

# Operator Amenability of the Fourier Algebra in the cb-Multiplier Norm

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*Abstract.* Let  $G$  be a locally compact group, and let  $A_{cb}(G)$  denote the closure of  $A(G)$ , the Fourier algebra of  $G$ , in the space of completely bounded multipliers of  $A(G)$ . If  $G$  is a weakly amenable, discrete group such that  $C^*(G)$  is residually finite-dimensional, we show that  $A_{cb}(G)$  is operator amenable. In particular,  $A_{cb}(\mathbb{F}_2)$  is operator amenable even though  $\mathbb{F}_2$ , the free group in two generators, is not an amenable group. Moreover, we show that if  $G$  is a discrete group such that  $A_{cb}(G)$  is operator amenable, a closed ideal of  $A(G)$  is weakly completely complemented in  $A(G)$  if and only if it has an approximate identity bounded in the cb-multiplier norm.

## Introduction

The Fourier algebra  $A(G)$  of a locally compact group  $G$  was introduced by P. Eymard [E]; for abelian  $G$ , the Fourier transform yields an isometric isomorphism of  $A(G)$  and  $L^1(\widehat{G})$ , where  $\widehat{G}$  is the dual group of  $G$ . Quite soon afterward, H. Leptin [Le] showed that the amenable locally compact groups can be characterized in terms of the Banach algebra  $A(G)$ : the group  $G$  is amenable if and only if  $A(G)$  has a bounded approximate identity.

B. E. Johnson [J1] introduced the notion of an amenable Banach algebra and showed that a locally compact group  $G$  is amenable if and only if  $L^1(G)$  is amenable (this result motivates the choice of terminology). Since every amenable Banach algebra has a bounded approximate identity, Leptin's theorem immediately yields that for any locally compact group  $G$ , the amenability of  $A(G)$  necessitates that of  $G$ . The question for precisely which locally compact groups  $G$  the Fourier algebra  $A(G)$  is amenable remained open for a surprisingly long period of time. Johnson [J3] showed that  $A(G)$  may fail to be amenable for certain compact groups  $G$ . Eventually, Forrest and Runde [FR] proved that  $A(G)$  is amenable if and only if  $G$  has an abelian subgroup of finite index.

The study of the Fourier algebra gained new momentum in 1995 [R]. As the predual of the group von Neumann algebra,  $A(G)$  is an operator space in a canonical manner for every locally compact group  $G$ . Z.-J. Ruan used this to add operator space overtones to Johnson's notion of an amenable Banach algebra and introduced the concept of operator amenability. He showed that a locally compact group  $G$  is amenable if and only if  $A(G)$  is operator amenable [R]. Since then it has become apparent that the theory of operator spaces [ER] provides powerful tools for the study

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of the Fourier algebra of a locally compact group and of related algebras ([A,ARS,FW,IS,LNR,Ru6,RS,S1,W1,W2]; for a detailed overview, see [Ru5]). Even if one is mainly interested in the Fourier algebra as a mere Banach algebra, operator space methods provide new insights: the main results of both [FKLS] and [FR] do not make any reference to operator spaces, but their respective proofs depend on operator space techniques.

In this paper, we investigate another Banach algebra associated with a locally compact group  $G$ , mostly from an operator space point of view. The Fourier algebra  $A(G)$  embeds canonically into the algebra of completely bounded multipliers on  $A(G)$ . For amenable  $G$ , the norm on  $A(G)$  inherited from this algebra (the cb-multiplier norm) is the given norm; for non-amenable groups, however, the two norms are inequivalent. We denote the completion of  $A(G)$  with respect to the cb-multiplier norm by  $A_{cb}(G)$ . Unlike the Fourier algebra,  $A_{cb}(G)$  may well have a bounded approximate identity for non-amenable  $G$  (such groups are called weakly amenable). The main result of this paper is that for certain discrete, non-amenable groups — among them  $F_2$ , the free groups in two generators —  $A_{cb}(G)$  not only has a bounded approximate identity, but is operator amenable. We then move on to study complementation properties of closed ideals of both  $A_{cb}(G)$  and  $A(G)$ , where  $G$  is discrete with  $A_{cb}(G)$  operator amenable. In particular, we show that for such  $G$ , a closed ideal of  $A(G)$  is (weakly) completely complemented in  $A(G)$  if and only if it has a bounded approximate identity.

## 1 Preliminaries

Our reference for the theory of operator spaces is the monograph [ER], whose notation and choice of terminology we adopt unless explicitly stated otherwise.

We begin by introducing basic definitions.

**Definition 1.1** A *quantized Banach algebra* is an algebra  $\mathfrak{A}$  which is also an operator space such that the multiplication of  $\mathfrak{A}$  is completely bounded.

**Remark** We do not require the multiplication of a quantized Banach to be completely contractive: this extra bit of generality can be convenient sometimes [LNR].

### Examples

1. If  $\mathfrak{A}$  is any Banach algebra, then  $\max \mathfrak{A}$  (the maximal operator space over  $\mathfrak{A}$ ; see [ER]) is a quantized Banach algebra.
2. If  $\mathfrak{H}$  is a Hilbert space, then every closed subalgebra of  $\mathcal{B}(\mathfrak{H})$  with its concrete operator space structure is a quantized Banach algebra.
3. Let  $E$  be an operator space. Then  $\mathcal{CB}(E)$  is a quantized Banach algebra.
4. Let  $G$  be a locally compact group, let  $VN(G)$  denote its group von Neumann algebra, and let  $C^*(G)$  and  $C_r^*(G)$  denote its full and reduced group  $C^*$ -algebra, respectively. The dualities

$$A(G) = VN(G)_*, \quad B(G) = C^*(G)^*, \quad \text{and} \quad B_r(G) = C_r^*(G)^*$$

equip  $A(G)$  as well as  $B(G)$  and  $B_r(G)$ , the *Fourier–Stieltjes algebra* and the *reduced Fourier–Stieltjes algebra* [E], respectively, with an operator space structure. With these operator space structures,  $A(G)$ ,  $B(G)$ , and  $B_r(G)$  are quantized Banach algebras.

5. Let  $G$  be a locally compact group. A *multiplier* of  $A(G)$  is a (necessarily bounded and continuous) function  $f : G \rightarrow \mathbb{C}$  such that  $fA(G) \subset A(G)$ . For each such  $f$ , multiplication with  $f$  is a linear operator on  $A(G)$  — bounded by the closed graph theorem — which we denote by  $M_f$ ; it is straightforward that  $M_f : A(G) \rightarrow A(G)$  is an  $A(G)$ -module homomorphism. Alternatively, the term multiplier is also used to refer to an  $A(G)$ -module homomorphism on  $A(G)$ . Both usages are equivalent: whenever  $T : A(G) \rightarrow A(G)$  is an  $A(G)$ -module homomorphism, there is (a necessarily unique)  $f : G \rightarrow \mathbb{C}$  with  $fA(G) \subset A(G)$  such that  $T = M_f$  [Da, p. 422]. The *multiplier algebra* of  $A(G)$  is the closed subalgebra

$$\mathcal{M}(A(G)) := \{M_f : f \text{ is a multiplier of } A(G)\}$$

of  $\mathcal{B}(A(G))$ . For notational convenience, we shall simply identify a multiplier  $f$  of  $A(G)$  and the corresponding  $M_f$ . Finally, the *cb-multiplier algebra* of  $A(G)$  is defined as  $\mathcal{M}_{cb}(A(G)) := \mathcal{CB}(A(G)) \cap \mathcal{M}(A(G))$ ; it is a closed subalgebra of  $\mathcal{CB}(A(G))$  and thus a quantized Banach algebra.

**Definition 1.2** Let  $\mathfrak{A}$  be a quantized Banach algebra. A *quantized Banach  $\mathfrak{A}$ -bimodule* is an  $\mathfrak{A}$ -bimodule  $E$  which is also an operator space such that the module actions

$$\mathfrak{A} \times E \rightarrow E, \quad (a, x) \mapsto a \cdot x \quad \text{and} \quad E \times \mathfrak{A} \rightarrow E, \quad (x, a) \mapsto x \cdot a$$

are completely bounded.

**Remark** Our quantized Banach bimodules are not to be confused with the operator bimodules studied, for instance, in [BL]: every operator bimodule over an operator algebra is a quantized Banach bimodule in the sense of Definition 1.2, but the converse is false.

If  $\mathfrak{A}$  is a quantized Banach algebra and  $E$  is a quantized Banach  $\mathfrak{A}$ -bimodule, then  $E^*$  becomes a quantized Banach  $\mathfrak{A}$ -bimodule in a canonical way through

$$\langle x, a \cdot \phi \rangle := \langle x \cdot a, \phi \rangle \quad \text{and} \quad \langle x, \phi \cdot a \rangle := \langle x, a \cdot \phi \rangle \quad (a \in \mathfrak{A}, \phi \in E^*, x \in E).$$

**Definition 1.3** A quantized Banach algebra  $\mathfrak{A}$  is said to be *operator amenable* if, for every quantized Banach  $\mathfrak{A}$ -bimodule  $E$ , every completely bounded derivation  $D : \mathfrak{A} \rightarrow E^*$  is inner.

**Examples**

1. Let  $G$  be a locally compact group. Then  $A(G)$  is operator amenable if and only if  $G$  is amenable [R, Theorem 3.6].

2. Let  $\mathfrak{A}$  be a Banach algebra. Then the quantized Banach algebra  $\max \mathfrak{A}$  is amenable if and only if  $\mathfrak{A}$  is amenable [R].
3. A  $C^*$ -algebra is amenable if and only if it is operator amenable [R, Theorem 5.1].

We also require a modification of Definition 1.3. Runde [Ru2] considered a class of Banach algebras — suggestively named *dual Banach algebras* — which are dual Banach spaces (with a fixed, but not necessarily unique predual) such that multiplication is separately  $w^*$ -continuous. Runde and Spronk [RS] extended this notion to a quantized setting.

**Definition 1.4** A quantized Banach algebra  $\mathfrak{A}$  is called *dual* if  $\mathfrak{A} = (\mathfrak{A}_*)^*$  for some operator space  $\mathfrak{A}_*$  such that the multiplication of  $\mathfrak{A}$  is separately  $\sigma(\mathfrak{A}, \mathfrak{A}_*)$ -continuous.

**Examples**

1. If  $\mathfrak{A}$  is a dual Banach algebra in the sense of [Ru2], then  $\max \mathfrak{A}$  is a dual, quantized Banach algebra.
2. Every von Neumann algebra is a dual, quantized Banach algebra.
3. Let  $G$  be a locally compact group. Then  $B(G)$  and  $B_r(G)$  are dual, quantized Banach algebras [RS].
4. Let  $G$  be a locally compact group. In both [dCH] and [S2], a predual space of  $\mathcal{M}_{cb}(A(G))$  is constructed. *A priori*, it is not clear that these two predual spaces are identical. However, from [dCH, Lemma 1.9] and [S2, Corollary 6.6], it follows that on norm bounded subsets of  $\mathcal{M}_{cb}(A(G))$ , the  $w^*$ -topology on  $\mathcal{M}_{cb}(A(G))$  induced by either predual space is the relative topology of  $\sigma(L^\infty(G), L^1(G))$ . The Krein–Šmulian theorem then yields that the predual spaces from both [dCH] and [S2] are identical. Since multiplication in  $L^\infty(G)$  is separately  $\sigma(L^\infty(G), L^1(G))$ -continuous, we obtain that multiplication in  $\mathcal{M}_{cb}(A(G))$  is separately  $w^*$ -continuous, first on norm-bounded sets, and then, by virtue of the Krein–Šmulian theorem again, on all of  $\mathcal{M}_{cb}(A(G))$ . Hence,  $\mathcal{M}_{cb}(A(G))$  is a dual, quantized Banach algebra.

In [Ru2], a weaker variant of amenability, called *Connes-amenability*, was introduced for dual Banach algebras. Generally, Connes-amenability seems to be better suited for dual Banach algebras than the original definition from [J1] (*cf.* [DGH, Ru3], for example). Runde and Spronk [RS] extended the notion of Connes-amenability to the quantized setting.

Let  $\mathfrak{A}$  be a quantized Banach algebra, and let  $E$  be a dual, quantized Banach  $\mathfrak{A}$ -bimodule, *i.e.*, a quantized Banach  $\mathfrak{A}$ -bimodule which is the canonical dual module of some other quantized Banach  $\mathfrak{A}$ -bimodule. Suppose that  $\mathfrak{A}$  is dual. Then we say that  $E$  is *normal* if the module actions

$$\mathfrak{A} \times E \rightarrow E, \quad (a, x) \mapsto a \cdot x \quad \text{and} \quad E \times \mathfrak{A} \rightarrow E, \quad (x, a) \mapsto x \cdot a$$

are separately  $w^*$ -continuous.

**Definition 1.5** A dual, quantized Banach algebra  $\mathfrak{A}$  is said to be *operator Connes-amenable* if, for every normal, dual, quantized Banach  $\mathfrak{A}$ -bimodule  $E$ , every  $w^*$ -continuous, completely bounded derivation  $D: \mathfrak{A} \rightarrow E$  is inner.

### Examples

1. A dual Banach algebra  $\mathfrak{A}$  is Connes-amenable in the sense of [Ru2] if and only if  $\max \mathfrak{A}$  is operator Connes-amenable.
2. A locally compact group  $G$  is amenable if and only if  $B_r(G)$  is operator Connes-amenable [RS, Theorem 4.4].
3. The free group in two generators, which we denote by  $F_2$ , is not amenable, but  $B(F_2)$  is operator Connes-amenable [RS].

## 2 Operator Amenability of $\mathcal{M}_{cb}(G)$ for Non-Amenable $G$

Let  $G$  be a locally compact group. Then we have the following completely contractive inclusions:

$$A(G) \subset B_r(G) \subset B(G) \subset \mathcal{M}_{cb}(G).$$

The first and the second inclusion are always complete isometries, whereas the embedding of  $A(G)$  into  $\mathcal{M}_{cb}(A(G))$  is bounded below only if  $G$  is amenable. In this case, we have completely isometric identifications  $B_r(G) = B(G) = \mathcal{M}_{cb}(A(G))$ , so that  $A(G)$  embeds into  $\mathcal{M}_{cb}(G)$  completely isometrically. For a discussion and further references, see [S2].

As  $\mathcal{M}_{cb}(A(G))$  is a dual quantized Banach algebra, it makes sense to ask for which locally compact groups  $G$  it is operator Connes-amenable. Of course, if  $G$  is amenable, then  $\mathcal{M}_{cb}(A(G)) = B(G) = B_r(G)$  is operator Connes-amenable [RS, Theorem 4.4]. The following proposition gives another sufficient condition.

Recall that a  $C^*$ -algebra is said to be *residually finite-dimensional* if its finite-dimensional, irreducible  $*$ -representations separate its points. Furthermore, following [HK], we say that a locally compact group  $G$  has the *approximation property* if there is a net in  $A(G)$  converging to the constant function 1 in the  $w^*$ -topology of  $\mathcal{M}_{cb}(A(G))$ .

**Proposition 2.1** *Let  $G$  be a locally compact group with the approximation property such that  $C^*(G)$  is residually finite-dimensional. Then  $\mathcal{M}_{cb}(A(G))$  is operator Connes-amenable.*

**Proof** Since  $G$  has the approximation property and since  $\mathcal{M}_{cb}(A(G))$  is a dual Banach algebra, it is clear that  $A(G)$  is  $w^*$ -dense in  $\mathcal{M}_{cb}(A(G))$ . Consequently,  $B(G) \supset A(G)$  is also  $w^*$ -dense in  $\mathcal{M}_{cb}(A(G))$ . Since  $C^*(G)$  is residually finite-dimensional,  $B(G)$  is operator Connes-amenable by [RS, Theorem 4.6]. As remarked earlier, the  $w^*$ -topologies on both  $B(G)$  and  $\mathcal{M}_{cb}(A(G))$  coincide on bounded sets with the relative topology induced by  $\sigma(L^\infty(G), L^1(G))$ , so that the inclusion  $B(G) \subset \mathcal{M}_{cb}(A(G))$  is  $w^*$ -continuous by the Krein–Šmulian theorem. From (the quantized analog of)

[Ru2, Proposition 4.2], it then follows that  $\mathcal{M}_{\text{cb}}(A(G))$  is also operator Connes-amenable. ■

**Example** By [Dav, Proposition VII.6.1],  $C^*(\mathbb{F}_2)$  is residually finite-dimensional, so that  $B(\mathbb{F}_2)$  and  $\mathcal{M}_{\text{cb}}(A(\mathbb{F}_2))$  are both operator Connes-amenable, even though  $\mathbb{F}_2$  fails to be amenable.

If  $G$  is an amenable, locally compact group, then  $A(G)$  embeds completely isometrically into  $\mathcal{M}_{\text{cb}}(A(G)) = B(G)$ . If  $G$  is not amenable, however,  $A(G)$  is not closed in  $\mathcal{M}_{\text{cb}}(A(G))$ . We denote the closure of  $A(G)$  in  $\mathcal{M}_{\text{cb}}(A(G))$  by  $A_{\text{cb}}(G)$ .

We collect a few basic properties of  $A_{\text{cb}}(G)$ .

**Proposition 2.2** *Let  $G$  be a locally compact group. Then  $A_{\text{cb}}(G)$  is a regular, commutative, Tauberian (quantized) Banach algebra whose character space is canonically identified with  $G$ .*

**Proof** It is clear that  $A_{\text{cb}}(G)$  is commutative and semisimple, and by [F1, Lemma 1], the character space of  $A_{\text{cb}}(G)$  is  $G$  in the canonical way. Let  $F \subset G$  be closed, and let  $x \in G \setminus F$ . Since  $A(G)$  is regular (see [E]), there is  $f \in A(G) \subset A_{\text{cb}}(G)$  such that  $f|_F \equiv 0$  and  $f(x) = 1$ . Hence,  $A_{\text{cb}}(G)$  is also regular.

To see that  $A_{\text{cb}}(G)$  is Tauberian, let  $f \in A_{\text{cb}}(G)$ , and let  $\epsilon > 0$ . Since  $A(G)$  is dense in  $A_{\text{cb}}(G)$ , there is  $g \in A(G)$  with  $\|f - g\|_{A_{\text{cb}}(G)} < \frac{\epsilon}{2}$ , and since  $A(G)$  is Tauberian, there is  $h \in A(G)$  with  $\text{supp}(h)$  compact and

$$\|g - h\|_{A_{\text{cb}}(G)} \leq \|g - h\|_{A(G)} < \frac{\epsilon}{2}.$$

It follows that  $\|f - h\| < \epsilon$ . Since  $\epsilon > 0$  is arbitrary, this means that  $A_{\text{cb}}(G)$  is Tauberian. ■

Leptin's theorem [Le], motivates the adverb “weakly” in the following definition.

**Definition 2.3** A locally compact group is said to be *weakly amenable* if  $A_{\text{cb}}(G)$  has a bounded approximate identity.

### Remarks

1. This definition of a weakly amenable, locally compact group is not quite the original one (*cf.* [CH]), but is easily seen to be equivalent [F2, Proposition 1].

2. Weakly amenable groups have the approximation property of [HK], whereas the converse is false.

3. In [BCD], a notion of weak amenability for Banach algebras was introduced. This Banach algebraic amenability, which can easily be adapted to the quantized setting, is related to Definition 2.3 only in the sense that both weak amenabilities are weaker than the notions of amenability for Banach algebras and locally compact groups, respectively. There are no analogs of [J1, Theorem 2.5] or [R, Theorem 3.6]:

$L^1(G)$  is weakly amenable [J2] and  $A(G)$  is operator weakly amenable [S1] for every locally compact group  $G$ .

**Examples**

1. By [Le], every amenable, locally compact group is weakly amenable.
2. Even though  $\mathbb{F}_2$  is not amenable, it is weakly amenable [dCH, Corollary 3.9].
3. It is shown in [D1] that  $SL(2, \mathbb{R}) \rtimes \mathbb{R}^N$  is not weakly amenable for  $N \geq 2$ . In [D2], this is used to show that every simple Lie group with real rank greater than or equal to two fails to be weakly amenable.

It is clear from [R, Theorem 3.6], combined with elementary hereditary properties of operator amenability, that  $A_{cb}(G)$  is operator amenable for every amenable, locally compact group. In the remainder of this section, we shall see that the converse need not be true.

We first present three lemmas. Let  $\mathfrak{A}$  be a Banach algebra, and recall that a Banach  $\mathfrak{A}$ -bimodule is called *pseudo-unital* (or *neo-unital*) if  $E = \{a \cdot x \cdot b : a, b \in \mathfrak{A}, x \in E\}$ .

**Lemma 2.4** *Let  $\mathfrak{A}$  be a quantized Banach algebra with a bounded approximate identity. Then  $\mathfrak{A}$  is operator amenable if and only if, for each pseudo-unital, quantized Banach  $\mathfrak{A}$ -bimodule  $E$ , every completely bounded derivation  $D: \mathfrak{A} \rightarrow E^*$  is inner.*

**Proof** The proof for the classical case [Ru1, Proposition 2.1.5] carries over nearly verbatim. ■

Let  $\mathfrak{A}$  be a Banach algebra, and let  $I$  be a closed ideal of  $\mathfrak{A}$ . The *I-strict* topology on  $\mathfrak{A}$  is the locally convex topology induced by the seminorms

$$\mathfrak{A} \rightarrow [0, \infty), \quad a \mapsto \|ax\| + \|xa\| \quad (x \in I).$$

(Note that this topology need not be Hausdorff.)

**Lemma 2.5** *Let  $\mathfrak{A}$  be a quantized Banach algebra, let  $I$  be a closed ideal of  $\mathfrak{A}$  with a bounded approximate identity, let  $E$  be a pseudo-unital, quantized Banach  $I$ -bimodule, and let  $D: I \rightarrow E^*$  be a completely bounded derivation. Then  $E$  is a quantized Banach  $\mathfrak{A}$ -bimodule in a canonical fashion, and there is a completely bounded derivation  $\tilde{D}: \mathfrak{A} \rightarrow E^*$  extending  $D$  which is continuous with respect to the  $I$ -strict topology on  $\mathfrak{A}$  and the  $w^*$ -topology on  $E^*$ .*

**Proof** By [Ru1, Proposition 2.1.6], the module action of  $I$  on  $E$  extends canonically to  $\mathfrak{A}$ , and  $D$  has a bounded extension  $\tilde{D}: \mathfrak{A} \rightarrow E^*$  which is continuous with respect to the  $I$ -strict topology on  $\mathfrak{A}$  and the  $w^*$ -topology on  $E^*$ . (Since  $I$  is dense in  $\mathfrak{A}$  in the  $I$ -strict topology,  $\tilde{D}$  is uniquely determined by its continuity properties.)

Two claims remain to be checked: that  $E$  is indeed a *quantized* Banach  $\mathfrak{A}$ -bimodule, and that  $\tilde{D}$  is *completely* bounded.

We first verify that  $E$  is a quantized Banach  $\mathfrak{A}$ -bimodule. Let  $(e_\alpha)_\alpha$  be an approximate identity for  $I$  bounded by  $C \geq 0$ . Note that since  $E$  is a pseudo-unital Banach  $\mathfrak{A}$ -bimodule, we have

$$\lim_\alpha [e_\alpha \cdot x_{j,k}] = [x_{j,k}] = \lim_\alpha [x_{j,k} \cdot e_\alpha] \quad (n \in \mathbb{N}, [x_{j,k}] \in M_n(E)).$$

Let  $\kappa \geq 0$  be the cb-norm of the completely bounded bilinear map  $I \times E \ni (b, x) \mapsto b \cdot x$ , and fix  $n \in \mathbb{N}$ . Let  $[a_{j,k}] \in M_n(\mathfrak{A})$  and let  $[x_{\nu,\mu}] \in M_n(E)$ . It follows that

$$\begin{aligned} \|[a_{j,k} \cdot \xi_{\nu,\mu}]\|_{M_{n^2}(E)} &= \lim_\alpha \|[a_{j,k} \cdot (e_\alpha \cdot x_{\nu,\mu})]\|_{M_{n^2}(E)} \\ &= \lim_\alpha \|[a_{j,k} e_\alpha \cdot x_{\nu,\mu}]\|_{M_{n^2}(E)} \\ &\leq \kappa \limsup_\alpha \|[a_{j,k} e_\alpha]\|_{M_n(I)} \|[x_{\nu,\mu}]\|_{M_n(E)} \\ &\leq \kappa C \|[a_{j,k}]\|_{M_n(\mathfrak{A})} \|[x_{\nu,\mu}]\|_{M_n(E)}, \end{aligned}$$

so that the extended module left module action  $\mathfrak{A} \times E \ni (a, x) \mapsto a \cdot x$  is completely bounded (by  $\kappa C$ ). Similarly, one sees that  $E \times \mathfrak{A} \ni (x, a) \mapsto x \cdot a$  is completely bounded. Consequently,  $E$  is indeed a quantized Banach  $\mathfrak{A}$ -bimodule.

Next, we turn to showing that the extension  $\tilde{D}: \mathfrak{A} \rightarrow E^*$  from [Ru1, Proposition 2.1.6] is not only bounded, but completely bounded. Let  $a \in \mathfrak{A}$ , let  $x \in E$ , and let  $b, c \in I$ , and note that

$$\begin{aligned} \langle b \cdot x \cdot c, \tilde{D}a \rangle &= \lim_\alpha \langle e_\alpha b \cdot x \cdot c, \tilde{D}a \rangle = \lim_\alpha \langle b \cdot x \cdot c, (\tilde{D}a) \cdot e_\alpha \rangle \\ &= \lim_\alpha \langle b \cdot x \cdot c, D(ae_\alpha) - a \cdot D(e_\alpha) \rangle. \end{aligned}$$

Since  $E$  is pseudo-unital, this means that

$$\tilde{D}a = \sigma(E^*, E\text{-}\lim_\alpha (D(ae_\alpha) - a \cdot D(e_\alpha))) \quad (a \in \mathfrak{A})$$

and, consequently,

$$\tilde{D}^{(n)}([a_{j,k}]) = \sigma(M_n(E^*), T_n(E)\text{-}\lim_\alpha (D([a_{j,k}e_\alpha]) - [a_{j,k} \cdot D(e_\alpha)]),$$

for  $n \in \mathbb{N}$ ,  $[a_{j,k}] \in M_n(\mathfrak{A})$ , where  $\tilde{D}^{(n)}: M_n(\mathfrak{A}) \rightarrow M_n(E^*)$  denotes the  $n$ -th amplification of  $\tilde{D}$  for  $n \in \mathbb{N}$ . To see that  $\tilde{D}$  is completely bounded, let  $n \in \mathbb{N}$  and  $[a_{j,k}] \in M_n(\mathfrak{A})$ , and note that by the foregoing,

$$\begin{aligned} \|\tilde{D}^{(n)}([a_{j,k}])\|_{M_n(E^*)} &\leq \limsup_\alpha \|(D([a_{j,k}e_\alpha]) - [a_{j,k} \cdot D(e_\alpha)])\|_{M_n(E^*)} \\ &\leq \limsup_\alpha (\|D\|_{cb} \| [a_{j,k}] \|_{M_n(\mathfrak{A})} \|e_\alpha\| + \tilde{\kappa} \| [a_{j,k}] \|_{M_n(\mathfrak{A})} \|D\| \|e_\alpha\|) \\ &\leq (C \|D\|_{cb} + \tilde{\kappa} C \|D\|) \| [a_{j,k}] \|_{M_n(\mathfrak{A})}, \end{aligned}$$

where  $\tilde{\kappa}$  is the cb-norm of the left module action  $\mathfrak{A} \times E^* \ni (a, \phi) \mapsto a \cdot \phi$ . Hence,  $\tilde{D}$  is indeed completely bounded (with  $\|\tilde{D}\|_{cb} \leq C \|D\|_{cb} + \tilde{\kappa} C \|D\|$ ). ■

Our final lemma is the following.

**Lemma 2.6** *Let  $G$  be a discrete group. Then the following topologies coincide on norm bounded subsets of  $\mathcal{M}_{cb}(A(G))$ :*

- (i) *the  $w^*$ -topology,*
- (ii) *the topology of pointwise convergence on  $G$ ,*
- (iii) *the  $A_{cb}(G)$ -strict topology.*

**Proof** That (i) and (ii) coincide on norm bounded subsets follows from [dCH, Lemma 1.9], and the fact that  $A_{cb}(G)$  is Tauberian yields the corresponding statement for (ii) and (iii). ■

We can now state and prove the main result of this section.

**Theorem 2.7** *Let  $G$  be a weakly amenable discrete group such that  $C^*(G)$  is residually finite-dimensional. Then  $A_{cb}(G)$  is operator amenable.*

**Proof** Let  $E$  be a quantized Banach  $A_{cb}(G)$ -bimodule, and let  $D: A_{cb}(G) \rightarrow E^*$  be a completely bounded derivation. Since  $G$  is weakly amenable, *i.e.*,  $A_{cb}(G)$  has a bounded approximate identity, we may invoke Lemma 2.4 and suppose without loss of generality that  $E$  is pseudo-unital. By Lemma 2.5,  $E$  is a quantized Banach  $\mathcal{M}_{cb}(A(G))$ -bimodule in a canonical way, and there is a completely bounded derivation  $\tilde{D}: \mathcal{M}_{cb}(A(G)) \rightarrow E^*$  that extends  $D$  and is continuous with respect to the  $A_{cb}(G)$ -strict topology on  $\mathcal{M}_{cb}(A(G))$  and the  $w^*$ -topology on  $E^*$ .

Due to Lemma 2.6, an argument as in the proof of [Ru4, Theorem 3.5] yields that the dual, quantized Banach  $\mathcal{M}_{cb}(A(G))$ -module  $E^*$  is actually normal and that  $\tilde{D}$  is  $w^*$ - $w^*$ -continuous.

Since  $C^*(G)$  is residually finite-dimensional and since  $G$  has the approximation property,  $\mathcal{M}_{cb}(A(G))$  is operator Connes-amenable by Proposition 2.1. Consequently,  $\tilde{D}$ , and therefore  $D$ , is inner. ■

With Theorem 2.7 proved, it is not hard to come up with examples of locally compact groups  $G$  that fail to be amenable, but for which  $A_{cb}(G)$  is nevertheless operator amenable.

**Example** Since  $\mathbb{F}_2$  is weakly amenable and  $C^*(\mathbb{F}_2)$  is residually finite-dimensional,  $A_{cb}(\mathbb{F}_2)$  is operator amenable by Theorem 2.7.

Even though we have exhibited non-amenable (discrete) groups  $G$  for which  $A_{cb}(G)$  is operator amenable, we are still far from a characterization of those locally compact groups  $G$  such that  $A_{cb}(G)$  is operator amenable. It may be that  $A_{cb}(G)$  is operator amenable whenever  $G$  is weakly amenable.

As in [ER],  $\hat{\otimes}$  stands for the projective tensor product of operator spaces. If  $\mathfrak{A}$  is a quantized Banach algebra,  $\mathfrak{A} \hat{\otimes} \mathfrak{A}$  becomes a quantized Banach  $\mathfrak{A}$ -bimodule via

$$a \cdot (x \otimes y) := ax \otimes y \quad \text{and} \quad (x \otimes y) \cdot a := x \otimes ya \quad (a, x, y \in \mathfrak{A}),$$

so that the multiplication operator

$$\Delta: \mathfrak{A} \widehat{\otimes} \mathfrak{A} \rightarrow \mathfrak{A}, \quad a \otimes b \mapsto ab$$

becomes a completely bounded homomorphism of  $\mathfrak{A}$ -bimodules.

The following definition arises naturally in A. Ya. Helemskiĭ's topological homology ([H1]), or rather in its quantized version (see [A, H2], for example).

**Definition 2.8** A quantized Banach algebra  $\mathfrak{A}$  is called *operator biprojective* if the multiplication operator  $\Delta: \mathfrak{A} \widehat{\otimes} \mathfrak{A} \rightarrow \mathfrak{A}$  has a completely bounded right inverse which is also a homomorphism of  $\mathfrak{A}$ -bimodules.

**Example** Let  $G$  be a locally compact group. As was shown independently by O. Yu. Aristov [A] and P. J. Wood [W2],  $A(G)$  is operator amenable if and only if  $G$  is discrete.

Let  $G$  be any locally compact group such that  $A_{cb}(G)$  is operator biprojective. Then  $G$  has to be discrete by (the quantized analogue of) [Da, Corollary 2.8.42]. It is possible that the converse implication holds as well.

Concluding this section, we shall see that  $A_{cb}(G)$  is operator biprojective at least for those groups  $G$  that satisfy the hypotheses of Theorem 2.7.

The key is the following lemma, whose straightforward proof we omit.

**Lemma 2.9** Let  $E_1, E_2, F_1, F_2$  be operator spaces, and let  $T_j \in \mathcal{CB}(E_j, F_j)$  be norm limits of finite rank operators for  $j = 1, 2$ . Then  $T_1 \otimes T_2 \in \mathcal{CB}(E_1 \widehat{\otimes} E_2, F_1 \widehat{\otimes} F_2)$  is a norm limit of finite rank operators and thus compact.

**Proposition 2.10** Let  $\mathfrak{A}$  be a commutative, semisimple, Tauberian quantized Banach algebra with discrete character space and a bounded approximate identity. Then  $\mathfrak{A}$  is operator biprojective if and only if  $\mathfrak{A}$  is operator amenable.

**Proof** Any operator biprojective quantized Banach algebra with a bounded approximate identity is operator amenable. Hence, only the “if” part needs proof.

Suppose that  $\mathfrak{A}$  is operator amenable. By [R, Proposition 2.4], it has an *approximate diagonal*, i.e., a bounded net  $(\mathbf{m}_\alpha)_{\alpha \in \Lambda}$  such that

$$a \cdot \mathbf{m}_\alpha - \mathbf{m}_\alpha \cdot a \rightarrow 0 \quad (a \in \mathfrak{A})$$

and

$$a \Delta \mathbf{m}_\alpha \rightarrow a \quad (a \in \mathfrak{A}).$$

For  $a \in \mathfrak{A}$ , let  $L_a, R_a \in \mathcal{CB}(\mathfrak{A})$  denote the operator of left and right multiplication by  $a$ , respectively. Since  $\mathfrak{A}$  is semisimple and Tauberian and has a discrete character space,  $L_a$  and  $R_a$  are norm limits of finite rank operators for each  $a \in \mathfrak{A}$ . Let  $\mathcal{U}$  be

an ultrafilter on  $\mathbb{A}$  dominating the order filter, and let  $a \in \mathfrak{A}$ . By Cohen’s factorization theorem [Da, Theorem 2.9.24], there are  $b, c \in \mathfrak{A}$  such that  $a = bc$ , and by Lemma 2.9,  $L_b \otimes R_c \in \mathcal{CB}(\mathfrak{A} \widehat{\otimes} \mathfrak{A})$  is compact. It follows that

$$\lim_{\mathcal{U}} a \cdot \mathbf{m}_\alpha = \lim_{\mathcal{U}} b \cdot \mathbf{m}_\alpha \cdot c = \lim_{\mathcal{U}} (L_b \otimes R_c) \mathbf{m}_\alpha$$

exists. Define

$$\rho: \mathfrak{A} \rightarrow \mathfrak{A} \widehat{\otimes} \mathfrak{A}, \quad a \mapsto \lim_{\mathcal{U}} a \cdot \mathbf{m}_\alpha.$$

Then  $\rho$  is completely bounded, and easily seen to be an  $\mathfrak{A}$ -bimodule homomorphism and a right inverse of  $\Delta$ . ■

**Remark** The proof of the non-obvious direction of Proposition 2.10 is very similar to that of [LRRW, Corollary 3.2]. However, we do not know if a straightforward quantization of [LRRW, Corollary 3.2] is possible. Unlike for the projective tensor product of Banach spaces, we do not know whether the tensor product of two compact, completely bounded maps between operator spaces is a compact map between the corresponding projective tensor products (of operator spaces).

In view of Proposition 2.1 and Theorem 2.7, we obtain the following.

**Corollary 2.11** *Let  $G$  be a weakly amenable discrete group such that  $C^*(G)$  is residually finite-dimensional. Then  $A_{\text{cb}}(G)$  is operator biprojective.*

### 3 Complementation of Ideals in $A(G)$ and $A_{\text{cb}}(G)$ : An Application

In this section, we will consider (complete) complementation properties of ideals in  $A(G)$  and  $A_{\text{cb}}(G)$ , where  $G$  is a discrete group such that  $A_{\text{cb}}(G)$  is operator amenable.

Let  $G$  be a locally compact (mostly discrete) group, and let  $F \subset G$  be closed. We set

$$I(F) := \{f \in A(G) : f|_F \equiv 0\} \quad \text{and} \quad I_{\text{cb}}(F) := \{f \in A_{\text{cb}}(G) : f|_F \equiv 0\}.$$

(Since  $A(G)$  and  $A_{\text{cb}}(G)$  have the same character space, we use different symbols when dealing with  $A(G)$  and  $A_{\text{cb}}(G)$ , respectively, in order to avoid confusion.) Similarly, we define

$$J(F) := \overline{\{f \in A(G) : \text{supp}(f) \text{ is compact, and } F \cap \text{supp}(f) = \emptyset\}} \cdot \|_{A(G)}$$

and

$$J_{\text{cb}}(F) := \overline{\{f \in A(G) : \text{supp}(f) \text{ is compact, and } F \cap \text{supp}(f) = \emptyset\}} \cdot \|_{A_{\text{cb}}(G)}.$$

We say that  $F$  is a *set of synthesis* for  $A(G)$  or  $A_{\text{cb}}(G)$ , respectively, if  $J(F) = I(F)$  or  $J_{\text{cb}}(F) = I_{\text{cb}}(F)$ , respectively.

We begin with a useful observation.

**Proposition 3.1** *Let  $G$  be a weakly amenable locally compact group, and let  $F \subset G$  be a set of synthesis for  $A(G)$ . Then  $F$  is a set of synthesis for  $A_{cb}(G)$ .*

**Proof** We first claim that  $I(F)$  is dense in  $I_{cb}(F)$ . To see this, let  $(e_\alpha)_{\alpha \in \mathbb{A}}$  be a bounded approximate identity for  $A_{cb}(G)$  contained in  $A(G)$ , and let  $f \in I_{cb}(F)$ , so that  $f = \lim_\alpha f e_\alpha$ . Since  $f e_\alpha \in A(G) \cap I_{cb}(F) = I(F)$  for each  $\alpha \in \mathbb{A}$ , this proves the claim.

Since  $F$  is a set of spectral synthesis for  $A(G)$ , it follows that  $I(F) \subset J_{cb}(F)$ , so that  $I_{cb}(F) = J_{cb}(F)$ . ■

**Corollary 3.2** *Let  $G$  be a discrete and weakly amenable group. Then every subset of  $G$  is a set of synthesis for  $A_{cb}(G)$ .*

**Proof** This follows immediately from the Proposition 3.1 and [KL, Proposition 2.2]. ■

Let  $E$  be an operator space, and let  $F$  be a closed subspace of  $E$ . We say that  $F$  is *completely complemented* in  $E$  if there exists a completely bounded projection  $P$  from  $E$  onto  $F$ , and we say that  $F$  is *completely weakly complemented* in  $E$  if there exists a completely bounded projection from  $E^*$  onto  $F^\perp$ . As in the classical situation [Ru1, Theorem 2.3.7], a closed ideal in an operator amenable, quantized Banach algebra is operator amenable if and only if it is weakly complemented and if and only if it has a bounded approximate identity (see [RS, Lemma 1.6]).

The following proposition adds two more equivalent statements in the case where the quantized Banach algebra is of the form  $A_{cb}(G)$  for a discrete group  $G$ .

**Proposition 3.3** *Let  $G$  be a discrete group such that  $A_{cb}(G)$  is operator amenable, and let  $I$  be a closed ideal of  $A_{cb}(G)$ . Then the following are equivalent:*

- (i)  $I$  is completely complemented;
- (ii)  $I$  is completely weakly complemented;
- (iii) there is  $F \subset G$  with  $1_F \in \mathcal{M}_{cb}(A(G))$  such that  $I = I_{cb}(F)$ ;
- (iv)  $I$  has an approximate identity bounded in the cb-multiplier norm;
- (v)  $I$  is operator amenable.

**Proof** As already stated, (ii)  $\Leftrightarrow$  (iv)  $\Leftrightarrow$  (v) are well known (and hold for any closed ideal in a quantized Banach algebra). Furthermore, (i)  $\Rightarrow$  (ii) is trivial.

(iv)  $\Rightarrow$  (iii): Let  $F$  be the hull of  $I$ , i.e.,  $F := \{x \in G : f(x) = 0 \text{ for all } f \in I\}$ . By Corollary 3.2, we have  $I = I_{cb}(F)$ . Let  $(e_\alpha)_\alpha$  be a bounded approximate identity for  $I$ . Since  $\mathcal{M}_{cb}(A(G))$  is a dual space, we can suppose that  $(e_\alpha)_\alpha$  converges in the  $w^*$ -topology to some  $f \in \mathcal{M}_{cb}(A(G))$ . Since  $w^*$ -convergence in  $\mathcal{M}_{cb}(A(G))$  entails pointwise convergence on  $G$ , it follows that  $f = 1_{G \setminus F}$ , so that  $1_F = 1 - 1_{G \setminus F} \in \mathcal{M}_{cb}(G)$ .

(iii)  $\Rightarrow$  (i): Since  $1_F \in \mathcal{M}_{cb}(A(G))$ , the map

$$A_{cb}(G) \rightarrow A_{cb}(G), \quad f \mapsto 1_{G \setminus F} f$$

is a completely bounded projection onto  $I$ . ■

**Remark** The first four equivalences of Proposition 3.3 can be viewed as extensions of the main results of [W1], which were primarily about the Fourier algebra of an amenable group. The proof of Proposition 3.3 is very similar to the corresponding arguments in [W1].

A somewhat more surprising result is that, under the same hypotheses as in Proposition 3.3, we can obtain the equivalence of (i) to (iv) for closed ideals of the Fourier algebra with its original norm (Corollary 3.5).

The crucial implication is the following.

**Theorem 3.4** *Let  $G$  be a discrete group such that  $A_{cb}(G)$  is operator amenable, and let  $I$  be a weakly completely complemented closed ideal of  $A(G)$ . Then there is  $F \subset G$  with  $1_F \in \mathcal{M}_{cb}(G)$  such that  $I = I(F)$ .*

**Proof** Since  $G$  is discrete and weakly amenable, [KL, Proposition 2.2] yields  $F \subset G$  such that  $I = I(F)$ . It remains to be shown that  $1_F \in \mathcal{M}_{cb}(G)$ .

Since  $I$  is an ideal of  $A(G)$ , it is a weakly completely complemented  $A_{cb}(G)$ -submodule of the (symmetric) quantized Banach  $A_{cb}(G)$ -module  $A(G)$ . By definition,  $I^\perp$  thus is a completely complemented, closed  $A_{cb}(G)$ -submodule of the dual  $A_{cb}(G)$ -module  $VN(G)$ . Since  $A_{cb}(G)$  is operator amenable, [W1, Theorem 1] implies that  $I^\perp$  is completely invariantly complemented, *i.e.*, there is a completely bounded projection  $P: VN(G) \rightarrow I^\perp$  which is an  $A_{cb}(G)$ -module homomorphism.

Define  $Q: A(G)^{**} \rightarrow A(G)^{**}$  as the complementary projection of  $P^*$ , *i.e.*,  $Q := \text{id}_{A(G)^{**}} - P^*$ . Then  $Q$  is a completely bounded projection from  $A(G)^{**}$  onto  $(I^\perp)^\perp = I^{**}$ , and is an  $A_{cb}(G)$ -module homomorphism.

Let  $x \in G$ , so that  $1_{\{x\}} \in A(G)$ , and note that

$$Q(1_{\{x\}}) = Q(1_{\{x\}}^2) = 1_{\{x\}} \cdot Q(1_{\{x\}}).$$

Since  $G$  is discrete,  $A(G)$  is an ideal in  $A(G)^{**}$ , so that  $Q(1_{\{x\}}) \in A(G)$ . Since  $A(G)$  is Tauberian, it follows that  $Q(A(G)) \subset A(G)$ .

All in all,  $Q$  is completely bounded, maps  $A(G)$  into itself, and is an  $A_{cb}(G)$ -module homomorphism. It follows that  $Q|_{A(G)}$  is a completely bounded multiplier of  $A(G)$ , *i.e.*, there is  $g \in \mathcal{M}_{cb}(A(G))$  such that  $Qf = gf$  for all  $f \in A(G)$ . Finally, as  $Q$  is a projection onto  $I^{**}$ , it is clear that  $g = 1_{G \setminus F}$ , so that  $1_F \in \mathcal{M}_{cb}(A(G))$ . ■

**Corollary 3.5** *Let  $G$  be a discrete group such that  $A_{cb}(G)$  is operator amenable, and let  $I$  be a closed ideal of  $A(G)$ . Then the following are equivalent:*

- (i)  $I$  is completely complemented;
- (ii)  $I$  is completely weakly complemented;
- (iii) there is  $F \subset G$  with  $1_F \in \mathcal{M}_{cb}(A(G))$  such that  $I = I(F)$ ;
- (iv)  $I$  has an approximate identity bounded in the  $cb$ -multiplier norm.

**Proof** (i)  $\Rightarrow$  (ii) is trivial, (ii)  $\Rightarrow$  (iii) follows from Theorem 3.4, and (iii)  $\Rightarrow$  (i) follows as in the proof of Proposition 3.3.

(iii)  $\Rightarrow$  (iv): Since  $A_{cb}(G)$  is operator amenable, it has a bounded approximate identity, so that  $A(G)$  has an approximate identity, say  $(e_\alpha)_\alpha$ , that is bounded in  $A_{cb}(G)$  [F2, Proposition 1]. Then  $(1_{G \setminus F} e_\alpha)_\alpha$  is the desired approximate identity.

(iv)  $\Rightarrow$  (iii): This is proven similarly to the corresponding implication of Proposition 3.3. ■

**Remark** The equivalence of Corollary 3.5(i) and (iii) was proved by Wood, first for amenable discrete groups [W1] and later for all discrete groups [W2]. Wood's techniques, however, do not allow proving the equivalence of (i) and (ii) or of (i) and (iii) with (iv) without the stronger hypothesis that  $G$  be amenable.

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