Canad. Math. Bull. Vol. **42** (4), 1999 pp. 463–477

A Generalized Characterization of Commutators of Parabolic Singular Integrals

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Abstract. Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\delta_\lambda x = (\lambda^{\alpha_1} x_1, \dots, \lambda^{\alpha_n} x_n)$, where $\lambda > 0$ and $1 \le \alpha_1 \le \cdots \le \alpha_n$. Denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We characterize those functions $A(x)$ for which the parabolic Calderón commutator

$$T_A f(x) \equiv \text{p.v.} \int_{\mathbb{R}^n} K(x-y)[A(x) - A(y)] f(y)\, dy$$

is bounded on $L^2(\mathbb{R}^n)$, where $K(\delta_\lambda x) = \lambda^{-|\alpha|-1} K(x)$, K is smooth away from the origin and satisfies a certain cancellation property.

0 Introduction

Multilinear singular integrals with parabolic homogeneity arise naturally in the theory of parabolic PDE (this is analogous to the elliptic case; see, for example [8] and [6]). The prototypical example in the elliptic case is the first Calderón commutator defined by

$$T_A f(x) = \text{p.v.} \int_{\mathbb{R}} \frac{A(x) - A(y)}{(x-y)^2} f(y)\, dy.$$

It was proved by A. P. Calderón in [1] that T_A is bounded on $L^2(\mathbb{R})$ if and only if A satisfies a Lipschitz condition.

Recently, the first author of this paper generalized this result to the case of the parabolic dilations:

$$x \longmapsto \delta_\lambda x = (\lambda x_1, \dots, \lambda x_{n-1}, \lambda^2 x_n), \quad \lambda > 0,$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$; more precisely, the author in [13] characterized those A such that the operator defined by

$$T_A f(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y)\big(A(x) - A(y)\big) f(y)\, dy \tag{0.1}$$

is bounded on $L^2(\mathbb{R}^n)$, where $K(\lambda x_1, \dots, \lambda x_{n-1}, \lambda^2 x_n) = \lambda^{-n-2} K(x_1, \dots, x_n)$ and K is smooth away from the origin and satisfies a certain cancellation condition (see (2.7) in [13], see also (1.15)). One of the typical examples of such K is given by

$$K(x_1, \dots, x_n) = x_n^{-1-n/2} \exp\Big(-\frac{|x_1|^2 + \cdots + |x_{n-1}|^2}{4x_n}\Big) \chi_{\{x_n > 0\}}. \tag{0.2}$$

Received by the editors February 5, 1996; revised September 29, 1998.
AMS subject classification: 42B20.
Keywords: parabolic singular integral, commutator, parabolic BMO sobolev space, homogeneous space, T1-theorem, symbol.

For this K, if we consider $x' = (x_1, \dots, x_{n-1})$ and x_n respectively as space variables and time variable, then the corresponding operator $T_A f$, defined as in (0.1), is the first order commutator which arises in the series expansion of the caloric double layer potential on a domain with a time dependent boundary $\big(x', x_n, A(x', x_n)\big) \in \mathbb{R}^{n+1}$, see [14]. In fact, $T_A = [H^{1/2}, A]$, where $H \equiv \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2} - \frac{\partial}{\partial x_n}$ is the heat operator on $\mathbb{R}^n$, see [13].

It turns out that, to obtain L^2 boundedness of the operator (0.1), it suffices that A satisfy a kind of "Lipschitz" condition, and furthermore, that A is expressed as the (parabolic) fractional integral operator of order 1 acting on a (parabolic) BMO function. And this condition is sharp in the sense that it is necessary for certain special cases, in particular for K as in (0.2) (see [13]).

In this note, we generalize the result of [13] to the space $\mathbb{R}^n$ endowed with arbitrary parabolic dilations (see (1.1)). We are motivated to treat this problem by our desire to develop a theory of singular integral operators applicable to the study of higher order parabolic operators such as $\Delta^2 - \frac{\partial}{\partial t}$, (corresponding to the case $1 = \alpha_1 = \cdots = \alpha_{n-1}$, $\alpha_n = 4$ of (1.1)), which we hope to treat in future papers. One might wonder whether the results of the present paper remain true in the more general setting of a space of homogeneous type, but at the present time, our problem is not well posed in that situation—see our comments following (1.9) below.

In the present paper, we obtain a necessary and sufficient condition for the L^2 boundedness of the Calderon commutator in the setting of generalized parabolic dilations. The said condition, consisting of two parts, comprises the hypotheses of our sufficiency result, Theorem 1 below. The necessity of this two part condition is the content of Theorems 2 and 3 below. The proof of Theorem 1 is a rather direct application of the non-isotropic T1 Theorem. For higher order multi-linear singular integrals of parabolic type, there is a problem of "roughness" of the kernel which arises due to the lack of a true product rule for fractional order derivatives. For such operators, the use of the T1 Theorem to establish L^2 bounds is less routine. We do not discuss the higher order case here, but we plan to treat it in future.

1 Notation, Statements and Preliminaries

We equip $\mathbb{R}^n$ with the group of parabolic dilations

$$\delta_\lambda x = (\lambda^{\alpha_1} x_1, \dots, \lambda^{\alpha_n} x_n), \quad \lambda > 0, \tag{1.1}$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and α_j's are real satisfying $1 \le \alpha_1 \le \cdots \le \alpha_n$. Denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We also denote $\alpha_0 = 0$ and $\alpha_{n+1} = \infty$ for convenience. For $\beta = (\beta_1, \dots, \beta_n)$ and $\gamma = (\gamma_1, \dots, \gamma_n)$ with β_j, γ_j reals, we denote $\beta \cdot \gamma = \sum_{j=1}^n \beta_j \gamma_j$, and $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ and $\frac{\partial^\beta}{\partial x^\beta} = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}}$ where β_j's are non-negative integers.

Without loss of generality, we can always assume $\alpha_1 = 1$. Let n_0 be the smallest integer such that $\alpha_{n_0} > 1$. Then $n_0 > 1$, and the case $n_0 = n + 1$ corresponds to the classical case $(\alpha_1, \dots, \alpha_n) = (1, \dots, 1)$.

Let $\rho(x)$ be a non-isotropic norm on $\mathbb{R}^n$ defined as the unique positive solution of the

https://doi.org/10.4153/CMB-1999-054-0 Published online by Cambridge University Press

following identity (see [9]):

$$1 = \sum_{j=1}^{n} \frac{x_j^2}{\rho(x)^{2\alpha_j}}. \tag{1.2}$$

It is immediate that $\rho(\delta_\lambda x) = \lambda\rho(x)$ and $\rho(x) \simeq \sum_{j=1}^n |x_j|^{1/\alpha_j}$, for all $\lambda > 0$ and $x \in \mathbb{R}^n$. With this norm, $\mathbb{R}^n$ is a space of homogeneous type in the sense of Coifman and Weiss [7] with homogeneous dimension $d = |\alpha|$. In particular, there is a constant $c_0 \geq 1$ such that $\rho(x+y) \leq c_0\big(\rho(x) + \rho(y)\big)$ for all $x, y \in \mathbb{R}^n$. We say a monomial $x_1^{\beta_1} \cdots x_n^{\beta_n}$ (β_j's are non-negative integers) is homogeneous of degree m if $\alpha_1\beta_1 + \cdots + \alpha_n\beta_n = m$, and the homogeneous degree of a polynomial is defined by the highest homogeneous degree of its monomials with non-zero coefficients.

One has the polar decomposition

$$x = \delta_r \sigma \tag{1.3}$$

with $\sigma \in S^{n-1}$, $r = \rho(x)$ and $dx = r^{d-1}\, dr\, J(\sigma)\, d\sigma$, where $J(\sigma)$ is a smooth and non-negative function of $\sigma \in S^{n-1}$ and is even in each of $\sigma_1, \ldots, \sigma_n$ separately.

We define the parabolic fractional integral of order 1 for suitable f by

$$(I_1 f)^\wedge(\xi) = \rho(\xi)^{-1} \hat{f}(\xi), \quad \xi \in \mathbb{R}^n, \tag{1.4}$$

and its inverse by

$$(P_1 f)^\wedge(\xi) = \rho(\xi) \hat{f}(\xi), \quad \xi \in \mathbb{R}^n. \tag{1.5}$$

By a parabolic cube $Q \subseteq \mathbb{R}^n$, we mean a set of the form

$$Q = I_1 \times I_2 \times \cdots \times I_n, \tag{1.6}$$

where $I_j \subseteq \mathbb{R}^1$ is an interval with the length r^{α_j}. The ball of radius r and center x_0 is defined by

$$B_r(x_0) = \{x_0 \in \mathbb{R}^n : \rho(x - x_0) < r\}. \tag{1.7}$$

And the parabolic BMO space is defined as the collection of all locally $L^1(\mathbb{R}^n)$ functions modulo constants with norm

$$\|b\|_{\mathrm{BMO}} \equiv \sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q|\, dx, \tag{1.8}$$

where Q is as in (1.7), $b_Q \equiv \frac{1}{|Q|}\int_Q b(y)\, dy$, $|Q|$ is the Lebesgue measure of Q and $|Q| = r^d$ (recall $d = |\alpha|$).

As in [13], we follow Strichartz [19] to define the parabolic BMO Sobolev space as follows

$$I_1(\mathrm{BMO}) = \{A(x) : A = I_1 a, a \in \mathrm{BMO}\}. \tag{1.9}$$

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As in [19], for all $b \in \mathrm{BMO}$, $I_1(b)$ is well-defined modulo polynomials of homogeneous degree ≤ 1. This space will play a crucial role in this paper. At present, it remains an open problem to define Sobolev spaces of order $\alpha = 1$ on a general space of homogeneous type (we are grateful to Y. S. Han for this observation). This obstacle restricts us to the more concrete setting considered here.

An interesting fact about an element A in $I_1(\mathrm{BMO})$ is the following lemma (see also Lemma 3.1 in [13]).

Lemma 1 *If $\alpha_1 = \cdots = \alpha_{n_0-1} = 1 < \alpha_{n_0}$, then for all $j \geq n_0$,*

$$|A(x_1, \dots, x_j + h, \dots, x_n) - A(x_1, \dots, x_j, \dots, x_n)| \leq c\|P_1 A\|_{\mathrm{BMO}} |h|^{1/\alpha_j}. \tag{1.10}$$

We sketch the proof here.

Proof of Lemma 1 For the sake of simplicity, we suppose $j = n$, the other cases being similar. Denote $x = (x_1, \dots, x_{n-1}, x_n) = (x', x_n)$. Then, with $a = P_1 A$, we have

$$\begin{aligned} A(x', x_n + h) - A(x', x_n) = \int_{-\infty}^{\infty} \int_{\mathbb{R}}^{n-1} a(y', y_n)[K(x' - y', x_n - y_n + h) \\ - K(x' - y', x_n - y_n)]\, dy'\, dy_n, \end{aligned} \tag{1.11}$$

for an appropriate even $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$, where K satisfies

$$K(\delta_\lambda x) = \lambda^{-d+1} K(x). \tag{1.12}$$

Since the expression in square brackets has mean value zero, we may assume that a has mean value zero on the parabolic ball $B_{|h|^{1/\alpha_n}}(x)$. Furthermore, (1.12) implies that the expression in square brackets is bounded in absolute value by $|h|\big(\rho(x-y)\big)^{-d-\alpha_n+1}$, whenever $|h|^{1/\alpha_n} < \big(\rho(x-y)\big)/2$, and by $\big(\rho(x-y)\big)^{-d+1} + \Big(\rho\big((x', x_n + h) - y\big)\Big)^{-d+1}$ otherwise. In the former case, one may then obtain the desired bound by adapting to the parabolic case a well known argument from [11]. In the latter case, one first dominates the corresponding piece of the integral in (1.11) by

$$\int_{\rho(x-y)<2|h|^{1/\alpha_n}} a(y)\big(\rho(x-y)\big)^{-d+1}\, dy + \int_{\rho(x-y)<2|h|^{1/\alpha_n}} a(y)\Big(\rho\big((x', x_n + h) - y\big)\Big)^{-d+1}\, dy.$$

The desired bound now follows for these last two integrals by applying Holder's inequality and the John-Nirenberg lemma. We omit the routine details. ■

Next we suppose that $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$ satisfies

$$K(\delta_\lambda x) = \lambda^{-d-1} K(x), \quad \lambda > 0, \quad x \in \mathbb{R}^n. \tag{1.13}$$

If we write $K(x) = \rho(x)^{-d-1}\Omega(x)$, then Ω is smooth on S^{n-1} and parabolically homogeneous of degree zero. We shall also impose a certain cancellation assumption on K. As above, we let n_0 be the smallest integer such that $a_{n_0} > 1$, *i.e.*,

$$1 = \alpha_1 = \cdots = \alpha_{n_0-1} < \alpha_{n_0}. \tag{1.14}$$

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Then we assume

$$\int_{S^{n-1}} \Omega(\sigma)\sigma_j J(\sigma)\, d\sigma = 0 \quad \text{for } j = 1, \dots, n_0 - 1. \tag{1.15}$$

In particular, this includes the case that $K(x)$ is even in $(x_1, \dots, x_{n_0})$. We know that (1.15) cannot be relaxed: consider the case $A(x) = x_j$, $j < n_0$ (see [10]).

We are interested in characterizing those A for which the following Calderón commutator

$$T_A f(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x - y)[A(x) - A(y)] f(y)\, dy \tag{1.16}$$

is bounded on $L^2(\mathbb{R}^n)$. Our main theorems are as follows.

Theorem 1 *Suppose that $A = I_1 a$, where $a \in \mathrm{BMO}$, and that $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$ satisfies (1.13) and (1.15). Suppose also that*

$$\Big\| \frac{\partial}{\partial x_j} A \Big\|_{L^\infty(\mathbb{R}^n)} \le B \quad \textit{for } j = 1, \dots, n_0 - 1, \tag{1.17}$$

then

$$\|T_A\|_{\mathrm{op}} \le c(n, K)(B + \|a\|_{\mathrm{BMO}});$$

In particular, when $n_0 = n$ and $\alpha_n = 2$, we recover Theorem 2.1 of [13].

Remark 1 The assumptions (1.10) and (1.17) imply the parabolic Lipschitz condition

$$|A(x) - A(y)| \le c(\|P_1 A\|_{\mathrm{BMO}} + B)\rho(x - y), \tag{1.18}$$

for all $x, y \in \mathbb{R}^n$. Thus, $K(x-y)[A(x) - A(y)]$ satisfies the standard (parabolic) Calderón-Zygmund kernel conditions.

Remark 2 The regularity assumption $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$ can certainly be relaxed—we impose it here only so that we may simplify certain arguments by invoking symbolic calculus results of [10]. Our primary purpose here is to establish the sharp condition on A, so we shall be content with smooth K.

As pointed out in [13], the sufficient conditions required for T_A to be bounded are also necessary in certain important special cases. The kernel K defined in (0.2) provides such an example (see Theorem 2.2 in [13]). Actually, there are more general results. Let e_j denote a unit vector in the x_j direction. In the next theorem, we shall assume that K satisfies a certain "non-degeneracy" condition; namely that there exists a j, $1 \le j \le n$, and a neighborhood $N_j \subseteq S^{n-1}$ of at least one of the poles e_j (or $-e_j$), such that for all $\sigma \in N_j$, either

$$|K(\sigma)| \ge c > 0, \quad \text{or} \quad |K(\sigma) + K(-\sigma)| \ge c > 0, \tag{1.19}$$

https://doi.org/10.4153/CMB-1999-054-0 Published online by Cambridge University Press

for some positive constant c. By continuity, this means that $K(\sigma)$ (resp. $K(\sigma) + K(-\sigma)$) has constant sign in N_j.

Theorem 2 *Let $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$ satisfy (1.13) and (1.19). Then, if T_A, defined as in (1.16), is bounded on $L^2(\mathbb{R}^n)$, we have*

$$|A(x) - A(y)| \le c\|T_A\|_{\text{op}}\,\rho(x-y).$$

Remark 3 The kernel $K(x_1, \dots, x_n)$ in (0.2) is an example satisfying Theorem 2. More examples include $K(x) = \frac{1}{\rho(x)^{d+1}}$ and also any kernel which is C^∞ and positive away from the origin, is homogeneous of degree $-d-1$, and is even in the variables $(x_1, \dots, x_{n_0-1})$.

In addition, the condition that A can be expressed as the fractional integral operator of order 1 acting on a BMO function is also necessary in some sense. This is the content of the following Theorem 3 and the Corollary (see also the example in [13]). Before stating the result, we make a definition.

Let $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and $K(\delta_\lambda x) = \lambda^{-d-1}K(x)$. Define

$$Hf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y)\big(f(y) - f(x)\big)\,dy \tag{1.20}$$

for all f in $\mathcal{S}(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz class on $\mathbb{R}^n$. Denote by $m(\xi)$ the symbol of H in the sense that

$$\int_{\mathbb{R}^n} (Hf)(x)\varphi(x)\,dx = \int_{\mathbb{R}^n} m(\xi)\hat{f}(\xi)\hat{\varphi}(\xi)\,d\xi \tag{1.21}$$

for all $f, \varphi \in \mathcal{S}(\mathbb{R}^n)$. We also call m the symbol associated with K.

The following proposition characterizes the relation between such a kernel K and its symbol m. The construction of m from K is crucial in proving the Corollary after Theorem 3. The proof of this proposition may be readily obtained by adapting the ideas from [10], and we therefore omit it.

Proposition *Let $m \in C^\infty(\mathbb{R}^n \setminus \{0\})$ such that $m(\delta_\lambda\xi) = \lambda m(\xi)$. Moreover m is even in each of the variables $\xi_1, \dots, \xi_{n_0-1}$. Then there exists a function $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$ satisfying*

1) *$K(\delta_\lambda x) = \lambda^{-d-1}K(x)$;*
2) *If $K(x) = \frac{\Omega(x)}{\rho(x)^{d+1}}$, then $\int_{S^{n-1}} \Omega(\sigma)\sigma_j J(\sigma)\,d\sigma = 0, \quad j = 1, \dots, n_0 - 1$;*
3) *The operator defined by*

$$Hf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y)\big(f(y) - f(x)\big)\,dy \quad \textit{for all } f \in \mathcal{S}(\mathbb{R}^n),$$

has symbol $m(\xi)$ in the sense (1.21).

Conversely, let $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$ satisfy (1.13) and (1.15). Then there exists a function $m \in C^\infty(\mathbb{R}^n \setminus \{0\})$ satisfying $m(\delta_\lambda\xi) = \lambda m(\xi)$ and (1.21).

Theorem 3 *Suppose that $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$, $K(\delta_\lambda x) = \lambda^{-d-1}K(x)$ and K is associated to a symbol $m(\xi) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ in the sense of (1.21), with $m(\xi) \neq 0$, for $\xi \neq 0$. If A satisfies the Lipschitz condition,*

$$|A(x) - A(y)| \leq B_0\rho(x-y), \quad \textit{for all } x, y \in \mathbb{R}^n,$$

and T_A, defined as in (1.16), is bounded on $L^2(\mathbb{R}^n)$, then

$$\|P_1 A\|_{\mathrm{BMO}} \leq c\|T_A\|_{\mathrm{op}} + CB_0.$$

As a consequence of the proposition, we shall also prove the following:

Corollary *Let $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$ satisfy (1.13), (1.15), and assume that $K(x) + K(-x) \neq 0$. Let m be the symbol of K given by the Proposition. Then $m(\xi) \neq 0$ whenever $\xi \neq 0$. In particular, such K satisfy the hypotheses of both Theorem 2 and Theorem 3.*

Remark 4 The last two examples in Remark 3 both satisfy the conclusion of Theorem 3. Another example satisfying Theorem 3 is the commutator $[T, A]$, where the kernel K of T has symbol $m(\xi) = \rho(\xi)$. By the proposition, K satisfies our basic assumptions (1.13) and (1.15).

2 Proof of Theorem 1

We shall require the following standard Lemma.

Lemma 2 *Let $m(\xi) \in L^\infty(\mathbb{R}^n)$, and assume m is continuously differentiable on $\mathbb{R}^n \setminus \{0\}$ up to order N with $N > |\alpha|/2$. Moreover, assume that*

$$\frac{1}{R^{|\alpha|}} \int_{R/2 \leq \rho(\xi) \leq 2R} \left| R^{\alpha \cdot \beta} \left(\frac{\partial}{\partial \xi}\right)^\beta m(\xi)\right|^2 d\xi \leq c_m,$$

for all $\beta = (\beta_1, \dots, \beta_n)$ with β_j's non-negative integers satisfying $|\beta| \leq N$, and $|m(\xi)| \leq c_m$ a.e., where constant c_m is independent of R. Then the operator T with symbol m is a bounded linear operator on H^1. Since the same is true for T^*, T is a bounded linear operator on BMO.

Remark 6 The $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ boundedness of T is proved in [9]. In the isotropic Euclidean space $\mathbb{R}^n$, the multiplier m in the Lemma is a Hörmander-Mihlin multiplier (see [18, Chapter 4]), and the proof of the boundedness of T on the standard $H^p(\mathbb{R}^n)$ $(0 < p \leq 1)$ can be found in [12, Chapter 7]. The latter proof adapts readily to the parabolic case, but one may also consult [4] and [17] for some more general results on parabolic Hardy spaces. Thus, we omit the proof here.

Remark 7 If $m(\xi) \in C^N(\mathbb{R}^n)$ $(N > |\alpha|/2)$ and is homogeneous of degree zero, it is easy to verify that $m(\xi)$ satisfies the conditions of Lemma 2 (see Chapter 4 in [18] for the isotropic case).

Now we turn to the proof of Theorem 1. The main idea of the proof is the same as before (see [13]): we use the (nonisotropic) $T1$-theorem; that is, since our kernels satisfy "standard" regularity estimates, it is enough to show that T_A and T_A^* map 1 into BMO, and that T_A satisfies the "Weak Boundedness Property" (defined below). The proof of the usual $T1$ theorem given in [6] is easily adapted to the non-isotropic setting.

We impose (as we may) the *a priori* assumption that A is smooth. Our estimates will, of course, not depend on this *a priori* regularization.

We first verify that $T_A 1 \in \text{BMO}$. The same argument can be used to show that $T_A^* 1 \in \text{BMO}$. We only give a formal proof of this fact. The proof can be made rigorous as in [13]. We use polar coordinates (1.3) and integrate by parts. Then,

$$\begin{aligned} T_A 1(x) &= \text{p.v.} \int_{\mathbb{R}^n} K(y)[A(x) - A(x-y)]\, dy \\ &= \text{p.v.} \int_{S^{n-1}} \Omega(\sigma) \int_0^\infty [A(x) - A(x - \delta_\rho \sigma)] \frac{d\rho}{\rho^2} J(\sigma)\, d\sigma \\ &= \text{p.v.} \int_{S^{n-1}} \Omega(\sigma) \sum_{j=1}^{n_0-1} \sigma_j \int_0^\infty \frac{\partial}{\partial x_j} A(x - \delta_\rho \sigma) \frac{d\rho}{\rho} J(\sigma)\, d\sigma \\ &\quad + \text{p.v.} \int_{S^{n-1}} \Omega(\sigma) \sum_{j=n_0}^{n} \sigma_j \alpha_j \int_0^\infty \frac{\partial}{\partial x_j} A(x - \delta_\rho \sigma) \rho^{\alpha_j - 2}\, d\rho\, J(\sigma)\, d\sigma \\ &\equiv \sum_{j=1}^{n_0-1} L_j + \sum_{j=n_0}^{n} J_j. \end{aligned}$$

By the cancellation condition (1.15), each L_j for $1 \le j \le n_0 - 1$ is an $L^2(\mathbb{R}^n)$-bounded parabolic singular integral operator acting on a bounded function. Thus, $\|L_j\|_{\text{BMO}} \le c\|\frac{\partial}{\partial x_j} A\|_{L^\infty(\mathbb{R}^n)}$, $1 \le j \le n_0 - 1$.

To handle J_j, $j = n_0, \ldots, n$, note that

$$\frac{\partial}{\partial x_j} A = \frac{\partial}{\partial x_j} I_1 a \tag{2.1}$$

has symbol $\xi_j/\rho(\xi)$, and is thus a "parabolic derivative" of order $\alpha_j - 1$. Moreover the operator defined by

$$\tilde{I}_1^j f(x) \equiv \text{p.v.} \int_{\mathbb{R}^n} \frac{\tilde{\Omega}_j(x-y)}{\rho(x-y)^{d-\alpha_j+1}} f(y)\, dy$$

is parabolically smoothing of order $\alpha_j - 1$, where $\tilde{\Omega}_j(\sigma) = \alpha_j \Omega(\sigma)\sigma_j$ is parabolically homogeneous of degree zero. Indeed, $\tilde{I}_1^j$ has symbol $\tilde{m}_j(\xi)/\rho(\xi)^{\alpha_j - 1}$, where $\tilde{m}_j(\xi) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and $\tilde{m}_j(\delta_\lambda \xi) = \tilde{m}_j(\xi)$, for all $\lambda > 0$. Thus, we have $(J_j)^\wedge(\xi) = \frac{\hat{m}_j(\xi)\xi_j}{\rho(\xi)^{\alpha_j}} \hat{a}(\xi)$, and the multiplier $\hat{m}_j(\xi)\xi_j/\rho(\xi)^{\alpha_j}$ satisfies the conditions of Lemma 2. Therefore, $\|J_j\|_{\text{BMO}} \le c\|a\|_{\text{BMO}}$, for $j = n_0, \ldots, n$.

Next, we will prove that T_A satisfies the "Weak Boundedness Property" (WBP). Following [13], we can define WBP as follows. Let $\Phi(x, \delta)$ denote the space of all $\varphi \in C_0^\infty(\mathbb{R}^n)$,

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supported in $B_\delta(x) \equiv \{y : \rho(x-y) < \delta\}$, and satisfying

$$\text{(2.2)}\quad \begin{aligned} &\text{(i)} && \|\varphi\|_{L^\infty(\mathbb{R}^n)} \le 1;\\ &\text{(ii)} && |\varphi(x)-\varphi(y)| \le \frac{1}{\delta}\rho(x-y);\\ &\text{(iii)} && \sup_{|\gamma|\le\alpha_n}\Big\{\delta^{\gamma_1+\cdots+\gamma_{n_0-1}+\alpha_{n_0}\gamma_{n_0}+\cdots+\alpha_n\gamma_n} \times \Big\|\Big(\frac{\partial}{\partial x}\Big)^\gamma\varphi\Big\|_{L^\infty(\mathbb{R}^n)}\Big\} \le 1. \end{aligned}$$

Then we say that T satisfies WBP if for all $x \in \mathbb{R}^n$, $\delta > 0$, and for all $\psi, \varphi \in \Phi(x,\delta)$, we have

$$|\langle\psi, T\varphi\rangle| \le c\delta^d. \tag{2.3}$$

Let us now show that this property holds. By dilation invariance, we may take $\delta = 1$. Then ψ, φ are supported in the parabolic ball $B_1 \equiv B_1(x_0)$, for some fixed $x_0 \in \mathbb{R}^n$. Again, we write $T_A\varphi$ in polar coordinates and integrate by parts so that

$$\begin{aligned} T_A\varphi(x) &= \text{p.v.}\int_{S^{n-1}}\Omega(\sigma)\int_0^\infty [A(x)-A(x-\delta_\rho\sigma)]\varphi(x-\delta_\rho\sigma)\frac{d\rho}{\rho^2}J(\sigma)\,d\sigma\\ &= \sum_{i=1}^{n_0-1}\text{p.v.}\int_{S^{n-1}}\Omega(\sigma)\sigma_i\int_0^\infty \frac{\partial}{\partial x_i}A(x-\delta_\rho\sigma)\varphi(x-\delta_\rho\sigma)\frac{d\rho}{\rho}J(\sigma)\,d\sigma\\ &\quad+\sum_{i=n_0}^{n}\alpha_i\int_{S^{n-1}}\Omega(\sigma)\sigma_i\int_0^\infty\frac{\partial}{\partial x_i}A(x-\delta_\rho\sigma)\\ &\qquad\times\varphi(x-\delta_\rho\sigma)\rho^{\alpha_i-2}\,d\rho\,J(\sigma)\,d\sigma\\ &\quad-\sum_{j=1}^{n_0-1}\int_{S^{n-1}}\Omega(\sigma)\sigma_j\int_0^\infty[A(x)-A(x-\delta_\rho\sigma)]\\ &\qquad\times\frac{\partial}{\partial x_j}\varphi(x-\delta_\rho\sigma)\frac{d\rho}{\rho}J(\sigma)\,d\sigma\\ &\quad-\sum_{j=n_0}^{n}\alpha_j\int_{S^{n-1}}\Omega(\sigma)\sigma_j\int_0^\infty[A(x)-A(x-\delta_\rho\sigma)]\\ &\qquad\times\frac{\partial}{\partial x_j}\varphi(x-\delta_\rho\sigma)\rho^{\alpha_j-2}\,d\rho\,J(\sigma)\,d\sigma\\ &\equiv\sum_{i=1}^n L_i-\sum_{j=1}^n J_j. \end{aligned}$$

As before, by the cancellation condition (1.15), each L_i $(1 \le i \le n_0-1)$ is a bounded parabolic singular integral operator acting on $(\frac{\partial}{\partial x_i}A)\varphi$, and thus, by Schwarz' inequality, $|\langle\psi, L_i\rangle| \le c\|(\frac{\partial}{\partial x_i}A)\varphi\|_{L^2(\mathbb{R}^n)} \le cB$, where B is the constant in the theorem.

To handle $J_1, \dots, J_n$, we note that $x \in B_1$ and also $x - \delta_\rho \sigma \in B_1$. From this, we deduce that $\rho \leq c$. Thus, by (1.17),

$$\Big|\sum_{j=1}^{n} J_j\Big| \leq c(\|PA\|_{\mathrm{BMO}} + B) \int_0^c \|\nabla\varphi\|_{L^\infty(\mathbb{R}^n)}\, d\rho$$
$$\leq c(\|PA\|_{\mathrm{BMO}} + B),$$

by (2.2) (iii), and the desired bound for $|\langle\psi, \sum_{j=1}^{n} J_j\rangle|$ follows immediately.

Finally, we consider L_i, $i = n_0, \dots, n$. Writing L_i in rectangular coordinates, we have

$$L_i(x) = \int_{\mathbb{R}^n} K_i(x-y) \frac{\partial}{\partial y_i} A(y)\varphi(y)\, dy \tag{2.4}$$

where $K_i \in C^\infty(\mathbb{R}^n \setminus \{0\})$ satisfies $K_i(\delta_\lambda x) = \lambda^{-d-1+\alpha_i} K_i(x)$. The convolution operator with kernel K_i is parabolically smoothing of order $\alpha_i - 1$, (because $\alpha_i > 1$) and it has symbol

$$\hat{K}_i(\xi) = \frac{\bar{m}_i(\xi)}{\rho(\xi)^{\alpha_i - 1}}, \tag{2.5}$$

where $\bar{m}_i \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and $\bar{m}_i(\delta_\lambda \xi) = \bar{m}_i(\xi)$.

As before, the symbol of the operator which maps a to $\frac{\partial}{\partial y_i} A(y)$ is $\frac{\xi_i}{\rho(\xi)}$ which is of homogeneous degree $\alpha_i - 1$. Following the same argument as [10, p. 111], we have that, for each $i = 1, \dots, n$, there exists $H^i(y) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ which is of homogeneous degree $-d - \alpha_i + 1$, and satisfies

$$\operatorname{sgn}\big(H^i(y)\big) = c \operatorname{sgn}(y_i), \tag{2.6}$$

with $c \equiv 1$ or -1, and

$$(H^i)^\wedge(\xi) = \frac{\xi_i}{\rho(\xi)}. \tag{2.7}$$

By (2.7), we have

$$\int_{S^{n-1}} \sigma^\beta H^i(\sigma) J(\sigma)\, d\sigma = 0, \tag{2.8}$$

for all $\beta = (\beta_1, \dots, \beta_n)$ where the β_j's are non-negative integers satisfying

$$\beta_j = 0 \quad \text{or} \quad |\beta| = \beta_1 + \cdots + \beta_n \text{ is even.} \tag{2.9}$$

Let $\phi(x) = (a(x) - a_{B_1})\eta(x)$, where $a_{B_1} = \frac{1}{|B_1|}\int_{B_1} a(x)\, dx$, and $\eta \in C_0^\infty\big(B_{20}(x_0)\big)$ with $\eta \equiv 1$ on $B_{10}(x_0)$. By Taylor's theorem, as $\rho(x - x_0) \to 0$, we have

$$\Big|\phi(x) - \Big[\sum_{\substack{|\beta| \leq \alpha_i - 1 \\ \beta \text{ is in (2.9)}}} \Big(\frac{\partial}{\partial x}\Big)^\beta \phi(x_0)(x - x_0)^\beta\Big]\eta_0\big(\rho(x)\big)\Big| \leq c\rho(x - x_0)^{\alpha_i}, \tag{2.10}$$

where $\eta_0(t)$ is a non-negative $C_0^\infty(\mathbb{R})$-function which is 1 on a neighbourhood of $\rho(x_0)$. Thus, by (2.8) and (2.10), we can write $\frac{\partial}{\partial y_i}A(y)$ (we recall that we have made the *a priori* assumption that A is smooth) as the sum of the following two well-defined integrals:

$$\begin{aligned}\frac{\partial}{\partial y_i}A(y) &= \text{p.v.}\int_{\mathbb{R}^n} H^i(y-z)(a(z)-a_{B_1})\eta(z)\,dz \\ &\quad + \text{p.v.}\int_{\mathbb{R}^n} H^i(y-z)\big(a(z)-a_{B_1}\big)\big(1-\eta(z)\big)\,dz \\ &\equiv A_ia_1(y)+A_ia_2(y).\end{aligned} \tag{2.11}$$

The term $A_ia_2(y)$ has a bound as follows:

$$|A_ia_2(y)| \le c\int_{\mathbb{R}^n}\frac{|a(z)-a_{B_1}|}{1+\rho(x_0-z)^{d+\alpha_i-1}}\,dz \le c\|a\|_{\text{BMO}}. \tag{2.12}$$

The last inequality follows from the parabolic version of the standard estimate in [11]. We then write

$$\begin{aligned}L_i &= \int_{\mathbb{R}^n} K_i(x-y)\text{p.v.}\int_{\mathbb{R}^n} H^i(y-z)a_1(z)\big(\varphi(y)-P_z(y-z)\big)\,dz\,dy \\ &\quad + \int_{\mathbb{R}^n} K_i(x-y)\text{p.v.}\int_{\mathbb{R}^n} H^i(y-z)a_1(z)P_z(y-z)\,dz\,dy \\ &\quad + \int_{\mathbb{R}^n} K_i(x-y)\varphi(y)A_ia_2(y)\,dy \equiv L_i^1+L_i^2+L_i^3,\end{aligned}$$

where $P_z(y-z)=\sum_{|\beta|\le\alpha_i-1}(\frac{\partial}{\partial z})^\beta\varphi(z)(y-z)^\beta$. Since supp $\varphi\subseteq B_1$, by (2.5) and (2.13), as $x\in$ supp $\psi\subseteq B_1$, $|L_i^3(x)|\le c\|a\|_{\text{BMO}}\int_{\rho(x-y)\le c}\frac{1}{\rho(x-y)^{d-\alpha_i+1}}\,dy\le c\|a\|_{\text{BMO}}$, where the second inequality follows from the fact that $i\ge n_0$, *i.e.*, $\alpha_i>1$. Thus, $|\langle\psi,L_i^3\rangle|\le c\|a\|_{\text{BMO}}$. For L_i^2, choose β such that $|\beta|\le\alpha_i-1$. Let $\tilde H_i(y-z)=H_i(y-z)(y-z)^\beta$, where $\tilde H_i$ is of homogeneous degree $-d-\alpha_i+1+|\beta|$. Denote $\tilde A_i$ the convolution operator associated with kernel $\tilde H_i$. Then the operator with symbol $\frac{\tilde m_i(\xi)}{\rho(\xi)^{\alpha_i-1}}\frac{\partial^\beta}{\partial\xi^\beta}(\frac{\xi_i}{\rho(\xi)})=\frac{\tilde m_i(\xi)}{\rho(\xi)^{|\beta|}}$, where $\tilde m_i(\xi)\in C_0^\infty(\mathbb{R}^n)$ and is of homogeneous degree zero, is a nice parabolic singular integral operator (bounded on $L^2(\mathbb{R}^n)$) when $\beta=0$, and is a parabolic fractional integral operator when $\beta\ne 0$. Moreover, by (2.6), $L_i^2(x)=\sum_{|\beta|\le\alpha_i-1}(K_i*\tilde A_i)(a_1D^\beta\varphi)(x)$, where D^β is the differential operator $\frac{\partial^\beta}{\partial x^\beta}$. For each β above, we can choose $1<p_\beta,q_\beta<\infty$ such that $\frac{1}{q_\beta}=\frac{1}{p_\beta}-\frac{|\beta|}{d}$, and $\|(K_i*\tilde A_i)(a_1D^\beta\varphi)\|_{L^{q_\beta}(\mathbb{R}^n)}\le c\|a_1D^\beta\varphi\|_{L^{p_\beta}(\mathbb{R}^n)}\le c\|a\|_{\text{BMO}}$. Here we are using the non-isotropic version of the Hardy-Littlewood-Sobolev theorem on fractional integration. The proof in [18] carries over easily to the non-isotropic setting. Thus, $|\langle\psi,L_i^2\rangle|\le c\|a\|_{\text{BMO}}$. For L_i^1, we write

$$\begin{aligned}L_i^1(x) &= \int_{\mathbb{R}^n} K_i(x-y)\text{p.v.}\int_{\rho(y-z)\le 1} H^i(y-z)a_1(z)\big(\varphi(y)-P_z(y-z)\big)\,dz\,dy \\ &\quad + \int_{\mathbb{R}^n} K_i(x-y)\text{p.v.}\int_{\rho(y-z)>1} H^i(y-z)a_1(z)\big(\varphi(y)-P_z(y-z)\big)\,dz\,dy \\ &\equiv L_{i,1}^1+L_{i,2}^1.\end{aligned}$$

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If $\rho(y-z) \le 1$, by Taylor's theorem and (2.2), we obtain

$$|\varphi(y) - P_z(y-z)| \le c\rho(y-z)^{\alpha_i}. \tag{2.13}$$

Thus,

$$\begin{aligned} |f_1(y)| &\equiv \left| \int_{\rho(y-z)\le 1} H^i(y-z) a_1(z) \big(\varphi(y) - P_z(y-z)\big)\, dz \right| \\ &\le c \int_{\rho(y-z)\le 1} \frac{1}{\rho(y-z)^{d-1}} |a_1(z)|\, dz, \end{aligned}$$

and therefore, $f_1 \in L^p(\mathbb{R}^n)$, where $1 < p < \infty$, and

$$\|f_1\|_{L^p(\mathbb{R}^n)} \le c\|a_1\|_{L^p(\mathbb{R}^n)} \le c\|a\|_{\mathrm{BMO}}. \tag{2.14}$$

Since $|L^1_{i,1}(x)| \le c \int_{\mathbb{R}^n} |K_i(x-y)|\, |f_1(y)|\, dy \le c \int_{\mathbb{R}^n} \frac{|f_1(y)|}{\rho(x-y)^{d-\alpha_i+1}}\, dy$, and we can choose $1 < p, q < \infty$ such that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha_i - 1}{d}$, thus, we obtain, by (2.14), $\|L^1_{i,1}\|_{L^q(\mathbb{R}^n)} \le c\|f_1\|_{L^p(\mathbb{R}^n)} \le c\|a\|_{\mathrm{BMO}}$, and the desired bound for $|\langle \psi, L^1_{i,1}\rangle|$ follows. For $L^1_{i,2}$, since $\rho(y-z) > 1$, we obtain the following trivial estimate:

$$|\varphi(y) - P_z(y-z)| \le c. \tag{2.15}$$

We have

$$\begin{aligned} |f_2(y)| &\equiv \left| \int_{\rho(y-z)>1} H^i(y-z) a_1(z) \big(\varphi(y) - P_z(y-z)\big)\, dz \right| \\ &\le c \int_{\rho(y-z)>1} \frac{1}{\rho(y-z)^{d+\alpha_i-1}} |a_1(z)|\, dz \le cM(a_1)(y), \end{aligned}$$

where M is the Hardy-Littlewood maximal function. Again, we obtain, for $1 < p, q < \infty$ with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha_i - 1}{d}$ as before, $\|L^1_{i,2}\|_{L^q(\mathbb{R}^n)} \le c\|M(a_1)\|_{L^p(\mathbb{R}^n)} \le c\|a_1\|_{L^p(\mathbb{R}^n)} \le c\|a\|_{\mathrm{BMO}}$. The estimate $|\langle \psi, L^1_{i,2}\rangle| \le c\|a\|_{\mathrm{BMO}}$ follows. Therefore, we finish the proof of WBP and of Theorem 1.

3 Proofs of Theorem 2 and Theorem 3

First, we prove Theorem 2. The proof is similar to that of [13, Theorem 2.2] (but see also Murray [16], where these ideas had previously appeared in the 1-dimensional setting).

Let K as in Theorem 2. We may assume that $K(\sigma) + K(-\sigma) > C > 0$ in a neighborhood of the "north pole" $(0, \dots, 0, 1)$, the other cases being similar, or simpler. We suppose that T_A is bounded on $L^2(\mathbb{R}^n)$. So by duality,

$$T_A^* f(x) \equiv \text{p.v.} \int_{\mathbb{R}^n} K(y-x)[A(y) - A(x)] f(y)\, dy \tag{3.1}$$

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is also bounded on $L^2(\mathbb{R}^n)$. Therefore $T_A - T_A^*$ is bounded and its kernel is

$$h(x,y) = \big(K(x-y) + K(y-x)\big)[A(x) - A(y)] \equiv \mathcal{K}(x-y)[A(x) - A(y)].$$

To prove the theorem, it suffices to show that for each parabolic Q with $|Q| = r^d$, there is a constant a_Q such that

$$\int_Q |A(x) - a_Q|\,dx \le c\|T_A - T_A^*\|_{\mathrm{op}} r^{d+1} \le c\|T_A\|_{\mathrm{op}} r^{d+1}. \tag{3.2}$$

Indeed, the well known result of N. Meyers [15] extends readily to the non-isotropic setting (we omit the details, but we note that the key tool is an appropriate version of the Calderon-Zygmund decomposition [9]).

Let $Q = J \times [a_n, a_n + r^{\alpha_n}]$, where J is an $n-1$ dimensional non-isotropic cube with side lengths r^{α_j} in the x_j direction, and let $\tilde{Q} = J \times [a_n + (Nr)^{\alpha_n}, a_n + (Nr)^{\alpha_n} + r^{\alpha_n}]$, with N a large number to be determined later (*i.e.*, $\tilde{Q}$ is a copy of Q, translated "upward" by a distance of $(Nr)^{\alpha_n}$). By Schwarz' inequality,

$$\frac{1}{|Q|}\int_Q \left| \int_{\tilde{Q}} h(x,y)\,dy \right| dx \le c\|T_A\|_{\mathrm{op}}.$$

On the other hand,

$$\begin{aligned}\int_{\tilde{Q}} h(x,y)\,dy &= [A(x) - (A)_{\tilde{Q}}] \int_{\tilde{Q}} \mathcal{K}(x-y)\,dy + \int_{\tilde{Q}} [(A)_{\tilde{Q}} - A(y)]\mathcal{K}(x-y)\,dy \\ &\equiv F(x) + G(x),\end{aligned}$$

where $(A)_{\tilde{Q}} = \frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} A(x)\,dx$.

For $x \in Q$ and $y \in \tilde{Q}$, we have $\rho(x-y) \approx Nr$. This implies that for sufficiently large N,

$$\begin{aligned}|F(x)| &\ge c_0 |A(x) - (A)_{\tilde{Q}}| \int_{\tilde{Q}} \rho(x_y)^{-d-1}\,dy \\ &\ge c_0 |A(x) - (A)_{\tilde{Q}}| \int_{\tilde{Q}} |x_n - y_n|^{-\frac{d+1}{\alpha_n}}\,dy \\ &\ge \frac{c_0 |A(x) - (A)_{\tilde{Q}}|}{N^{d+1} r}.\end{aligned} \tag{3.3}$$

Furthermore, using $\int_{\tilde{Q}} (A(x) - (A)_{\tilde{Q}})\,dx = 0$, and standard non-isotropic Calderon-Zygmund estimates, we obtain

$$|G(x)| \le \frac{c_1}{N^{d+2} r |\tilde{Q}|} \int_{\tilde{Q}} |(A)_{\tilde{Q}} - A(y)|\,dy. \tag{3.4}$$

Using (3.3) and (3.4), we can obtain (3.2) by following verbatim the argument in [13, pp. 207–208], and therefore, we finish the proof of Theorem 2.

We now proceed to prove Theorem 3. Define S by $(Sf)^\wedge(\xi) = \frac{\rho(\xi)}{m(\xi)}\hat{f}(\xi)$, $f \in \mathcal{S}(\mathbb{R}^n)$, where m is the symbol of the kernel K (by abuse of notation, we shall also use K to denote the operator of convolution with the kernel K). Since $\frac{\rho(\xi)}{m(\xi)}$ satisfies the conditions of Lemma 2, S is a bounded operator on BMO. On the other hand, since $K1 = 0$, we have $T_A 1 = [K, A]1 = KA$. By the hypotheses of the theorem, $K(x-y)[A(x)-A(y)]$ satisfies the standard size and smoothness condition of a parabolic Calderón-Zygmund kernel whose associated operator T_A is bounded on $L^2(\mathbb{R}^n)$. Hence $T_A 1 \in \mathrm{BMO}$, with appropriate bounds. Notice that $P_1 = SK$. Therefore, $\|P_1 A\|_{\mathrm{BMO}} = \|SKA\|_{\mathrm{BMO}} \le c\|KA\|_{\mathrm{BMO}} = c\|T_A 1\|_{\mathrm{BMO}}$. This finishes the proof of Theorem 3.

Finally, we give the proof of the Corollary. Without loss of generality, we may assume that $K(x) + K(-x) > 0$ for all $x \neq 0$. We keep the notation in the proof of the Proposition. From (3.5),

$$m(\xi) = \int_0^\infty \frac{1}{\lambda} \int_{\mathbb{R}^n} K(x)\psi\Big(\frac{\rho(x)}{\lambda}\Big)[e^{-ix\cdot\xi} - 1]\,dx\,d\lambda$$
$$= \int_0^\infty \frac{1}{\lambda} \int_{\mathbb{R}^n} K(-x)\psi\Big(\frac{\rho(x)}{\lambda}\Big)[e^{ix\cdot\xi} - 1]\,dx\,d\lambda$$

by changing x to $-x$. To prove $m(\xi) \neq 0$, whenever $\xi \neq 0$, we need only to show that $\operatorname{Re} m(\xi) \neq 0$. But,

$$2\operatorname{Re} m(\xi) = \int_0^\infty \frac{1}{\lambda} \int_{\mathbb{R}^n} \big(K(x) + K(-x)\big)\psi\Big(\frac{\rho(x)}{\lambda}\Big)\big(\cos(x\cdot\xi) - 1\big)\,dx\,d\lambda,$$

which is never zero away from $\xi = 0$ by the assumption. This finishes the proof of the Corollary.

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https://doi.org/10.4153/CMB-1999-054-0 Published online by Cambridge University Press