Canad. Math. Bull. Vol. **27** (2), 1984

ASYMPTOTIC THEORY OF SINGULAR SEMILINEAR ELLIPTIC EQUATIONS

BY

TAKAŜI KUSANO AND CHARLES A. SWANSON[(1)]

ABSTRACT. Necessary and sufficient conditions are found for the existence of two positive solutions of the semilinear elliptic equation $\Delta u + q(|x|)u = f(x, u)$ in an exterior domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, where q, f are real-valued and locally Hölder continuous, and $f(x, u)$ is nonincreasing in u for each fixed $x \in \Omega$. An example is the singular stationary Klein–Gordon equation $\Delta u - k^2 u = p(x)u^{-\lambda}$ where k and λ are positive constants. In this case NASC are given for the existence of two positive solutions $u_i(x)$ in some exterior subdomain of Ω such that both $|x|^m \exp[(-1)^{i-1}k\,|x|]u_i(x)$ are bounded and bounded away from zero in this subdomain, $m = (n-1)/2$, $i = 1, 2$.

1. **Introduction.** The semilinear elliptic equation

$$\Delta u + q(|x|)u = f(x, u), \qquad x \in \Omega \tag{1.1}$$

is under consideration in an exterior domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, where $q: I \to \mathbb{R}$, $I = (0, \infty)$ and $f: \Omega \times I \to I$ are locally Hölder continuous, and $\min_{|x|=t} f(x, u)$, $\max_{|x|=t} f(x, u)$ are both nonincreasing in $u \in I$ for each $t \in I$. The main theorems in §3 are necessary and sufficient conditions for the existence of two positive solutions of (1.1) in some exterior subdomain of Ω, with specific asymptotic behavior as $|x| \to \infty$. A prototype of (1.1) is the stationary Klein–Gordon equation

$$\Delta u + q(|x|)u = p(x)u^{-\lambda}, \qquad x \in \Omega, \tag{1.2}$$

where λ is a positive constant and $p: \Omega \to I$ is locally Hölder continuous.

The case that $f(x, u)$ is nonpositive and nondecreasing in u, e.g. $p(x) < 0$ and $\lambda < 0$ in (1.2), was solved earlier by Kreith and Swanson [5]. An essential difference in the case $\lambda > 0$ is that (1.2) (or (1.1)) can have a *singular solution*, i.e. a positive solution $u(x)$ such that $\lim_{x \to x_0} u(x) = 0$ for $x_0 \in \Omega$. In addition we prove the existence of *two* positive solutions u_1, u_2 of (1.1) such that $u_1(x)/u_2(x) \to 0$ as $|x| \to \infty$ uniformly in Ω.

Received by the editors March 25, 1983 and, in revised form, June 28, 1983.

1980 Mathematics Subject Classifications: Primary: 35B05 Secondary: 35J60, 34C11, 34EXX.

(1) Support from NSERC (Canada) under Grant A 3105 is acknowledged with gratitude.

Applications of (1.2) in the case $\lambda>0$, $q=0$ arise in particular from boundary layer theory of viscous fluids [1, 2].

The preliminary results for nonlinear ordinary differential equations in §2 also are new, extending theorems of Taliaferro [9, 10]. These are applied in §3 to the ordinary differential equations (3.5) and (3.6), arising when the linear part of (1.1) is restricted to the radial component (3.2) and $f(x, u)$ is replaced by the radial majorants (3.1). Solutions of (3.5), (3.6) obtained in this way then satisfy the partial differential inequalities (3.12), (3.13), implying [8, p. 125] the existence of positive solutions of (1.1) with appropriate asymptotic properties.

2. **Ordinary differential equations.** The existence of positive solutions $y(t)$ of the ordinary differential equation

$$(r(t)y')'=h(t, y), \qquad t\geq a \tag{2.1}$$

will be proved under the following assumptions:

(i) $r:[a,\infty)\to I=(0,\infty)$ is continuous and satisfies $\lim_{t\to\infty} R(t)=\infty$, where

$$R(t)=\int_a^t \frac{ds}{r(s)}; \quad \text{and} \tag{2.2}$$

(ii) $h:[a,\infty)\times I\to I$ is continuous and nonincreasing in the second variable.

A positive solution $y(t)$ of (2.1) defined in some half-line $[T_y,\infty)$ is called a *proper* positive solution. If $y(t)$ is a local solution of (2.1) near $t=a$ with positive initial values $y(a)$ and $y'(a)$, then $y'(t)>0$ throughout $[a,\infty)$ since $r(t)y'(t)$ is increasing, and it is easily seen from assumption (ii) that $y(t)$ can be continued to ∞. Furthermore, integration of $(ry')'>0$ twice yields $y(t)\geq r(a)y'(a)R(t)$, $t\geq a$. Hence equation (2.1) always has proper positive solutions which are unbounded as $t\to\infty$.

If $y(t)$ is a proper positive solution of (2.1) defined in an interval $[t_0,\infty)$, then there are only two possibilities: Either $y'(t)<0$ throughout $[t_0,\infty)$, or $y'(t)>0$ throughout $[t_1,\infty)$ for some $t_1\geq t_0$. If $y'(t)<0$ for all $t\geq t_0$, then $\lim_{t\to\infty} y(t)=k\geq 0$ exists and is finite. Moreover, $\lim_{t\to\infty} r(t)y'(t)=0$, for if $\lim_{t\to\infty} r(t)y'(t)=-m<0$, then $r(t)y'(t)\leq -m$ throughout $[t_0,\infty)$, and integration implies a contradiction of the positivity of $y(t)$. If $y'(t)>0$ for all $t\geq t_1$, integration of $(ry')'>0$ twice gives

$$y(t)-y(t_1)\geq r(t_1)y'(t_1)\int_{t_1}^t \frac{ds}{r(s)}, \qquad t\geq t_1,$$

and hence $y(t)$ is unbounded and $y(t)/R(t)$ is bounded from below by a positive constant for $t\geq t_1$.

These observations are summarized in the lemma below.

LEMMA 2.1. *Every proper positive solution* $y(t)$ *of* (2.1) *defined in an interval*

https://doi.org/10.4153/CMB-1984-032-1 Published online by Cambridge University Press

$[t_0, \infty)$ has exactly one of the following properties:

I. *$y(t)$ is strictly decreasing in $[t_0, \infty)$ with nonnegative finite limit at ∞, and $\lim_{t\to\infty} r(t)y'(t) = 0$;*

II. *$y(t)$ is strictly increasing in $[t_1, \infty)$ for some $t_1 \geq t_0$ and there exists a positive constant c such that $y(t) \geq cR(t)$ for all $t \geq t_1$.*

THEOREM 2.2. *Equation (2.1) has a proper positive solution which is eventually decreasing if and only if*

$$\int^{\infty} R(t)h(t, c)\,dt < \infty \tag{2.3}$$

for some positive constant c.

Proof. If (2.1) has a positive decreasing solution $y(t)$ in $[t_0, \infty)$, there exists a constant $k>0$ such that $y(t) \leq k$ in $[t_0, \infty)$. Integration of (2.1) twice gives

$$y(t_0) - y(t) = \int_{t_0}^{t} \frac{1}{r(s)} \int_s^{\infty} h(\sigma, y(\sigma))\,d\sigma\,ds,$$

which implies that

$$\int_{t_0}^{\infty} \left[\int_{t_0}^{\sigma} \frac{ds}{r(s)}\right] h(\sigma, y(\sigma))\,d\sigma = \int_{t_0}^{\infty} \frac{1}{r(s)} \int_s^{\infty} h(\sigma, y(\sigma))\,d\sigma\,ds < \infty$$

or

$$\int_{t_0}^{\infty} R(t)h(t, y(t))\,dt < \infty. \tag{2.4}$$

Since $y(t) \leq k$ and $h(t, y)$ is nonincreasing in y, (2.4) implies that $\int^{\infty} R(t)h(t, k)\,dt < \infty$.

Conversely, if (2.3) holds for some $c>0$, choose $T>a$ such that $\int_T^{\infty} R(t)h(t, c)\,dt < c$ and consider the set of continuous functions

$$\mathcal{Y} = \{y \in C[T, \infty) : c \leq y(t) \leq 2c, \quad t \geq T\}. \tag{2.5}$$

Clearly $\mathcal{Y}$ is a closed convex subset of the space of continuous functions $C[T, \infty)$ with the compact open topology. Let $M: \mathcal{Y} \to C[T, \infty)$ be the integral operator defined by

$$(My)(t) = c + \int_t^{\infty} \left[\int_t^{s} \frac{d\sigma}{r(\sigma)}\right] h(s, y(s))\,ds, \qquad t \geq T. \tag{2.6}$$

It is easily verified that (i) M maps $\mathcal{Y}$ into $\mathcal{Y}$; (ii) M is a continuous mapping; and (iii) $M\mathcal{Y}$ is relatively compact. Therefore M has a fixed point $y \in \mathcal{Y}$ by the Schauder-Tychonoff fixed point theorem. A standard proof shows that $y(t)$ is a solution of (2.1), which by (2.6) is necessarily positive and decreases to c as $t \uparrow \infty$.

REMARK 1. An open question is to characterize the existence of a proper positive solution $y(t)$ of (2.1) with the property $\lim_{t\to\infty} y(t)=0$.

THEOREM 2.3. *Equation* (2.1) *has an eventually positive solution* $y(t)$ *such that* $y(t)/R(t)$ *has a finite positive limit at* ∞ *if and only if there exists a positive constant* c *such that*

$$\int^{\infty} h(t, cR(t))\, dt<\infty. \tag{2.7}$$

Proof. If $y(t)$ is a positive solution of (2.1) in $[t_0, \infty)$ such that $y(t)/R(t)$ has a finite positive limit at ∞, there exist positive constants k_1 and k_2 such that

$$k_1R(t)\le y(t)\le k_2R(t) \quad \text{for} \quad t\ge t_1, \tag{2.8}$$

where $t_1\ge t_0$ is sufficiently large. Since $r(t)y'(t)$ is increasing, it is not difficult to see that $\lim_{t\to\infty} r(t)y'(t)=\lim_{t\to\infty} y(t)/R(t)$, and hence integration of (2.1) yields

$$\int_{t_1}^{\infty} h(t, y(t))\, dt<\infty. \tag{2.9}$$

It follows from (2.8) and (2.9) that $\int_{t_1}^{\infty} h(t, k_2R(t))\, dt<\infty$, proving the necessity of (2.7).

The sufficiency proof is similar to that of Theorem 2.2. We can choose $T>a$ large enough so that $\int_T^{\infty} h(t, cR(t))\, dt<c$, and replace (2.5) and (2.6), respectively, by

$$\mathcal{Y}=\{y\in C[T, \infty): cR(t)\le y(t)\le 2cR(t), \quad t\ge T\}, \tag{2.5'}$$

$$(My)(t)=2cR(t)-\int_T^t \frac{1}{r(s)}\int_s^{\infty} h(\sigma, y)(\sigma))\, d\sigma\, ds, \qquad t\ge T. \tag{2.6'}$$

THEOREM 2.4. *Every proper positive solution* $y(t)$ *of* (2.1) *satisfies* $\lim_{t\to\infty}[y(t)/R(t)]=+\infty$ *if and only if*

$$\int^{\infty} h(t, cR(t))\, dt=+\infty \quad \textit{for all} \quad c>0. \tag{2.10}$$

The necessity of (2.10) is proved as in Theorem 2.3. Conversely, if (2.10) holds, then $\int^{\infty} R(t)h(t, c)\, dt=+\infty$ for all $c>0$ since $h(t, y)$ is nonincreasing in y. By Theorem 2.2, equation (2.1) cannot have a proper positive solution which is eventually decreasing, and hence every proper positive solution $y(t)$ must be unbounded with $\lim_{t\to\infty}[y(t)/R(t)]>0$. This limit cannot be finite because of Theorem 2.3.

REMARK 2. Since (2.3) implies (2.7), condition (2.3) is sufficient for equation (2.1) to have *two* proper positive solutions $y_1(t)$ and $y_2(t)$ such that both $y_1(t)$ and $y_2(t)/R(t)$ have finite positive limits at ∞; and so in particular

https://doi.org/10.4153/CMB-1984-032-1 Published online by Cambridge University Press

$\lim_{t\to\infty} y_1(t)/y_2(t)=0$. On the other hand, if (2.7) is satisfied but *not* (2.3), then every proper positive solution $y(t)$ is unbounded with $\lim_{t\to\infty}[y(t)/R(t)]$ finite and positive.

In the case of the specialization

$$(2.11) \qquad (r(t)y')'=p(t)y^{-\lambda}, \qquad \lambda>0$$

of (2.1), where r is as before and $p:[a,\infty)\to I$ is continuous, conditions (2.3), (2.7) and (2.10) reduce to, respectively

$$(2.12) \qquad \int^{\infty} R(t)p(t)\,dt<\infty,$$

$$(2.13) \qquad \int^{\infty} [R(t)]^{-\lambda}p(t)\,dt<\infty,$$

$$(2.14) \qquad \int^{\infty} [R(t)]^{-\lambda}p(t)\,dt=+\infty.$$

COROLLARY 2.5. *Condition* (2.12) *is sufficient for* (2.11) *to have two eventually positive proper solutions* $y_1(t)$ and $y_2(t)$ *such that both* $y_1(t)$ *and* $y_2(t)/R(t)$ *have finite limits at* ∞.

COROLLARY 2.6. *If* (2.13) *holds but* $\int^{\infty} R(t)p(t)\,dt=+\infty$, *then* $y(t)/R(t)$ *has a finite positive limit at* ∞ *for every proper positive solution* $y(t)$ *of* (2.11).

COROLLARY 2.7. *Condition* (2.13) *is necessary and sufficient for* (2.11) *to have an eventually positive solution* $y(t)$ *such that* $y(t)/R(t)$ *has a finite positive limit at* ∞.

COROLLARY 2.8. *Condition* (2.14) *is sufficient for every proper positive solution* $y(t)$ *of* (2.11) *to have the property that* $\lim_{t\to\infty}[y(t)/R(t)]=+\infty$.

It is possible for a solution $y(t)$ of (2.11) to be *singular* at $t_1>a$, i.e. $y(t)>0$ in $[t_0, t_1)$, $t_0\geq a$, but $\lim_{t\to\infty} t_1-y(t)=0$. It is not difficult to prove

THEOREM 2.9. *If* $0<\lambda<1$, *then for any* $t_1>a$ *there exists a singular solution of* (2.11) *at* t_1. *If* $\lambda>1$, *no singular solution of* (2.11) *exists, i.e. every positive solution of* (2.11) *is continuable to* ∞.

The results of this section can be proved similarly in the case that $h(t, y)$ is *nondecreasing* in y. (In particular, the first inequality in (2.8) is used instead of the second inequality).

3. **Elliptic equations.** The following notation will be used:

$$\Omega_t=\{x\in\mathbb{R}^n:|x|>t\}, \qquad t>0.$$

Since Ω is an exterior domain, there exists $a>0$ such that $\Omega_t\subset\Omega$ for all $t\geq a$.

https://doi.org/10.4153/CMB-1984-032-1 Published online by Cambridge University Press

Let

$$\phi(t,u)=\min_{|x|=t} f(x,u),\qquad \Phi(t,u)=\max_{|x|=t} f(x,u). \tag{3.1}$$

We consider equation (1.1) in Ω under the following standing hypotheses.

HYPOTHESES FOR (1.1).

(H_1) The functions $q: I \to \mathbb{R}$ and $f: \Omega \times I \to I$ are locally Hölder continuous, where $I=(0,\infty)$;

(H_2) Both $\phi(t,u)$ and $\Phi(t,u)$ are nonincreasing in $u \in I$ for each $t \in I$;

(H_3) The linear differential equation (3.2) below is nonoscillatory in $[a,\infty)$:

$$Lz=t^{1-n}\frac{d}{dt}\left(t^{n-1}\frac{dz}{dt}\right)+q(t)z=0. \tag{3.2}$$

Equation (3.2) is the radial component of the linear part of (1.1), i.e. $f(x,u)$ in (1.1) is replaced by 0, and Δ is replaced by the radial component in spherical polar coordinates.

By hypothesis (H_3), it is well-known that (3.2) has two eventually positive solutions $z_1(t)$ and $z_2(t)$ with $\lim[z_1(t)/z_2(t)]=0$ as $t\to\infty$, and furthermore that L has the factorized form [4, 11]

$$Lz=\frac{1}{p_2(t)}\frac{d}{dt}\left[\frac{1}{p_1(t)}\frac{d}{dt}\left(\frac{z}{p_0(t)}\right)\right], \tag{3.3}$$

where

$$p_0(t)=z_1(t),\qquad p_1(t)=[z_2(t)/z_1(t)]',\qquad p_2(t)=[z_1(t)p_1(t)]^{-1}. \tag{3.4}$$

In view of (3.2) and (3.3), the ordinary differential equations

$$Ly=\phi(t,y) \tag{3.5}$$

$$Lz=\Phi(t,z) \tag{3.6}$$

have the equivalent forms

$$(p_1^{-1}(t)Y')'=p_2(t)\phi(t,p_0(t)Y),\qquad Y=y/p_0(t) \tag{3.7}$$

$$(p_1^{-1}(t)Z')'=p_2(t)\Phi(t,p_0(t)Z),\qquad Z=z/p_0(t) \tag{3.8}$$

of the form (2.1) in the case $r(t)=p_1^{-1}(t)$, $R(t)=z_2(t)/z_1(t)$. (An additive constant can be ignored.) Conditions (2.3) and (2.7) applied to (3.8) are, respectively

$$\int^{\infty} p_2(t)\frac{z_2(t)}{z_1(t)}\Phi(t,cz_1(t))\,dt<\infty \tag{3.9}$$

$$\int^{\infty} p_2(t)\Phi(t,cz_2(t))\,dt<\infty \tag{3.10}$$

for some positive constants c.

THEOREM 3.1. *Condition* (3.9) *is sufficient for equation* (1.1) *to have a positive solution* $u \in C^{2+\alpha}_{\text{loc}}(\Omega_T)$ *for some* $T \geq a$, $0 < \alpha < 1$, *such that* $u(x)/z_1(|x|)$ *is bounded and bounded away from zero in* Ω_T.

Proof. The sufficiency proof of Theorem 2.2 shows that equation (3.8) has an eventually decreasing positive solution $Z(t)$ such that $\lim_{t\to\infty} Z(t) = c$, c as in (3.9). Since $\phi(t, u)$ is nonincreasing in u, (3.1) and (3.9) imply that

$$\int^{\infty} p_2(t) \frac{z_2(t)}{z_1(t)} \phi(t, c^* z_1(t))\, dt < \infty \tag{3.11}$$

for arbitrary $c^* > c$. By Theorem 2.2 again, equation (3.7) has an eventually decreasing positive solution $Y(t)$ such that $\lim_{t\to\infty} Y(t) = c^*$. Then there exists $T \geq a$ such that $Y(t) > Z(t)$ for all $t \geq T$. Define $v(x) = z_1(|x|)Y(|x|)$ and $w(x) = z_1(|x|)Z(|x|)$ for $x \in \Omega_T$. Since (3.5), (3.6) are equivalent to (3.7), (3.8), respectively, it follows from (3.1) that $v(x)$ and $w(x)$ satisfy the differential inequalities

$$\Delta v + q(|x|)v \leq f(x, v), \qquad x \in \Omega_T \tag{3.12}$$

$$\Delta w + q(|x|)w \geq f(x, w), \qquad x \in \Omega_T \tag{3.13}$$

respectively. Furthermore $w(x) \leq v(x)$ throughout Ω_T and $v, w \in C^{2+\alpha}_{\text{loc}}(\Omega_T)$ for some α in $0 < \alpha < 1$ by standard regularity theory [6; §4.8] for equations (3.7), (3.8), in view of the assumed local Hölder continuity of the coefficients in (3.2), (3.7), (3.8). A theorem of Noussair and Swanson [8, p. 125] applied to (3.12), (3.13) shows that equation (1.1) has a positive solution $u(x) \in C^{2+\alpha}_{\text{loc}}(\Omega_T)$ satisfying $w(x) \leq u(x) \leq v(x)$, $x \in \Omega_T$. This solution $u(x)$ evidently has the stated boundedness properties in Theorem 3.1.

The following theorem is proved by virtually the same procedure, applying Theorem 2.3 instead of Theorem 2.2.

THEOREM 3.2. *Condition* (3.10) *is sufficient for equation* (1.1) *to have a positive solution* $u \in C^{2+\alpha}_{\text{loc}}(\Omega_T)$ *for some* $T \geq a$, $0 < \alpha < 1$, *such that* $u(x)/z_2(|x|)$ *is bounded and bounded away from zero in* Ω_T.

COROLLARY 3.3. *Condition* (3.9) *is sufficient for the existence of two positive solutions* $u_i \in C^{2+\alpha}_{\text{loc}}(\Omega_T)$ *for some* $T \geq a$, $0 < \alpha < 1$ *such that each of* $u_i(x)/z_i(|x|)$ *is bounded and bounded away from zero in* Ω_T, $i = 1, 2$.

As an example of (1.1), consider the semilinear equation

$$\Delta u - k^2 u = p(x) u^{-\lambda}, \qquad x \in \Omega, \tag{3.14}$$

where $k \geq 0$ and $\lambda > 0$ are constants and $p \in C^{\alpha}_{\text{loc}}(\Omega)$. In this case (3.1) reduces to

$$\phi(t, u) = p_*(t) u^{-\lambda}, \qquad \Phi(t, u) = p^*(t) u^{-\lambda},$$

where

$$p_*(t) = \min_{|x|=t} p(x), \qquad p^*(t) = \max_{|x|=t} p(x).$$

Case I: $k=0$. The solutions $z_1(t)$, $z_2(t)$ of (3.2) can then be taken to be

$$z_1(t) = 1, \qquad z_2(t) = \log t \quad \text{if} \quad n=2;$$
$$z_1(t) = t^{2-n}, \qquad z_2(t) = 1 \quad \text{if} \quad n \ge 3.$$

Condition (3.9) reduces to

$$\int^\infty t \log t\, p^*(t)\, dt < \infty \quad \text{if} \quad n=2; \tag{3.15}$$

$$\int^\infty t^\sigma p^*(t)\, dt < \infty \quad \text{if} \quad n \ge 3, \tag{3.16}$$

where $\sigma = (n-1) + \lambda(n-2)$, and condition (3.10) reduces to

$$\int^\infty t(\log t)^{-\lambda} p^*(t)\, dt < \infty \quad \text{if} \quad n=2; \tag{3.17}$$

$$\int^\infty t\, p^*(t)\, dt < \infty \quad \text{if} \quad n \ge 3. \tag{3.18}$$

Theorems 3.1 and 3.2 imply the following results.

COROLLARY 3.4 ($k=0, n=2$). *Condition* (3.15) *is sufficient for equation* (3.14) *to have a positive solution which is bounded and bounded away from zero in some domain* $\Omega_T \subset \Omega$. *Condition* (3.17) *is sufficient for* (3.14) *to have a positive solution* $u(x)$ *in some domain* $\Omega_T \subset \Omega$ *such that* $u(x)/\log|x|$ *is bounded and bounded away from zero in* Ω_T.

COROLLARY 3.5. ($k=0$, $n \ge 3$). *Condition* (3.18) *is sufficient for* (3.14) *to have a positive solution which is bounded and bounded away from zero in some domain* $\Omega_T \subset \Omega$. *Condition* (3.16) *implies that* (3.14) *has a positive solution* $u(x)$ *in some domain* Ω_T *such that* $|x|^{n-2}\, u(x)$ *is bounded and bounded away from zero in* Ω_T.

Case II: $k>0$. The solutions $z_1(t)$, $z_2(t)$ of (3.2) can be taken as

$$z_1(t) = t^{-\nu} K_\nu(kt), \qquad z_2(t) = t^{-\nu} I_\nu(kt), \qquad \nu = \frac{n}{2} - 1,$$

where I_ν and K_ν denote the modified Bessel functions of order ν. In view of the well-known asymptotic behavior of these Bessel functions [3, p. 86], it is easily verified that (3.9) and (3.10) reduce to, respectively.

$$\int^\infty t^\rho e^{\sigma t} p^*(t)\, dt < \infty \tag{3.19}$$

https://doi.org/10.4153/CMB-1984-032-1 Published online by Cambridge University Press

$$\int^{\infty} t^{\rho} e^{-\sigma t} p^*(t)\, dt < \infty, \tag{3.20}$$

where $\rho = (n-1)(\lambda+1)/2$, $\sigma = k(\lambda+1)$.

COROLLARY 3.6. ($k>0$). *Condition* (3.20) *is sufficient for* (3.14) *to have a positive solution which is bounded by a constant multiple of* $|x|^{(1-n)/2} e^{k|x|}$ *in some domain* Ω_T. *Condition* (3.19) *implies that* (3.14) *has two positive solutions* $u_1(x)$ *and* $u_2(x)$ *in* Ω_T *for which both*

$$|x|^m e^{k|x|} u_1(x) \quad \textit{and} \quad |x|^m e^{-k|x|} u_2(x), \qquad m = \frac{n-1}{2}$$

are bounded and bounded away from zero in Ω_T.

REMARK 3. A question naturally arises: If the integral in (3.18) diverges to ∞, does there always exist an unbounded positive solution of equation (3.14) ($k=0$, $n\geq 3$)? There are analogous questions in the cases $k>0$ and $n=2$. We note that the answer is affirmative in the radially symmetric case $p(x)=\tilde{p}(|x|)$ because of Theorem 2.4. Consider the example

$$\Delta u = 2\,|x|^{\lambda-1} u^{-\lambda}, \qquad \lambda>0 \tag{3.21}$$

in $\Omega = \{x \in \mathbb{R}^3 : |x| > 1\}$. Since the integral in (3.18) diverges, Theorem 2.4 shows that every positive radially symmetric solution of (3.21) is unbounded. One such solution is $u(x)=|x|$.

If the hypothesis that $\phi(t,u)$ in (3.1) is *convex* in u for each fixed $t>0$ is added, conditions (3.9) and (3.10) become *necessary* conditions for the conclusions of Theorems 3.1 and 3.2, respectively, provided Φ is replaced by ϕ. The proof based on spherical means, Jensen's inequality for convex functions, and our results in §2, is essentially the same as in [7, 8]. If we make the additional mild assumption that

$$\sup_{t\geq a,\, u>0} \frac{\Phi(t,u)}{\phi(t,u)} < \infty,$$

it follows that (3.9) and (3.10) *characterize* equations (1.1) for which solutions exist satisfying the conclusions of Theorems 3.1 and 3.2, respectively. In particular, since $p(x)u^{-\lambda}$ is convex in u, (3.15)–(3.18) are necessary and sufficient conditions for the existence of positive solutions of (3.14) with the properties stated in Corollaries 3.4 and 3.5.

REFERENCES

1. A. J. Callegari and A. Nachman, *Some singular, nonlinear differential equations arising in boundary layer theory*, J. Math. Anal. Appl. **64** (1978), 96–105.

2. ——, *A nonlinear singular boundary value problem in the theory of pseudoplastic fluids*, SIAM J. Appl. Math. **38** (1980), 275–281.

3. A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, and Bateman Project Staff, *Higher Transcendental Functions*, Vol. 2, McGraw-Hill, New York-Toronto-London, 1953.

4. Yuichi Kitamura and Takaŝi Kusano, *Nonlinear oscillations of higher-order functional differential equations with deviating arguments*, J. Math. Anal. Appl. **77** (1980), 100–119.

5. Kurt Kreith and C. A. Swanson, *Asymptotic solutions of semilinear elliptic equations*, J. Math. Anal. Appl., **98** (1984).

6. O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.

7. E. S. Noussair and C. A. Swanson, *Oscillation theory for semilinear Schrödinger equations and inequalities*, Proc. Roy. Soc. Edinburgh **A75** (1975/76), 67–81.

8. ——, *Positive solutions of quasilinear elliptic equations in exterior domains*, J. Math. Anal. Appl. **75** (1980), 121–133.

9. Steven Taliaferro, *On the positive solutions of* $y''+\phi(t)y^{-\lambda}=0$, Nonlinear Anal. **2** (1978) 437–466.

10. ——, *Asymptotic behavior of solutions of* $y''=\phi(t)y^{\lambda}$, J. Math. Anal. Appl. **66** (1978), 95–134.

11. W. F. Trench, *Canonical forms and principal systems for general disconjugate equations*, Trans. Amer. Math. Soc. **189** (1974), 319–327.

Department of Mathematics,
Hiroshima University,
Hiroshima, Japan.

Department of Mathematics,
University of British Columbia,
Vancouver B.C.,
Canada, V6T 1Y4

https://doi.org/10.4153/CMB-1984-032-1 Published online by Cambridge University Press