T\,.\,W_2)^n$ contains a subword by which it is Ω-reducible, must be false, that is, $(T\,.\,W_2)^n$ is Ω-reduced for all positive integers n. This completes the proof of the lemma.

The following corollary will be used in a later paper.

COROLLARY 2. *Let W be cyclically Ω-reduced and $l(W) > 1$. Suppose that W^2 is not cyclically Ω-reduced, so that there exist a cyclic arrangement U of W, an element V of Π and an element S of Ω_1 such that $U^2 = SV$, $R = ST^{-1} \in \Omega$* and *$l^0(T\,.\,V) < l(U^2)$. Suppose moreover that U and V are chosen so that $\beta(T, V) = \beta(V, T) = 0$. Then, if $|W|$ has infinite order in $\Pi/[\Omega]$,* (iii) *of the lemma holds with $U = W_1 W_2$ and $V = W_2$.*

Proof. It is clear that W satisfies (iii) of the lemma, and an examination of the proof of (B), with particular reference to 4^0, shows that we must have $U = W_1 W_2$ and $V = W_2$.

We now come to the proof of the main theorems.

Proof of Theorem 2. Let $\Pi/[\Omega]$ be an FPS-group such that Ω contains a proper power R, $R = J^\lambda (\lambda > 1)$. We note that $l(J) \geqq 2$ and $\alpha(J, J) = 0$, since $l(R) \geqq 7$ and $\alpha(R, R) = 0$.

We have $J^\lambda \approx I$, so that the order r of $|J|$ is less than or equal to λ. Suppose that $r < \lambda$. Then

$$J^{\lambda - r} = J^\lambda (J^r)^{-1} \approx I \quad \text{and} \quad \lambda - r > 0, \quad \text{so that} \quad r \leqq \lambda - r.$$

Since $J^r \approx I$, it must contain a subword Q by which it is Ω-reducible, $J^r = XQY$ say, where $QT^{-1} = R_1 \in \Omega$ and $l(Q) \geqq [\tfrac{1}{2} l(R_1)]$. Clearly we can find a cyclic arrangement J_1 of J such that $J_1^r = QYX$. We note that $J_1^\lambda = QYXJ_1^{\lambda - r} \in \Omega$ since $J^\lambda \in \Omega$. Also, since $Q \in \Omega_1$, we have, by (2.21) and property (4) of Ω, that

$$QT^{-1} = QYXJ_1^{\lambda - r}.$$

Hence $T^{-1} = YXJ_1^{\lambda - r}$, and replacing Q in XQY we obtain, since $XQY = J^r$ is reducible by Q,

$$l(J^r) > l\{X\,.\,(J_1^{r-\lambda} X^{-1} Y^{-1})\,.\,Y\} \geqq l^0\{X\,.\,(J_1^{r-\lambda} X^{-1} Y^{-1})\,.\,Y\} = l(J_1^{r-\lambda}) = l(J^{\lambda - r}).$$

This is a contradiction, since $r \leqq \lambda - r$. Hence we must have $r = \lambda$, which proves the theorem.

Proof of Theorem 3. Let $\Pi/[\Omega]$ be an FPS-group and let A be an element of Π such that $A \neq I$. Then it is easy to see, by an argument similar to that used in the proof of (iii) of Lemma 3, that there exists a cyclically Ω-reduced element W of Π such that $|A|$ is conjugate to $|W|$, so that $|A|$ and $|W|$ have the same order. Now, if $l(W) = 1$, then by (ii) of Theorem 1 the order of $|W|$ is equal to the order of W, while if $l(W) > 1$, then by Lemma 5 either $|W|$ has infinite order or (ii) of Lemma 5 holds. (See the note following the statement of Lemma 5.)

It follows that, if each constituent group of Π is torsion-free and Ω does not contain a proper power, then $\Pi/[\Omega]$ is torsion-free, while if either Π has a constituent group containing elements of finite order or Ω contains a proper power, then $\Pi/[\Omega]$ has elements of finite order. This proves the theorem.

Proof of Corollary 1. We use the notation of the above proof and suppose that $|A|$ has finite order. It is clear that, if $l(W)=1$, then (i) of the corollary holds, since then we have $|A|=|P|^{-1}|W||P|$ for some $P\in\Pi$, so that A is the image under ϕ of $P^{-1}.W.P$. Thus we can suppose that $l(W)>1$. Since $|A|$ has finite order, it follows that (ii) of Lemma 5 must hold, so that there exist $J\in\Pi$, $R\in\Omega$ and integers m, n such that $W=J^m, R=J^n$. We may assume that both m and n are positive, since, if m is negative, then $W=(J^{-1})^{-m}$ and $R=(J^{-1})^{-n}$, while if n is negative, then $W=J^m$ and $R^{-1}=J^{-n}\in\Omega$. Let t be the highest common factor of m and n. Then $|J|$ has order n by Theorem 2, and, since $|W|=|J|^m$, it follows that $|W|$, and therefore $|A|$, has order n/t. This proves the corollary.

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