THE PURITY OF A COMPLETION

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This note establishes two statements from R. M. Fossum's review [1] of a paper by E. A. Magarian [2]. Firstly, if $A \to B$ is a pure homomorphism (of commutative rings) then $A[[x_1, \ldots, x_s]] \to B[[x_1, \ldots, x_s]]$ is pure. Secondly, if $R_n \to R$ is a directed family of pure homomorphisms then $\cup R_n \to R$ is pure. A consequence is that if $R_n \to R$ is a directed family of pure homomorphisms and if R is Noetherian, then $\cup R_n[[x_1, \ldots, x_s]]$ is Noetherian.

A homomorphism $A \to B$ is said to be *pure* (respectively *n-pure*) if for every A-module M (respectively generated by n elements) the natural bimodule map $M \to M \otimes B$ is injective. Clearly a morphism is pure if and only if it is n-pure for each n. The notion of n-pure is equivalent to what Gilmer and Mott [3] called condition ξ_n, namely each system of n linear equations over A which has a solution in B already has a solution in A. This equivalence is easily seen by observing that each of n-pure and ξ_n is equivalent to the condition that for each n and each submodule L of A^n the diagram

$$\begin{array}{ccc} L & \to & A^n \\ \downarrow & & \downarrow \\ LB & \to & B^n \end{array}$$

be a pullback. Moreover, it is clear that it suffices to check that those diagrams with L finitely generated are pullbacks. Thus $A \to B$ is n-pure if $M \to M \otimes B$ is injective for each finitely presented A-module M generated by n elements. The condition that $f: A \to B$ be 1-pure is equivalent to $f^{-1}(f(I)B) = I$ for each ideal I of A. For this reason it is sometimes called (C) Here n-pure (or for $n = 1$ *cyclic pure*) is used since most of the arguments are module-, rather than ideal-, theoretic. The corollary above generalizes a result from [2] where it is assumed that each of the homomorphisms $R_n \to R$ admits an R_n-linear retraction $R \to R_n$. In that case for each R_n-module M there is a retraction of $M \to M \otimes R$ induced by $R \to R_n$. Maps admitting retractions are always injective, hence $M \to M \otimes R$ is injective for each M and therefore $R_n \to R$ is pure. So the corollary above is a more general statement than [2, Theorem 2]. In the same paper Magarian proves that $\cup R_n[[x]]$ is Noetherian (one variable) if $R_n \to R$ is a directed family of cyclic pure homomorphisms. The technique suggested by Fossum in [1] cannot be modified to give an n-variable generalization of this last theorem; nor does an induction on the number of variables seem possible. The obstruction is that $A \to B$ can be cyclic pure while $A[[x]] \to B[[x]]$ is not. To see this one can replace by power series rings the polynomial rings in Enoch's construction [4] of a cyclic pure homomorphism $A \to B$ such that

$$A[x_1, \ldots, x_m] \to B[x_1, \ldots, x_m]$$

is not cyclic pure. Then one obtains $A \to B$ cyclic pure, but $A[[x_1, \ldots, x_m]] \to B[[x_1, \ldots, x_m]]$ not cyclic pure. Finally by choosing m minimal and replacing A by $A[[x_1, \ldots, x_{m-1}]]$ we find that $A \to B$ is cyclic pure but $A[[x]] \to B[[x]]$ is not.

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THEOREM 1. *Let $u: A \to B$ be a unitary homomorphism of commutative rings and I a finitely generated ideal of A. Let super $\wedge$ denote the I-adic completion functor. Suppose A and B are Hausdorff in the I-adic topologies. If $u: A \to B$ is pure (respectively n-pure) then $\hat{u}: \hat{A} \to \hat{B}$ is pure (respectively n-pure). If A is a Zariski ring and if $\hat{u}: \hat{A} \to \hat{B}$ is pure (respectively n-pure) then $u: A \to B$ is pure (respectively n-pure).*

Proof. Assume $u: A \to B$ is n-pure. Let M be an A-module generated by n elements. Note that M and $B \otimes M$ are complete and Hausdorff in the I and IB-adic topologies [5, p. 58]. Hence $u_M: M \to M \otimes B$ is the inverse limit of the mappings $u_r: M_r \to (M \otimes B)_r$ where $M_r = M \otimes A_r$, $A_r = A/(I^r A) = A/I^r$, [5, §2, no. 6]. However u_r is also obtained by tensoring $A_r \to B_r$ with M_r over A_r. Also M_r is generated by n elements over A_r. Therefore, by the left exactness of inverse limit it suffices for the injectivity of u_M that $A_r \to B_r$ be n-pure. This follows from the next proposition. The converse implication is valid for a Zariski ring A (i.e. A is Noetherian with Jacobson radical containing I) because in this case $A \to \hat{A}$ is faithfully flat [5, p. 72] thence pure. In effect the composite $(A \to B \to \hat{B}) = (A \to \hat{A} \to \hat{B})$ is pure so $A \to B$ is pure.

PROPOSITION 2. *If $A \to B$ is n-pure and if $A \to C$ is either a surjection or a flat epimorphism of commutative rings then $C \to C \otimes B$ is n-pure. If $A \to B$ is pure and $A \to C$ is any homomorphism, then $C \to C \otimes B$ is pure.*

Proof. Suppose that $A \to C$ is a surjection. Given an n-generated C-module M, M is also generated by n elements over A. Hence $M \to M \otimes_A B$ is injective. But $M \otimes_A B = M \otimes_C C \otimes_A B$. Now suppose $A \to C$ is a flat epimorphism. Given a C-module M generated by $m_1, \ldots, m_n$ let L be the A-submodule of M generated by the m's. Then, since L is n-generated, $L \to L \otimes_A B$ is injective. Since C is A-flat $L \otimes_A C \to L \otimes_A C \otimes_A B$ is injective. Since $A \to C$ is a flat epimorphism the multiplication mapping $L \otimes_A C \to M$ is an isomorphism. Hence $M \to M \otimes_A B = M \otimes_C C \otimes_A B$ is injective. The analog for pure is trivial since no cardinality problems arise.

COROLLARY 3. (Fossum [1]) *If $A \to B$ is pure then $A[[x_1, \ldots, x_s]] \to B[[x_1, \ldots, x_s]]$ is pure.*

Proof. Since $A \to B$ is pure then by Proposition 2 $A[x_1, \ldots, x_s] \to B[x_1, \ldots, x_s]$ is pure. Then Theorem 1 applies with I taken to be the ideal generated by the x's.

PROPOSITION 4. *If $R_t \to R$ is a directed family of pure (respectively n-pure) homomorphisms then $\cup R_t \to R$ is pure (respectively n-pure).*

Proof. Let $A = \cup R_t$ and let M be a finitely presented A-module generated by n elements. Choose a presentation $A^m \xrightarrow{f} A^n \to M \to 0$ and matrix representing f. Define $M_t = \text{coker}$ $(R_t^m \to R_t^n)$ if the elements of the matrix of f are in R_t and $M_t = 0$ otherwise. For each $s \geqq t$ let $M_t \to M_s$ be the natural bimodule homomorphism obtained by extension of scalars $R_t \to R_s$. Then $\varinjlim M_t = M$ and M_t is generated by n elements over R_t. Hence $M_t \to M_t \otimes_{R_t} R$ is injective for each t; the injectivity of $M \to M \otimes_A R$ follows from the exactness of direct limits.

COROLLARY 5. (Fossum [1]) *If* $R_t \to R$ *is a directed family of pure homomorphisms and if* R *is Noetherian then* $\cup R_t[[x_1, \ldots, x_s]]$ *is Noetherian.*

Proof. By the preceding results the union A of the $R_t[[x_1, \ldots, x_s]]$ is pure in $R[[x_1, \ldots, x_s]]$. However if $A \to R$ is cyclic pure and R Noetherian then A is Noetherian by [6, Theorem 4].

In [3] Gilmer and Mott and in [7] Gilmer study under what conditions a cyclic pure homomorphism is pure. The following provides a simple proof of one of the results from [3].

PROPOSITION 6. *Let* $A \to B$ *be a cyclic pure homomorphism such that* B *is a torsionfree* A*-module and* A *is an integral domain. If* M *is a finitely presented* A*-module of projective dimension* $\leqq 1$, *then* $M \to M \otimes B$ *is injective.*

Proof. By hypothesis there is an exact sequence $0 \to A^m \to A^n \to M \to 0$. Given y in B^m such that $u(y) = x$ and x in A^n, it must be shown that for some z in A^m, $u(z) = x$. Since u is injective some m by m minor of u is not zero. Let S be the corresponding set of linear equations. By Cramer's Rule $dy_i = d_i$ where d is the (nonzero) determinant of the coefficient matrix of S and d_i is the determinant obtained by replacing the i-th column of S by the appropriate entries of x. Now d, d_i are in A and so d_i is in $A \cap dB = dA$. Thus $d_i = z_i d$ for z_i in A. Since d is a nonzero-divisor on B then $y_i = z_i$. So the original solution y actually lies in A^m. This establishes Proposition 6.

COROLLARY 7. (Gilmer and Mott [3]) *If* $A \to B$ *is a cyclic pure homomorphism with* B A*-torsionfree and if* A *is a Prüfer domain then* $A \to B$ *is pure.*

Proof. By Proposition 6 since over a Prüfer domain each finitely presented module has projective dimension $\leqq 1$.

REFERENCES

1. R. M. Fossum, Review number 290, *Math. Reviews* **48** (1974), 58–59.

2. E. A. Magarian, Direct limits of power series rings, *Math. Scand.* **31** (1972), 103–110.

3. R. Gilmer and J. Mott, Some results on contracted ideals, *Duke Math. J.* **37** (1970), 751–767.

4. E. E. Enoch, On absolutely pure modules, Preprint, University of Kentucky.

5. N. Bourbaki, *Algébre Commutative* Chapitre 3 (Hermann, 1967).

6. R. Gilmer, Contracted ideals with respect to integral extensions, *Duke Math. J.* **34** (1967), 561–572.

7. R. Gilmer, Contracted ideals in Krull domains, *Duke Math. J.* **37** (1970), 769–774.

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