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Approximative compactness and continuity of metric projections

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In the paper "Some remarks on approximative compactness", *Rev. Roumaine Math. Pures Appl.* 9 (1964), Ivan Singer proved that if K is an approximatively compact Chebyshev set in a metric space, then the metric projection onto K is continuous. The object of this paper is to show that though, in general, the continuity of the metric projection supported by a Chebyshev set does not imply that the set is approximatively compact, it is indeed so in a large class of Banach spaces, including the locally uniformly convex spaces. It is also proved that in such a space X the metric projection onto a Chebyshev set is continuous on a set dense in X.

Introduction

The important concept of the approximative compactness of a set, which was first introduced by Efimov and Stečkin [2], arises quite naturally in the theory of best approximation in normed linear spaces. We recall that a set K in a normed linear space X is called approximatively compact if, for any x in X and any sequence $\{g_n\}$ in K, the relation $\|x-g_n\| \to \inf\{\|x-g\| : g \in K\}$ implies that $\{g_n\}$ is compact in K in the sense that there is a subsequence of $\{g_n\}$ which converges in K. Every boundedly compact set is approximatively compact, and in a uniformly convex Banach space every weakly sequentially closed set is approximatively

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compact. Singer [8] characterised those spaces which share this property of uniformly convex Banach spaces. In the same paper Singer proved that an approximatively compact Chebyshev set supports a continuous metric projection ([8], Corollary 2). This came as a generalization of various continuity properties of the metric projection given by Klee ([5], Propositions 2.3 and 2.4) and Ky Fan and Glicksberg ([3], Theorem 8). In the present paper we prove that in a large class of Banach spaces which we call "spaces with property (M) " a converse of Singer's result holds: namely, the continuity of the metric projection onto a Chebyshev set K implies the approximative compactness of the set K. We also prove that in such a space X the metric projection supported by a Chebyshev set is always continuous on a subset dense in X. Again it is shown that a proximal sun in such a space is approximatively compact and hence in a Banach space with property (M) every Chebyshev subspace supports a continuous metric projection.

Throughout this paper we shall denote a real normed linear space by X and its conjugate space by X^*. Let

$$B[x, r] = \{y \in X : \|x-y\| \leq r\} , \quad U(X) = \{y \in X : \|y\| \leq 1\} ,$$

and

$$S(X) = \{y \in X : \|y\| = 1\} .$$

The distance from a point x in X to a subset K of X is defined by $d_K(x) = \inf\{\|x-y\| : y \in K\}$. The metric projection P_K supported by a subset K of X is a mapping defined by

$$P_K(x) = \{y \in K : \|x-y\| = d_K(x)\} \quad \text{for } x \text{ in } X .$$

K is a *proximal* (respectively *Chebyshev*) set provided $P_K(x)$ contains at least (respectively exactly) one point for each $x \in X$. It is called a *sun* if for each $x \in X$, there is a $v \in P_K(x)$ such that $v \in P_K(v+\lambda(x-v))$ for every $\lambda \geq 0$. P_K is said to be *upper semicontinuous* at x if for every open set $M \subset K$ such that $P_K(x) \subset M$, there exists in M an open neighborhood V of x such that $P_K(y) \subset M$ for every $y \in V$.

A *subgradient* of a proper convex function f at a point $x \in X$ is an $x^* \in X^*$ such that

$$f(y) \geq f(x) + x^*(y-x) \text{ , for all } y \in X .$$

In particular, the norm being a proper convex function, an element $x^* \in X^*$ is a subgradient of the norm at $x \neq \theta$ (θ will denote the null element of X), if and only if $\|x^*\| = 1$ and $x^*(x) = \|x\|$.

A normed linear space X is said to have *property* (M) if the following holds:

For any x in $S(X)$, each sequence $\{g_n\}$ contained in $U(X)$ with $\|g_n - x\| \to 2$ has a convergent subsequence in $U(X)$.

Clearly every locally uniformly convex normed linear space has property (M). Indeed it can be easily proved that X is locally uniformly convex if and only if it is strictly convex and has property (M). The spaces with property (M) have been discussed in another paper by the authors [7]. Lemma 2 of the present paper contains one geometric property of the unit sphere of such spaces.

The main result

We now prove our main result, the 'only if' part of which was proved by Singer as mentioned above.

THEOREM 1. *Let* K *be a Chebyshev subset of a normed linear space* X *which has property* (M). *Then* K *is approximatively compact if and only if the metric projection* P_K *is continuous on* X.

For the proof of the theorem we need the following lemma due to Vlasov ([9], Lemma 1), a proof of which is given below for the sake of completeness.

LEMMA 1. *Let* X *be a normed linear space, and let* K *be a Chebyshev subset of* X. *If* $x \in X$ *is a point of continuity of the metric projection* P_K *then*

$$\lim_{t \to 0} \frac{d_K\left(x+t\left(x-P_K(x)\right)\right)-d_K(x)}{t} = d_K(x) .$$

Proof of Lemma 1. Let $z \in X$ and let $\{t_n\}$ be any sequence of positive real numbers converging to zero. Let ϕ_n be a subgradient of the

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norm at $x - P_K(x+t_nz)$. Then for all $n \geq 1$,

(1) $$\phi_n\big(x-P_K(x+t_nz)\big) = \|x-P_K(x+t_nz)\| ,$$

and

$$\begin{aligned}\|x+t_nz-P_K(x+t_nz)\| &\geq \|x-P_K(x+t_nz)\| + \phi_n(t_nz) \\ &\geq d_K(x) + \phi_n(t_nz) ,\end{aligned}$$

so that

(2) $$d_K(x+t_nz) - d_K(x) \geq \phi_n(t_nz) .$$

Again let ϕ'_n be a subgradient of the norm at $x + t_nz - P_K(x)$. Then for all $n \geq 1$,

(3) $$\phi'_n\big(x+t_nz-P_K(x)\big) = \|x+t_nz-P_K(x)\|$$

and

$$\begin{aligned}\|x-P_K(x)\| &\geq \|x+t_nz-P_K(x)\| - \phi'_n(t_nz) \\ &\geq d_K(x+t_nz) - \phi'_n(t_nz)\end{aligned}$$

so that

(4) $$d_K(x+t_nz) - d_K(x) \leq \phi'_n(t_nz) .$$

Now let ϕ and ϕ' be any weak*-cluster points of $\{\phi_n\}$ and $\{\phi'_n\}$ respectively. Then by (1) and (3) and the continuity of P_K we have

(5) $$\phi\big(x-P_K(x)\big) = \|x-P_K(x)\| = \phi'\big(x-P_K(x)\big)$$

and thus ϕ and ϕ' are subgradients of the norm at $x - P_K(x)$. From (2) and (4) we get

$$\phi_n(z) \leq \frac{d_K(x+t_nz)-d_K(x)}{t_n} \leq \phi'_n(z) .$$

Putting $z = x - P_K(x)$ and taking account of (5) we conclude that

$$\lim_{t\to 0^+} \frac{d_K\big(x+t\big(x-P_K(x)\big)\big)-d_K(x)}{t} = d_K(x) .$$

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In the same way we can prove that

$$\lim_{t\to 0^-} \frac{d_K\big(x+t(x-P_K(x))\big)-d_K(x)}{t} = d_K(x) ,$$

and hence the result is proved.

Proof of Theorem 1. Let $\{g_n\} \subset K$, $x \in X \sim K$ and $\|x-g_n\| \to d_K(x)$. Suppose that P_K is continuous. Set $t_n = \big(\|x-g_n\|-d_K(x)\big)^{\frac{1}{2}}$. Clearly t_n tends to zero through nonnegative values. We can assume without loss of generality that $t_n > 0$ by passing on to a subsequence, if necessary. Let ϕ_n be a subgradient of the norm at $x + t_n\big(x-P_K(x)\big) - g_n$ so that

$$(6) \qquad \phi_n\big(x+t_n(x-P_K(x))-g_n\big) = \|x+t_n(x-P_K(x))-g_n\| ,$$

and

$$\begin{aligned}\|x-g_n\| &\geq \|x+t_n(x-P_K(x))-g_n\| - t_n\phi_n\big(x-P_K(x)\big) \\ &\geq d_K\big(x+t_n(x-P_K(x))\big) - t_n\phi_n\big(x-P_K(x)\big) .\end{aligned}$$

Hence

$$(7) \qquad \frac{d_K\big(x+t_n(x-P_K(x))\big)-d_K(x)}{t_n} \leq t_n + \phi_n\big(x-P_K(x)\big) .$$

Applying Lemma 1 to (7) we see that $\|x-P_K(x)\| \leq \phi_0\big(x-P_K(x)\big)$, where ϕ_0 is any weak*-cluster point of the sequence $\{\phi_n\}$. This implies that $\phi_0\big(x-P_K(x)\big) = \|x-P_K(x)\|$. We assume without loss of generality that $\{\phi_n\}$ converges to ϕ_0 in the weak*-topology. Now set

$$z_n = \frac{x+t_n\big(x-P_K(x)\big)-g_n}{\|x+t_n\big(x-P_K(x)\big)-g_n\|} \quad\text{and}\quad z = \frac{x-P_K(x)}{\|x-P_K(x)\|} .$$

Then

$$2 = \lim_{n\to\infty} \phi_n\big(z_n+z\big) \leq \liminf_{n\to\infty} \|z_n+z\| \leq \limsup_{n\to\infty} \|z_n+z\| \leq 2$$

and hence $\|z_n+z\| \to 2$ as $n \to \infty$. Since X has property (M) , the sequence $\{z_n\}$ has a convergent subsequence. Moreover,

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$$\lim_{n\to\infty} \|x+t_n(x-P_K(x))-g_n\| = \lim_{n\to\infty} \|x-g_n\| = d_K(x) \ , \quad x \in X \sim K \ ,$$

and hence $\{g_n\}$ also has a convergent subsequence.

The converse is well known and holds even in a metric space setting (see [8]).

The following example shows that, in general, continuity of the metric projection does not imply that the Chebyshev set is approximatively compact.

EXAMPLE. Let X be the dual space of the Banach space constructed by Klee [6] by suitably renorming l^2. Lambert (unpublished) has shown that in the space X the metric projection P_K supported by any Chebyshev subspace K of X is continuous. However, X does not satisfy the Efimov-Stečkin property and hence contains a closed hyperplane X which is not approximatively compact (Singer, [8], Theorem 3). Since X is strictly convex and reflexive, K is a Chebyshev subspace and thus supports a continuous metric projection. The space X of course would not have property (M) (see also Holmes [4], p. 165).

Some further results

We give next a geometric property of the unit sphere of spaces with property (M). This property will be used to obtain some more results about continuity behaviour of metric projections.

LEMMA 2. *Let X be a normed linear space with property (M). Let x be a nonzero element of $U(X)$, and let*

$$K_n = B[x, 1-\|x\|+1/n] \sim \operatorname{int}U(X) \ .$$

Then each sequence $\{g_n\}$ with $g_n \in K_n$ has a convergent subsequence in $X \sim \operatorname{int}U(X)$.

Proof. From

$$1 - \|x\| \le \|g_n\| - \|x\| \le \|x-g_n\| \le 1 - \|x\| + 1/n \ ,$$

it follows that $\|g_n\| \to 1$ as $n \to \infty$. Now there exists $\psi_n \in S(X^*)$ such that $\psi_n(g_n) = \|g_n\|$ and hence

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$$\|g_n\| = \psi_n(g_n - x + x) \leq \|x - g_n\| + \psi_n(x) \ . \tag{8}$$

If ψ_0 is any weak*-cluster point of the sequence $\{\psi_n\}$ then from (8) we get $1 \leq 1 - \|x\| + \psi_0(x)$; that is, $\|x\| \leq \psi_0(x)$. As $\|\psi_0\| \leq \liminf_{n\to\infty} \|\psi_n\| = 1$, this implies that $\psi_0(x) = \|x\|$ and $\|\psi_0\| = 1$. Now set $z_n = g_n/\|g_n\|$ and $z = x/\|x\|$. Then taking a subsequence $\{\psi_{n_i}\}$ such that $\psi_{n_i} \to \psi_0$ in the weak*-topology, we get

$$2 = \lim_{i\to\infty} \psi_{n_i}\left(z_{n_i}+z\right) \leq \liminf_{i\to\infty} \left\|z_{n_i}+z\right\| \leq \limsup_{i\to\infty} \left\|z_{n_i}+z\right\| \leq 2 \ ,$$

and thus $\lim_{i\to\infty} \left\|z_{n_i}+z\right\| = 2$. As X has property (M) the sequence $\{z_{n_i}\}$ has a convergent subsequence in $U(X)$. Hence $\{g_{n_i}\}$ has a convergent subsequence in $X \sim \operatorname{int}U(X)$. This in turn implies that the original sequence has a convergent subsequence in $X \sim \operatorname{int}U(X)$. This proves the lemma.

THEOREM 2. *Let X be a space with property (M), and let K be a proximal sun in X. Then*

(i) K is approximatively compact, and

(ii) P_K is upper semicontinuous on X.

Proof. Let $\{g_m\} \subset K$ be a sequence such that $\|g_m - x\| \to d_K(x)$. Since K is a sun, there is a point $y \in P_K(x)$ such that all points on the ray $\overrightarrow{yx}$ are projected on y . Take $z_0 = x + \lambda_0(x-y)$ where $\lambda_0 > 0$ and set

$$K_n = B[x, \|x-y\|+1/n] \sim \operatorname{int}B[z_0, \|z_0-y\|] \ .$$

Since $\|g_m - x\| \to \|x-y\| = d_K(x)$, there exists for each n a g_{m_n} such that $g_{m_n} \in K_n$. By Lemma 2, there exists a subsequence of $\{g_{m_n}\}$ which is convergent in K (K is closed). Thus K is approximatively compact.

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Since the metric projection supported by an approximatively compact set is upper semi-continuous (Singer [8], Theorem 1), *(i)* implies *(ii)*.

REMARK. As a consequence of Theorem 2 all proximal subspaces in a space X with property (M) are approximatively compact. Thus all reflexive spaces having property (M) satisfy the Efimov-Stečkin property (see Singer [8]). However, it is known that the converse is not true (see [7]).

In the following we use Lemma 2 to obtain another result about the continuity behaviour of the metric projection onto a Chebyshev set which may not be a sun.

THEOREM 3. *Let K be a Chebyshev set in a normed linear space X with property (M). Then P_K is continuous on a subset dense in X.*

Proof. Let $x \in X \sim K$ and let $z \neq x$ be any point on the line segment between x and $P_K(x)$; that is, $z = \lambda x + (1-\lambda)P_K(x)$ for $0 \leq \lambda < 1$. Let $\{z_n\}$ be any sequence converging to z. It can be easily proved that $\|P_K(z_n)-z\| \to \|P_K(z)-z\|$. Now an application of Lemma 2 in the manner it was done in Theorem 2 shows that the sequence $\{P_K(z_n)\}$ is compact in K. As K is Chebyshev this means that $P_K(z_n) \to P_K(z)$ in norm. On the other hand, if $x \in K$ then $P_K(x) = x$ and hence P_K is continuous on K. Combining these two we get the required result.

The assumption in Theorem 3 that K is Chebyshev can be relaxed provided the norm in X has some additional properties.

THEOREM 4. *Let K be either*

(a) a closed subset of a uniformly convex Banach space, or

(b) a proximal subset of a locally uniformly convex Banach space.

Then there exists a subset G dense in X such that the restriction of the metric projection P_K to the set G is singlevalued and continuous.

Proof. First, let K be a closed subset of a uniformly convex Banach space. Then by a result of Edelstein [1], there exists a subset D dense in X such that every point in D admits a nearest point in K. The

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union G of all sets $\{\lambda x+(1-\lambda)y : 0 \leq \lambda < 1,\ x \in D,\ y \in P_K(x)\}$ is clearly dense in X. Moreover, every element of G admits a unique nearest point in K. An application of Lemma 2 in the way it was done in Theorem 3 yields the required result.

In the case of *(b)* the set D is the whole space X and the proof is the same. Hence the theorem is proved.

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