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REALISABLE SETS OF CATENARY DEGREES OF NUMERICAL MONOIDS

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Abstract

The catenary degree is an invariant that measures the distance between factorisations of elements within an atomic monoid. In this paper, we classify which finite subsets of $\mathbb{Z}_{\geq 0}$ occur as the set of catenary degrees of a numerical monoid (that is, a co-finite, additive submonoid of $\mathbb{Z}_{\geq 0}$). In particular, we show that, with one exception, every finite subset of $\mathbb{Z}_{\geq 0}$ that can possibly occur as the set of catenary degrees of some atomic monoid is actually achieved by a numerical monoid.

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1. Introduction

Nonunique factorisation theory aims to classify and quantify the failure of elements of a cancellative commutative monoid M to factor uniquely into irreducibles [9]. This is often achieved using arithmetic quantities called factorisation invariants. We consider here the catenary degree invariant (Definition 2.3), which is a nonnegative integer $\mathsf{c}(m)$ measuring the distance between the irreducible factorisations of an element $m \in M$.

Many problems in factorisation theory involve characterising which possible values an invariant can take, given some minimal assumptions on the underlying monoid M. Solutions to such problems often take the form of a 'realisation' result in which a family of monoids achieving all possible values is identified. One such problem of recent interest concerns the possible values of the delta set invariant $\Delta(M)$, comprised of successive differences in lengths of factorisations of elements. It can be shown easily that $\min \Delta(M) = \gcd \Delta(M)$ under minimal assumptions [9], but no further restrictions were known. The so-called 'delta set realisation problem' [3, 6, 7] was recently solved by Geroldinger and Schmid [10] by identifying a family of finitely generated Krull monoids whose delta sets achieve every finite set $D \subset \mathbb{Z}_{\geq 1}$ satisfying $\min(D) = \gcd(D)$.

In a similar vein, the catenary degree realisation problem [11, Problem 4.1] asks which finite sets can occur as the set $\mathsf{C}(M)$ of catenary degrees achieved by elements of some atomic monoid M. A recent paper by Fan and Geroldinger [5] proves that, with minimal assumptions, the product $M = M_1 \times M_2$ of two monoids M_1 and M_2 has set of catenary degrees $\mathsf{C}(M) = \mathsf{C}(M_1) \cup \mathsf{C}(M_2)$. While this does provide a complete solution to the catenary degree realisation problem, the brevity of the proof raises the question 'is such a result possible without appealing to Cartesian products?'.

In this paper, we prove that with only a single exception, any finite set that can occur as the set of catenary degrees of some atomic monoid occurs as the set of catenary degrees of a numerical monoid (Theorem 4.2). Since numerical monoids are submonoids of a reduced rank-one free monoid (namely, $\mathbb{Z}_{\geq 0}$), they never contain a Cartesian product of two nontrivial monoids.

Our method of constructing numerical monoids with prescribed sets of catenary degrees involves carefully gluing smaller numerical monoids (Definition 2.6) in such a way as to retain complete control of the catenary degree of every element (Theorem 3.3). The relationship between the catenary degree and gluings has been studied before [1], though notable subtleties arise. Remark 3.2 clarifies some ambiguity in [1], and is another primary contribution of this manuscript.

2. Background

In what follows, $\mathbb{N} = \mathbb{Z}_{\geq 0}$ denotes the set of nonnegative integers.

Definition 2.1. A *numerical monoid* S is an additive submonoid of $\mathbb{N}$ with a finite complement. When we write $S = \langle n_1, \ldots, n_k \rangle$, we assume $n_1 < \cdots < n_k$, and the chosen generators $n_1, \ldots, n_k$ are minimal with respect to set-theoretic inclusion. These minimal generators are called *irreducible* elements or *atoms*.

Definition 2.2. Fix $n \in S = \langle n_1, \ldots, n_k \rangle$ as above. A *factorisation* of n is an expression $n = u_1 + \cdots + u_r$ of n as a sum of atoms $u_1, \ldots, u_r$ of S. Write

$$\mathsf{Z}_S(n) = \{(a_1, \ldots, a_k) : n = a_1 n_1 + \cdots + a_k n_k\} \subset \mathbb{Z}_{\geq 0}^k$$

for the *set of factorisations* of $n \in S$. Given $\mathbf{a} \in \mathsf{Z}_S(n)$, we denote by $|\mathbf{a}|$ the number of irreducibles in the factorisation $\mathbf{a}$, that is, $|\mathbf{a}| = a_1 + \cdots + a_k$.

We are now ready to define the catenary degree.

Definition 2.3. Fix an element $n \in S = \langle n_1, \ldots, n_k \rangle$ and factorisations $\mathbf{a}, \mathbf{a}' \in \mathsf{Z}_S(n)$. The *greatest common divisor of* $\mathbf{a}$ *and* $\mathbf{a}'$ is given by

$$\gcd(\mathbf{a}, \mathbf{a}') = (\min(a_1, b_1), \ldots, \min(a_k, b_k)) \in \mathbb{Z}_{\geq 0}^k,$$

and the *distance between* $\mathbf{a}$ *and* $\mathbf{a}'$ (or the *weight of* $(\mathbf{a}, \mathbf{a}')$) is given by

$$\mathsf{d}(\mathbf{a}, \mathbf{a}') = \max(|\mathbf{a} - \gcd(\mathbf{a}, \mathbf{a}')|, |\mathbf{a}' - \gcd(\mathbf{a}, \mathbf{a}')|).$$

Given $N \geq 0$, an *N-chain from* $\mathbf{a}$ *to* $\mathbf{a}'$ is a sequence $\mathbf{a} = \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k = \mathbf{a}' \in \mathsf{Z}_S(n)$ of factorisations such that $\mathsf{d}(\mathbf{a}_{i-1}, \mathbf{a}_i) \leq N$ for all $i \leq k$. The *catenary degree of* n, denoted $\mathsf{c}_S(n)$, is the smallest $N \geq 0$ such that an N-chain exists between any two factorisations of n. The *set of catenary degrees of* S is the set $\mathsf{C}(S) = \{\mathsf{c}(m) : m \in S\}$, and the *catenary degree of* S is the supremum $\mathsf{c}(S) = \sup \mathsf{C}(S)$.

We conclude this section by defining Betti elements, which are crucial in computing the catenary degree of numerical monoids (Theorem 2.5), and the gluing operation on numerical monoids, under which Betti elements can be easily described (Theorem 2.7). For a more thorough introduction to gluing, see [12, Ch. 8].

Definition 2.4. Fix a numerical monoid S. For each nonzero $n \in S$, consider the graph ∇_n with vertex set $\mathsf{Z}(n)$ in which two vertices $\mathbf{a}, \mathbf{a}' \in \mathsf{Z}(n)$ share an edge if $\gcd(\mathbf{a}, \mathbf{a}') \neq \mathbf{0}$. If ∇_n is not connected, then n is called a *Betti element* of S. We write

$$\mathrm{Betti}(S) = \{b \in S : \nabla_b \text{ is disconnected}\}$$

for the set of Betti elements of S.

Theorem 2.5 [2, Theorem 3.1]. *For any numerical monoid S,*

$$\mathsf{c}(S) = \max\{\mathsf{c}(b) : b \in \mathrm{Betti}(S)\}.$$

Definition 2.6. Fix numerical monoids S_1 and S_2, and positive integers d_1 and d_2. The monoid $S = d_1S_1 + d_2S_2$ is a *gluing of S_1 and S_2 by d* if $d = \mathrm{lcm}(d_1, d_2) \in S_1 \cap S_2$.

Theorem 2.7 [1, Corollary 2]. *With the notation from Definition 2.6,*

$$\mathrm{Betti}(S) = d_2\,\mathrm{Betti}(S_1) \cup d_1\,\mathrm{Betti}(S_2) \cup \{d\}.$$

3. Catenary degree of numerical monoid gluings

Theorem 3.3 provides the primary technical result used in Theorem 4.2 to construct a numerical monoid with a prescribed set of catenary degrees, and is closely related to Theorem 3.1, an ambiguously worded result (see Remark 3.2) appearing in [1].

Theorem 3.1 [1, Corollary 4]. *If S is a gluing of S_1 and S_2 by d, then*

$$\mathsf{c}(S) \leq \max\{\mathsf{c}(S_1), \mathsf{c}(S_2), \mathsf{c}_S(d)\}.$$

Moreover,

$$\mathsf{c}(S) = \max\{\max\{\mathsf{c}_S(n) : n \in S_1\}, \max\{\mathsf{c}_S(n) : n \in S_2\}, \mathsf{c}_S(d)\}.$$

Remark 3.2. The original statement of Theorem 3.1 as [1, Corollary 4] contained only the second claim above (starting 'Moreover, ...'), although it was stated as

$$\mathsf{c}(S) = \max\{\mathsf{c}(S_1), \mathsf{c}(S_2), \mathsf{c}_S(d)\},$$

with the assumption that '$\mathsf{c}(S_1)$' and '$\mathsf{c}(S_2)$' are computed (in a nonstandard way) by taking the maximum of $\mathsf{c}_S(n)$ over elements of S_1 and S_2, respectively. If these values are computed in the usual way, then the inequality in Theorem 3.1 can indeed be strict (see, for instance, Examples 3.4 and 3.5); the unambiguous statement given in Theorem 3.1 is the result of discussions with the second author of [1] that took place after such a monoid S was found.

Remark 3.2 is an example of one of the many subtleties one encounters when working with the catenary degree. It remains an interesting question to determine for which S equality is achieved in Theorem 3.1; Theorem 3.3 gives one such setting.

Theorem 3.3. *Suppose $S = \langle n_1, \ldots, n_k \rangle$ is a numerical monoid. Fix $c > \mathsf{c}(S)$ and $b \in S$ nonzero with $\gcd(b, c) = 1$, and let $T = \langle cn_1, \ldots, cn_k, b \rangle$. Then*

$$\mathsf{c}_T(n) = \begin{cases} \mathsf{c}_S(n/c) & \text{if } n - cb \notin T, \\ c & \text{if } n - cb \in T, \end{cases}$$

for any $n \in T$. In particular, if $\mathsf{C}(S) = \{\mathsf{c}_S(n) : n < b\}$, then $\mathsf{C}(T) = \mathsf{C}(S) \cup \{c\}$.

Proof. Since $b \in S$ and $\gcd(b, c) = 1$, it follows from [12, Theorem 8.2] that T is a gluing of S and $\mathbb{N}$ by cb, meaning that $\mathrm{Betti}(T) = c\,\mathrm{Betti}(S) \cup \{cb\}$ by Theorem 2.7. As such, if $n - cb \notin T$, then a_{k+1} is constant among all factorisations $\mathbf{a} \in \mathsf{Z}_T(n)$, and thus $\mathsf{c}_T(n) = \mathsf{c}_S((n - a_{k+1}b)/c)$.

Now, suppose $n - cb \in T$. By Theorem 2.5,

$$\mathsf{c}(T) = \max\{\mathsf{c}_T(m) : m \in \mathrm{Betti}(T)\} = \mathsf{c}_T(cb) = c,$$

so it suffices to prove that $\mathsf{c}_T(n) \geq c$. To this end, we will show that if $\mathbf{a}, \mathbf{a}' \in \mathsf{Z}_T(n)$ with $a_{k+1} \neq a'_{k+1}$, then $\mathsf{d}(\mathbf{a}, \mathbf{a}') \geq c$. Without loss of generality, it suffices to assume $\gcd(\mathbf{a}, \mathbf{a}') = 0$ and $a_{k+1} > 0$. Since

$$a_{k+1}b + c(a_1 n_1 + \cdots + a_k n_k) = c(a'_1 n_1 + \cdots + a'_k n_k)$$

and $\gcd(b, c) = 1$, we must have $c \mid a_{k+1}$. This implies

$$\mathsf{d}(\mathbf{a}, \mathbf{a}') \geq a_{k+1} \geq c,$$

which completes the proof. □

We conclude with examples demonstrating that no hypotheses in Theorem 3.3 can be omitted. Both examples can be verified using the `GAP` package `numericalsgps` [4].

Example 3.4. The numerical monoid $T = \langle 6, 9, 10, 14 \rangle = 2S + 9\mathbb{N}$ is a gluing, where $S = \langle 3, 5, 7 \rangle$. In this case, $\mathsf{c}(T) = 3$, even though $\mathsf{c}(S) = 4$.

Example 3.5. The numerical monoid $T = \langle 15, 25, 35, 18, 27 \rangle = 5S_1 + 9S_2$ is a gluing, where $S_1 = \langle 3, 5, 7 \rangle$ and $S_2 = \langle 2, 3 \rangle$. In this case, $\mathsf{c}(T) = 3$, even though $\mathsf{c}(S_1) = 4$, $\mathsf{c}(S_2) = 3$, and both scaling factors 5 and 9 are strictly larger than $\mathsf{c}(S_1)$ and $\mathsf{c}(S_2)$.

4. Realisable sets of catenary degrees

In this section, we apply Theorem 3.3 to characterise the finite subsets of $\mathbb{Z}_{\geq 0}$ which are realised as the set of catenary degrees of a numerical monoid, thus providing an alternative answer to [11, Problem 4.1] from that appearing in [5].

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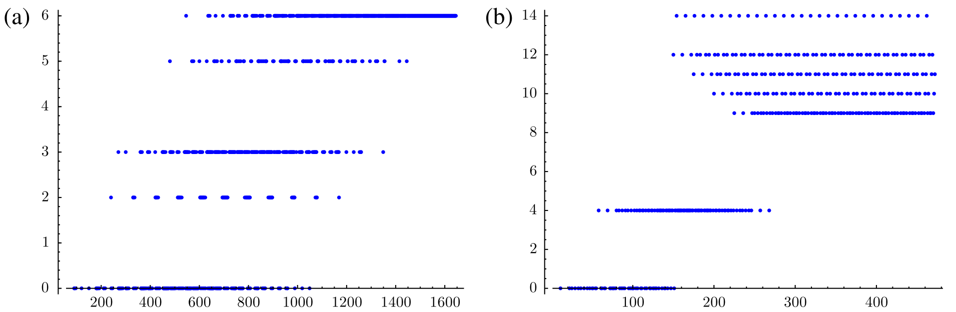

FIGURE 1. Catenary degrees of elements in $S_1 = \langle 90, 91, 96, 120, 150\rangle$ (a) and $S_2 = \langle 11, 25, 29\rangle$ (b) from Example 4.1.

EXAMPLE 4.1. Theorem 4.2 implies $S_1 = \langle 90, 91, 96, 120, 150\rangle$ has the set of catenary degrees $\{0, 2, 3, 5, 6\}$. The plot in Figure 1(a) depicts the catenary degrees of the elements of S_1. The Betti element $480 \in \mathrm{Betti}(S_1)$ has catenary degree $\mathsf{c}(480) = 5$, and the elements $n \in S_1$ with catenary degree $\mathsf{c}(n) \geq 5$ are precisely those divisible by 480 (in the monoid-theoretic sense, that is, $n - 480 \in S_1$).

Not all numerical monoids have catenary degrees so closely tied to their Betti elements. For instance, $S_2 = \langle 11, 25, 29\rangle$, whose catenary degree plot is shown alongside S_1 in Figure 1, has Betti elements β_1, β_2 and β_3, with catenary degrees 4, 12 and 14, respectively. The remaining catenary degrees first occur at elements of the form $\beta_2 + 25k$ for $k = 1, 2, 3$; it is this phenomenon which the monoids constructed in Theorem 4.2 avoid.

THEOREM 4.2. *Fix a finite set $C \subset \mathbb{Z}_{\geq 0}$. Then there exists a numerical monoid with the set of catenary degrees C if and only if (i) $0 \in C$, (ii) $1 \notin C$ and (iii) $c = \max C \geq 3$.*

PROOF. For the backward direction, (i) and (ii) both clearly follow from Definition 2.3, and (iii) follows from the fact that no numerical monoid is half-factorial [8].

For the converse direction, fix a finite set C satisfying conditions (i), (ii) and (iii). We will inductively build a monoid with set of catenary degrees C. If $C = \{0, c\}$, then C is the set of catenary degrees of $\langle c + 1, 2c + 1\rangle$ by [11, Remark 4.2], and if $C = \{0, 2, c\}$, then C is the set of catenary degrees of $\langle 3, 3 + (c - 2), 3 + 2(c - 2)\rangle$ by [11, Theorem 4.3]. In all other cases, $C' = C \cap [0, c)$ satisfies (i), (ii) and (iii) above, so we can inductively assume that C' is the set of catenary degrees of some numerical monoid $S = \langle n_1, \ldots, n_k\rangle$. Choosing $b \in S$ sufficiently large with $\gcd(b, c) = 1$ and applying Theorem 3.3 yields a monoid T with set of catenary degrees C, as desired. □

EXAMPLE 4.3. Let $C = \{0, 2, 7, 20, 26, 57\}$. Following the proof of Theorem 4.2, we begin with the monoid $S = \langle 3, 8, 13\rangle$, which has the set of catenary degrees $\{0, 2, 7\}$. Subsequent choices of b outlined in Table 1 yield a numerical monoid with set of catenary degrees C. Note that each choice of b lies in the monoid S from the previous row, as required by Theorem 4.2 (for instance, $1301 = 11 \cdot 51 + 7 \cdot 60 + 2 \cdot 160$). This can be readily checked with the GAP package `numericalsgps` [4].

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TABLE 1. Computation of catenary degrees for Example 4.3.

c	b	S	Catenary degrees
7		$\langle 3, 8, 13\rangle$	$\{0, 2, 7\}$
20	51	$\langle 51, 60, 160, 260\rangle$	$\{0, 2, 7, 20\}$
26	1301	$\langle 1301, 1326, 1560, 4160, 6760\rangle$	$\{0, 2, 7, 20, 26\}$
57	57001	$\langle 57001, 74157, 75582, 88920, 237120, 385320\rangle$	$\{0, 2, 7, 20, 26, 57\}$

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