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Co-maximal Graphs of Subgroups of Groups

Saieed Akbari, Babak Miraftab, and Reza Nikandish

Abstract. Let H be a group. The co-maximal graph of subgroups of H, denoted by $\Gamma(H)$, is a graph whose vertices are non-trivial and proper subgroups of H and two distinct vertices L and K are adjacent in $\Gamma(H)$ if and only if $H = LK$. In this paper, we study the connectivity, diameter, clique number, and vertex chromatic number of $\Gamma(H)$. For instance, we show that if $\Gamma(H)$ has no isolated vertex, then $\Gamma(H)$ is connected with diameter at most 3. Also, we characterize all finite groups whose co-maximal graphs are connected. Among other results, we show that if H is a finitely generated solvable group and $\Gamma(H)$ is connected, and moreover, the degree of a maximal subgroup is finite, then H is finite. Furthermore, we show that the degree of each vertex in the co-maximal graph of a general linear group over an algebraically closed field is zero or infinite.

1 Introduction

Recently, using graph theoretical tools in the investigation of algebraic structures attracted many researchers. There are many papers which apply combinatorial methods to obtain algebraic results. There are many papers on assigning a graph to algebraic structures; see for example [1, 7, 8, 23]. Also, the concept of co-maximal graph associated with algebraic structures was first introduced by Sharma and Bhatwadekar [18] and developed by many authors; see, for instance [3, 15].

Let G be a graph with the vertex set $V(G)$. If u and v are two adjacent vertices of G, then we write u—v. The degree of a vertex v is denoted by $\deg(v)$. The minimum degree of vertices of G is denoted by $\delta(G)$. Assume that n is a positive integer. By nG, we mean the disjoint union of n copies of G. For every pair of distinct vertices x and y of G, let $d(x, y)$ be the length of the shortest path from x to y and if there is no such a path, then we define $d(x, y) = \infty$. The *diameter* of G, $\operatorname{diam}(G)$, is the supremum of the set $\{d(x, y) : x \text{ and } y \text{ are distinct vertices of } G\}$. The complete graph of order n and the complete bipartite graph with sizes m and n are denoted by K_n and $K_{m,n}$, respectively. A *clique* of G is a complete subgraph of G and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the *clique number* of G. For a graph G, let $\chi(G)$ denote the *chromatic number* of G, *i.e.*, the minimum number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. A graph G is called *perfect* if $\omega(S) = \chi(S)$, for every induced subgraph S of G.

The *semidirect product* of groups H and K denoted by $H \rtimes K$. Let H be a group and $h \in H$ and K be a subgroup of H. By H^n, $\langle h \rangle$, $N_H(K)$, K^h, and $o(h)$ we mean

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the n-th derived subgroup of H, the subgroup generated by $h \in H$, the normalizer of K in H, $h^{-1}Kh$, and the order of h, respectively. A subgroup K of a group H is said to be *supplemented* in H if there exists a subgroup L of H such that $KL = H$. In this case, L is called a *supplement* of K. If $K \cap L = 1$, then K is called *complemented*. If every subgroup of H has a proper supplement (complement), then H is called an *aS-group* (*aC-group*), see [14]. A *metabelian* group is a group whose commutator subgroup is abelian. The group H is called *factorizable* if and only if there exist proper subgroups K and L such that $H = KL$ (for more detail see [13]). The set of all maximal subgroups of the group H is denoted by $\mathrm{Max}(H)$. The intersection of all maximal subgroups of a group H, denoted by $\Phi(H)$, is called the *Frattini subgroup* of H. If H has no maximal subgroup, then we define $\Phi(H) = H$.

Let R be a non-zero commutative ring with identity. Sharma and Bhatwadekar [18]defined the co-maximal graph $\Gamma(R)$ whose vertex set consists of all elements of R and where two distinct vertices a and b are adjacent if and only if $Ra + Rb = R$. Wang [21] considered a subgraph $\Gamma_2(R)$ of $\Gamma(R)$ with vertices as non-unit elements of R. The co-maximal graph of a non-commutative ring was also defined and studied in [3, 20]. The ideal version of co-maximal graph can be found in [6, 22]. Moreover, the co-maximal graph of a lattice was introduced in [2]. In this paper, we define a co-maximal graph of subgroups of an arbitrary group. Let H be a group. The *co-maximal graph of subgroups of H*, denoted by $\Gamma(H)$, is a graph whose vertices are non-trivial proper subgroups of H and where two distinct vertices L and K are adjacent in $\Gamma(H)$ if $H = KL$. For abbreviation, we use co-maximal graphs of H instead of co-maximal graphs of subgroups of H. Obviously, if either $H = 1$ or $H = \mathbb{Z}_p$, where p is a prime number, then $\Gamma(H)$ is empty. Throughout this paper, we suppose that $H \neq 1$ and $H \neq \mathbb{Z}_p$. In this paper, we study the connectivity, diameter, clique number, and vertex chromatic number of $\Gamma(H)$. For example, we show that the co-maximal graph of each aS-group is connected. Among other results, we show that if $\Gamma(H)$ is connected for a finitely generated solvable group H and the degree of a vertex assigned to a maximal subgroup is finite, then H is a finite group. Moreover, we show that the degree of each vertex of the co-maximal graph of a general linear group over an algebraically closed field is zero or infinite.

2 The Connectivity of Co-maximal Graphs

In this section, we investigate the connectivity of a co-maximal graph of a group. We prove that if $\delta(\Gamma(G)) \geq 1$, then $\Gamma(G)$ is a connected graph and $\mathrm{diam}(\Gamma(G)) \leq 3$. We also show that for a nilpotent group G, the graph $\Gamma(G)$ is connected if and only if $\Phi(G) = 1$ or $G \cong \mathbb{Z}_{p^2}$.

We start with the following theorem.

Theorem 2.1 *Let G be a group. If $\delta(\Gamma(G)) \geq 1$, then every vertex of $\Gamma(G)$ is adjacent to a maximal subgroup of finite index and moreover $\Phi(G) = 1$.*

Proof Let L be a vertex of $\Gamma(G)$ and $1 \neq l \in L$. First, we show that L is adjacent to a maximal subgroup. There exists a non-trivial subgroup H such that $\langle l \rangle H = G$. Set $\sum = \{K < G \mid H \leq K, l \notin K\}$. By Zorn's lemma, $\sum$ has a maximal element, say

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M. We show that M is a maximal subgroup of G. Assume that there exists a proper subgroup M_1 such that $M \subsetneqq M_1$. Since $H \subseteq M_1$ and $l \in M_1$, we find that $M_1 = G$, a contradiction. Therefore, M is a maximal subgroup of G and it is adjacent to L. Now we show L is adjacent to a maximal subgroup of finite index. With no loss of generality, we can assume that L is a torsion free subgroup. There exists a maximal subgroup M such that $\langle l \rangle M = \langle l^2 \rangle M = G$, for some $l \in L$. Therefore, M is a maximal subgroup of finite index of G, as $l^n \in M$, for a natural number n. Also, $\Phi(G) = 1$ follows from [14, Proposition 3.4]. ∎

Next we show that the diameter of the co-maximal graph of every aS-group does not exceed 3.

Theorem 2.2 *Let G be a group. If $\delta(\Gamma(G)) \geq 1$, then* $\operatorname{diam}(\Gamma(G)) \leq 3$.

Proof Let H and K be two non-adjacent vertices of $\Gamma(G)$. By Theorem 2.1, H is adjacent to a subgroup L of G such that $[G:L] < \infty$. If there exists $k \in K$ of infinite order, then $K \cap L \neq 1$. So if S is a proper supplement of $K \cap L$, then we have the path $H—L—S—K$. Thus one may assume that H and K are torsion subgroups of G. Suppose that $h \in H$ and $k \in K$ such that $o(h) = p_1$ and $o(k) = p_2$, where p_1 and p_2 are primes. First assume that $p_1 \neq p_2$. Let H_1 and K_1 be complements of $\langle h \rangle$ and $\langle k \rangle$, respectively. Thus $([G:H_1], [G:K_1]) = 1$ and so by [19, 3.13], we have $H_1K_1 = G$. So consider the path $H—H_1—K_1—K$. Now suppose that $o(h) = o(k) = p$ for some prime number p. Thus there are maximal subgroups M_1 and M_2 of index p that are complements of $\langle h \rangle$ and $\langle k \rangle$, respectively. Assume that N is a normal subgroup that is contained in M_1 of finite index. If $N \nsubseteq M_2$, then $NM_2 = G$ and $M_1M_2 = G$. Now consider the path $H—M_1—M_2—K$. Let $N \subseteq M_2$. If $hN \in M_2/N$, then $h \in M_2$. Thus we have the path $H—M_1—M_2—K$. Next, suppose that $hN \notin M_2/N$. Since $o(hN) = p$, we conclude that $\langle hN \rangle M_2/N = G/N$. So we find the path $H—M_2—K$. ∎

Remark 2.3 Note that $\Gamma(\mathbb{Z}_{pqr})$ is a graph with diameter 3, where p, q, and r are distinct prime numbers.

Now we characterize all finite groups whose co-maximal graphs are connected.

Theorem 2.4 *Let G be a finite group and $|V(\Gamma(G))| \geq 2$. Then the following statements are equivalent.*

(i) *$\Gamma(G)$ is connected.*
(ii) $\delta(\Gamma(G)) \geq 1$.
(iii) *G is an aC-group.*
(iv) *Every cyclic subgroup of prime order of G is complemented in G.*
(v) *G is supersolvable and its Sylow subgroups are all elementary abelian.*
(vi) *G is isomorphic to a subgroup of a direct product of groups of squarefree order.*

Proof By Theorem 2.2, $\Gamma(G)$ is connected if and only if $\delta(\Gamma(G)) \geq 1$. The other cases follow from [14, Corollary 3.7], [9, Corollary 2], and [11, Theorems 1 and 2]. ∎

Now we have the following remark.

Remark 2.5 Let G be a nilpotent group and $\delta(\Gamma(G)) \geq 1$. If M is a maximal subgroup, then by [12, Lemma 7.4], M is normal. The equality $G/M \cong \mathbb{Z}_p$ implies that $G' \subseteq M$. By Theorem 2.1, $\Phi(G) = 1$ and so $G' = 1$. Thus G is an abelian group. Moreover, if $|\operatorname{Max}(G)| = n$, then G is isomorphic to a direct product of cyclic groups of prime orders.

By Theorems 2.1 and 2.2, we have the following corollary.

Corollary 2.6 *Let G be a nilpotent group. Then $\Gamma(G)$ is connected if and only if $\Phi(G) = 1$ or $G \cong \mathbb{Z}_{p^2}$, for some prime number p.*

Proof Let $\Gamma(G)$ be connected. If $|V(\Gamma(G))| = 1$, then $G \cong \mathbb{Z}_{p^2}$. So we can assume that $\delta(\Gamma(G)) \geq 1$. It follows from Theorem 2.1 that $\Phi(G) = 1$. Hence, for every vertex H there exists a maximal subgroup M such that $H \nsubseteq M$. By [12, Lemma 7.4], every maximal subgroup is normal and so $HM = G$. Thus, $\delta(\Gamma(G)) \geq 1$ and so by Theorem 2.2, the proof is complete. ∎

The requirement that G be nilpotent is necessary. For example, consider the alternating group on 4 letters, A_4. Then $\Phi(A_4) = 1$, but $\Gamma(A_4)$ has three isolated vertices. Indeed $\Gamma(A_4) = K_{1,4} \cup 3K_1$. The isolated vertices of $\Gamma(A_4)$ correspond to $\langle(1\,2)(3\,4)\rangle$, $\langle(1\,4)(2\,3)\rangle$, and $\langle(1\,3)(2\,4)\rangle$. Even if G is a supersolvable group and $\Phi(G) = 1$, $\Gamma(G)$ may not be connected. For instance, consider the Frobenius group of order 20, $\mathrm{Fr}_{20} = \langle a, b \mid ab = ba^2\rangle$, where $a = (1\,2\,3\,4\,5)$ and $b = (1\,4\,2\,3)$. It is not hard to see that the isolated vertices of $\Gamma(\mathrm{Fr}_{20})$ correspond to $\langle(1\,5)(2\,4)\rangle$, $\langle(1\,3)(4\,5)\rangle$, $\langle(1\,4)(2\,3)\rangle$, $\langle(2\,5)(3\,4)\rangle$, and $\langle(1\,2)(3\,5)\rangle$.

In the next theorem, we investigate the connectedness of a co-maximal graph of subgroups of a group.

Theorem 2.7 *Let G be a group. If $\Gamma(G)$ is a connected graph, then for every subgroup H of G, $\Gamma(H)$ is connected.*

Proof It follows from [14, Proposition 3.5] and Theorem 2.2. ∎

The converse of the previous result does not hold in general. For example, $\Gamma(H)$ is connected for every subgroup H of $\mathbb{Z}_{12}$, but $\Gamma(\mathbb{Z}_{12}) = K_1 \cup K_{1,2}$.

Corollary 2.8 *Let G be a torsion nilpotent group and $\Gamma(G)$ be a connected graph. Then $G \cong \mathbb{Z}_{p^2}$ or G is isomorphic to a direct sum of cyclic groups of prime orders.*

Proof If $\Gamma(G)$ has exactly one vertex, then $G \cong \mathbb{Z}_{p^2}$. So we can assume that

$$\delta(\Gamma(G)) \geq 1.$$

By Remark 2.5, G is an abelian torsion group, and by [17, 5.1.6], $G = \bigoplus A_{p_i}$, where A_{p_i} is a p_i-component for $i \in I$. Since $\delta(\Gamma(G)) \geq 1$, by [14, Proposition 3.5], the order of each element of A_{p_i} is p_i. ∎

Note that $\Gamma(G)$ is a null graph, *i.e.,* it has no edge, if and only if G is not a factorizable group.

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Remark 2.9 Let G be an abelian group.

(i) The vertex K of $\Gamma(G)$ is isolated if and only if $K \subseteq \Phi(G)$ and K has no divisible quotient group isomorphic to $\mathbb{Z}_{p^\infty}$. [10, Exercise D 10.5]

(ii) $\Gamma(G)$ is a null graph if and only if $G \cong \mathbb{Z}_{p^k}$, where $k \in \{1, 2, \ldots, \infty\}$. [17, 13.1.6]

Remark 2.10 There are some non-abelian groups G such that $\Gamma(G)$ are null. For instance $\Gamma(PSL(2, 13))$ is a null graph, (for more details, see [17, 13.1.11]) and for the infinite case $\Gamma(\mathrm{Tr})$ is a null graph, where Tr is a *Tarski monster* group, see [16, Chapter 9, §28.1].

3 Diameter of Co-maximal Graphs

In this section, we study the diameter of a co-maximal graph of a group. For instance, we show that if G is a free abelian group, then $\mathrm{diam}(\Gamma(G)) = 2$. Furthermore, we characterize all groups whose co-maximal graphs are complete graph.

The following theorem characterizes those nilpotent torsion groups for which the diameter of co-maximal graphs is 2.

Lemma 3.1 *Suppose G is a nilpotent torsion group. Then* $\mathrm{diam}(\Gamma(G)) = 2$ *if and only if for some prime p, G is an elementary abelian p-group and $G \ncong \mathbb{Z}_p \oplus \mathbb{Z}_p$.*

Proof First, assume that $G \cong \bigoplus_{i \in I} \mathbb{Z}_p$ and A, B are two non-adjacent vertices of $\Gamma(G)$. Since G is a vector space over $\mathbb{Z}_p$, we have $\delta(\Gamma(G)) \geq 1$. If A and B are comparable, then $d(A, B) = 2$. So suppose that they are not comparable. Hence there are linearly independent elements $\alpha \in A$ and $\beta \in B$. Now one can extend $\{\alpha, \beta\}$ to a basis $\{\alpha, \beta\} \cup \{\gamma_i\}_{i \in \Gamma}$ for G. If $W = \langle \alpha + \beta, \gamma_i \rangle_{i \in \Gamma}$, then obviously, $W + A = W + B = G$ and so $d(A, B) = 2$. Conversely, assume that $\mathrm{diam}(\Gamma(G)) = 2$. Since $\Gamma(G)$ is connected, by Corollary 2.8, G is a direct sum of cyclic groups of prime orders. We show that G is a p-group. Let H and K be two non-isomorphic cyclic subgroups of G whose orders are prime. If H and K are adjacent, then $|G| = pq$ and $\Gamma(G) = K_2$, a contradiction. So suppose that there exists a subgroup L such that $H \oplus L = K \oplus L = G$. So we find that $H \cong K$, a contradiction. Thus G is an elementary abelian p-group, as desired. ■

Now we show that for a free abelian group, the diameter of a co-maximal graph is 2.

Theorem 3.2 *If G is an abelian free group, then* $\mathrm{diam}(\Gamma(G)) = 2$.

Proof We have $G \cong \bigoplus_{i \in I} \mathbb{Z}$. Consider the standard basis $\{e_i\}_{i \in I}$ for G. Let A and B be two vertices of $\Gamma(G)$. First suppose that I is finite. If $G \cong \mathbb{Z}$, then $\mathrm{diam}(\Gamma(G)) = 2$. So suppose that $|I| = n \geq 2$. If $a = (a_1, \ldots, a_n) \in A$ and $b = (b_1, \ldots, b_n) \in B$ are two non-zero elements of G, then we show that there exists a proper subgroup W of G such that $W + \langle a \rangle = W + \langle b \rangle = G$. If there exists i, $1 \leq i \leq n$, such that $a_i \neq 0$ and $b_i \neq 0$, then define $W = \langle e_1, \ldots, e_{i-1}, (3a_i b_i + 1)e_i, e_{i+1}, \ldots, e_n \rangle$. It is not hard to see that W is a proper subgroup of G and $W + \langle a \rangle = W + \langle b \rangle = G$. Otherwise, there are

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two distinct indices $i, j, 1 \le i, j \le n$, such that $a_i \ne 0$ and $b_j \ne 0$. Set

$$W = \langle e_1, \ldots, e_{i-1}, e_i + a_i b_j e_j, e_{i+1}, \ldots, e_{j-1}, e_j + a_i b_j e_i, e_{j+1}, \ldots, e_n \rangle.$$

Let M be a matrix such that the columns of M are generators of W. If $W = G$, then the columns of M are generators for $\mathbb{Z}^n$ and so $M \in GL_n(\mathbb{Z})$. But this contradicts $\det M \ne \pm 1$. It is not hard to check that $W + \langle a \rangle = W + \langle b \rangle = G$, as desired. Now suppose that I is infinite. Let $a = (a_i)_{i \in I} \in A$ and $b = (b_i)_{i \in I} \in B$ be two non-zero elements. Let J be a finite subset of I such that for every $i \in I \setminus J$, $a_i = b_i = 0$ and $|J| = m$. Again we show that there exists a subgroup W such that $W + \langle a \rangle = W + \langle b \rangle = G$. Thus $\langle a \rangle, \langle b \rangle \subseteq \mathbb{Z}^m$. By the previous case, there exists a subgroup W_1 such that $W_1 + \langle a \rangle = W_1 + \langle b \rangle = \mathbb{Z}^m$. Let $W = \langle W_1, e_{m+1}, e_{m+2}, \ldots \rangle$. Obviously, $W \ne \bigoplus_{i \in I} \mathbb{Z}$ and $W + \langle a \rangle = W + \langle b \rangle = \bigoplus_{i \in I} \mathbb{Z}$. Thus $\operatorname{diam}(\Gamma(G)) \le 2$. Since $\langle 2e_i \rangle$ and $\langle 2e_j \rangle$ are not adjacent, we find that $\operatorname{diam}(\Gamma(G)) \ne 1$, as desired. ■

In the next theorem, we characterize all finitely generated nilpotent groups whose co-maximal graphs have diameter 2.

Theorem 3.3 *Let G be a finitely generated nilpotent group. Then* $\operatorname{diam}(\Gamma(G)) = 2$ *if and only if for some prime p, $G \cong (\bigoplus \mathbb{Z}_p) \oplus (\bigoplus \mathbb{Z})$ and $G \ncong \mathbb{Z}_p \oplus \mathbb{Z}_p$.*

Proof Suppose that $\operatorname{diam}(\Gamma(G)) = 2$. Since $\delta(G) \ge 1$, by Remark 2.5, G is an abelian group. It follows from the fundamental theorem of finitely generated abelian groups [17, 5.4.2] that G is a direct sum of finitely many cyclic groups. If G has no torsion part, then we are done. Now we claim that the torsion part of G is a p-group. Assume that H and K are two non-isomorphic cyclic subgroups of G whose orders are distinct primes. If H and K are adjacent, then $|G| = pq$ and $\Gamma(G) = K_2$, a contradiction. So suppose that there exists a subgroup L such that $H \oplus L = K \oplus L = G$. Then we find that $H \cong K$, a contradiction, and the claim is proved. Since $\Gamma(G)$ is connected by Theorem 2.7, the co-maximal graph of torsion part of G is connected, too. It follows from Corollary 2.8 that the torsion part of G is an elementary abelian p-group. The converse follows from Lemma 3.1 and Theorem 3.2. ■

The following theorem characterizes all finite groups whose co-maximal graphs have diameter 2.

Theorem 3.4 *Let G be a finite group. Then* $\operatorname{diam}(\Gamma(G)) = 2$ *if and only if $G \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p$, where p and q are distinct primes or G is isomorphic to an elementary abelian p-group and $G \ncong \mathbb{Z}_p \oplus \mathbb{Z}_p$.*

Proof Assume that $\operatorname{diam}(\Gamma(G)) = 2$. Let p be the smallest prime number such that p divides $|G|$. If there exists another prime number q that divides $|G|$, then there exist subgroups H and K such that $|H| = p$ and $|K| = q$. If $HK = G$, then by [12, Proposition 6.1] we have $G \cong \mathbb{Z}_q \oplus \mathbb{Z}_p$ or $G \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p$. Note that $\Gamma(\mathbb{Z}_q \oplus \mathbb{Z}_p) = K_2$. Now we can assume that H and K are not adjacent vertices. We claim that G is a p-group. Since $\operatorname{diam}(\Gamma(G)) = 2$, there exists a complement subgroup L of H and K. By [12, Corollary 4.10] L is a normal subgroup and we have $H \cong K$, a contradiction. So G is a p-group and the claim is proved. It follows from Lemma 3.1 that G is an elementary abelian

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p-group. Conversely, assume that G is an elementary abelian p-group. By Lemma 3.1, $\text{diam}(\Gamma(G)) = 2$. Now assume that $G \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p$, where p and q are distinct prime numbers. It is not hard to see that $\text{diam}(\Gamma(G)) = 2$. ∎

In the sequel, we study those groups whose co-maximal graphs are complete. We show that if G is a nilpotent group and $\Gamma(G)$ has a vertex adjacent to all other vertices, then $\Gamma(G)$ is a complete graph.

Theorem 3.5 *Let G be a nilpotent group. Then the following are equivalent.*

(i) *There exists a vertex adjacent to all other vertices of $\Gamma(G)$.*
(ii) *$G \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$, where p and q are (not necessarily distinct) primes.*
(iii) *The graph $\Gamma(G)$ is complete.*

Proof (i) ⇒ (ii). Let H be a vertex of $\Gamma(G)$ adjacent to all other vertices. We claim that every vertex of $\Gamma(G)$ is a maximal with prime order. It is easy to see that H is a maximal subgroup of prime order of G. Now we show that every other vertex is a maximal subgroup. Let $H \neq K$ be an arbitrary vertex. Since K is a complement of H, by [12, Lemma 7.4], we have $K \cong G/H$ and so K is a maximal subgroup of prime order and the claim is proved. So $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$, where p and q are primes.

(ii) ⇒ (iii) and (iii) ⇒ (i) are clear. ∎

It is worth mentioning that $\Gamma(\mathbb{Z}_p \oplus \mathbb{Z}_q) = K_2$, where $p \neq q$ and $\Gamma(\mathbb{Z}_p \oplus \mathbb{Z}_p) = K_{p+1}$. The condition of G to be nilpotent in the previous theorem is necessary; see the next example.

Example 3.6 Consider the symmetric group S_3. Then $\Gamma(S_3)$ is $K_{1,3}$ and the subgroup $\langle(1\,2\,3)\rangle$ is adjacent to every other vertex in $\Gamma(S_3)$.

Corollary 3.7 *The graph $\Gamma(G)$ is complete if and only if $G \in \{\mathbb{Z}_{p^2}, \mathbb{Z}_p \oplus \mathbb{Z}_q\}$, where p and q are (not necessarily distinct) primes.*

4 Finiteness of Degree in Co-maximal Graphs

In this section, we show that if the minimum degree is at least 1 and the degree of each vertex corresponding to a maximal subgroup of a co-maximal graph is finite, then the co-maximal graph is finite. Also, we prove that if G is a finitely generated solvable group, $\delta(\Gamma(G)) \geq 1$ and $\deg(M) < \infty$ for a maximal subgroup M, then G is a finite group.

Theorem 4.1 *Let G be a group and $\delta(\Gamma(G)) \geq 1$. If $\deg(M) < \infty$ for every maximal subgroup M, then G is a finite group.*

Proof Let M be an arbitrary maximal subgroup. We claim that M is finite. Assume to the contrary that $\{H_i\}_{i\geq 1}$ is an infinite family of non-trivial subgroups of M; see [12, Exercise 8, p. 37]. Since $\delta(\Gamma(G)) \geq 1$ and $\deg(M) < \infty$, there exists at least one vertex K_i that is adjacent to H_i and each K_i is contained in a maximal subgroup M_i, for $i \geq 1$. If the number of these maximal subgroups is infinite, then $\deg(M) = \infty$, which

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is impossible. If the number of these maximal subgroups is finite, then $\deg(M_i) = \infty$, for some i, a contradiction. Thus M is finite and the claim is proved. Since $\deg(M)$ is finite, there exists another maximal subgroup M_1 such that $MM_1 = G$. So we deduce that G is a finite group, as desired. ∎

Remark 4.2 The condition of $\delta(\Gamma(G)) \geq 1$ is necessary. For instance, consider a Tarski Monster group. The degree of every maximal subgroup is finite but a Tarski Monster is not a finite group.

Theorem 4.3 *Let G be a group. If $\deg(M) < \infty$ for a normal maximal subgroup M, then G is a finite group.*

Proof Let $a \in G \setminus M$ and $o(a) = \infty$. Then there exist infinitely many natural numbers $\{n_i\}_{i\geq 1}$ such that $a^{n_i} \in G \setminus M$, for $i \geq 1$. Hence, there exist infinitely many distinct subgroups which are not contained in M which contradicts $\deg(M) < \infty$. Hence $o(a) < \infty$, for every $a \in G \setminus M$. We show that $|G \setminus M| < \infty$. Assume to the contrary that $|G \setminus M| = \infty$. Since $o(a) < \infty$ for every $a \in G \setminus M$, we deduce that there exist infinitely many subgroups which are adjacent to M, a contradiction. ∎

The requirement that M be a maximal subgroup in Theorem 4.3 is necessary. For instance, consider the subgroup $\mathbb{Z}_2 \times 0$ of $\mathbb{Z}_2 \times \mathbb{Z}$. Then $\deg(\mathbb{Z}_2 \times 0) = 1$ and $\Gamma(\mathbb{Z}_2 \times \mathbb{Z})$ is connected.

It is interesting to investigate the correctness of Theorem 4.3 if the maximal subgroup M is not normal. First, we need the following lemma.

Lemma 4.4 *Let G be a group and $\delta(\Gamma(G)) \geq 1$. If $\deg(M) < \infty$ for some maximal subgroup M, then M is a torsion subgroup.*

Proof Assume to the contrary that $a \in M$ has infinite order. Let $A = \{K_l\}_{l\in L}$ be the set of all vertices adjacent to $\langle a^i \rangle$, for $i \geq 1$. With no loss of generality, one can suppose that each K_l is maximal. Since $\deg(M) < \infty$, we conclude that A is finite. We claim that there exist i and l such that K_l is a complement of $\langle a^i \rangle$. Suppose that K_1 is adjacent to $\langle a \rangle$. If K_1 is a complement of $\langle a \rangle$, then we are done. So assume that $\langle a^{i_1} \rangle = \langle a \rangle \cap K_1$. Since $\delta(\Gamma(G)) \geq 1$, there exists a vertex K_2 which is adjacent to $\langle a^{i_1} \rangle$. If K_2 is a complement of $\langle a^{i_1} \rangle$, then we are done. So assume that $\langle a^{i_2} \rangle = \langle a^{i_1} \rangle \cap K_2$. By repeating this procedure, since A is finite, we find that there exists $\langle a^{i_j} \rangle$ such that $\langle a^{i_j} \rangle$ is contained in K_l, for every $l \in L$, which is impossible. So the claim is proved. By [19, Exercise 7, p.29], $MK_l^x = G$, for every $x \in G$ and so $[G : N_G(K_l)] < \infty$. Since K_l is a maximal subgroup, $[G : K_l]$ is a finite number and so there exist distinct integers n_1 and n_2 such that $a^{in_1}K_l = a^{in_2}K_l$. Thus $a^{in_1 - in_2} \in \langle a^i \rangle \cap K_l$, a contradiction. So M is torsion, as desired. ∎

Theorem 4.5 *Let G be a finitely generated solvable group. If $\delta(\Gamma(G)) \geq 1$ and $\deg(M) < \infty$ for some maximal subgroup M, then G is finite.*

Proof Let M be a maximal subgroup of G such that $\deg(M) < \infty$. Assume that n is the smallest number such that $G^n = 1$. So G^{n-1} is an abelian normal subgroup of

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G. If $G^{n-1} \nsubseteq M$, then $G^{n-1}M = G$ and so $[G:G^{n-1}] < \infty$. It follows from [17, Exercise 8.4.33, Part (a)] that G^{n-1} is finitely generated and hence $G^{n-1} \cong \mathbb{Z}^r \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$. If $r \geq 1$, then $p_i\mathbb{Z}^r$ is a characteristic subgroup of G^{n-1}, where p_i is the i-th prime number and so by [17, Exercise 2.11.13], G has infinitely many normal subgroups. By Lemma 4.4, M is torsion and hence $\deg(M) = \infty$, a contradiction. Thus $r = 0$. Since $[G:G^{n-1}] < \infty$, we conclude that G is finite. Now let $G^{n-1} \subseteq M$. Assume that $M_1 = M/G^{n-1}$ and $G_1 = G/G^{n-1}$. Now consider G^{n-2}/G^{n-1}. If $G^{n-2} \nsubseteq M$, then by a similar argument we conclude that $[G:G^{n-1}] < \infty$. It follows from [17, Exercise 8.4.33, Part (a)] that G^{n-1} is finitely generated and so by [17, Exercise 8.4.36], G^{n-1} is finite. Thus G is finite. Now if $G^{n-2} \subseteq M$, then by repeating the previous procedure, we find that $G' \subseteq M$. Thus M is a normal subgroup of G. Now Theorem 4.3 implies that G is a finite group. ■

Theorem 4.6 *Let G be a metabelian group. If* $\delta(\Gamma(G)) \geq 1$ *and* $\deg(M) < \infty$ *for a maximal subgroup M, then G is a finite group.*

Proof If $G' \subseteq M$, then by Theorem 4.3 G is finite. Now suppose that $G' \nsubseteq M$. Thus $G'M = G$ and so $[G:G'] < \infty$. Therefore, $[M:M \cap G'] < \infty$. If $M \cap G' = 1$, then M is finite and so G is finitely generated. Now the result follows from Theorem 4.5. Hence assume that $M \cap G' \neq 1$. We claim that $|M \cap G'| < \infty$. Assume to the contrary that $|M \cap G'| = \infty$. Lemma 4.4 implies that M is torsion. We show that the set of prime divisors of the orders of elements of M is finite. Let $o(m_i) = p_i$, where p_i is a prime number and $m_i \in M$. There exists M_i such that $\langle m_i \rangle M_i = G$ and so $[G:M_i] = p_i$. Since $\deg(M) < \infty$, we can deduce that the set of prime divisors of the orders of elements of M is finite. By [17, 5.1.6], $M \cap G' = A_{p_1} \oplus \cdots \oplus A_{p_t}$, where every A_{p_i} is a p_i-component. Since $\delta(\Gamma(G)) \geq 1$, it follows from Theorem 2.2 and Theorem 2.7 that $\Gamma(A_{p_i})$ is connected. By Corollary 2.8, A_{p_i} is an elementary abelian p_i-group. Since $|M \cap G'| = \infty$, we deduce that A_{p_i} is infinite for some i. Since $\deg(M) < \infty$, we have $\deg(\langle a \rangle) < \infty$ for every $1 \neq a \in A_{p_i}$. Thus there exists a vertex M_1 which is adjacent to infinitely many vertices $\langle x_i \rangle$, where $x_i \in A_{p_i}$. Let $X = \{x_i\}_{i=1}^{\infty}$, $a_1 = x_1$, and $a_{k+1} \in X \setminus \langle a_1 \rangle \cdots \langle a_k \rangle$, for every $k \geq 1$. Since $G = \langle a_k \rangle M_1$, $k \geq 1$, we deduce that there exists i_k such that $a_1^{-1}a_k^{i_k} \in M_1$. Set $b_k = a_1^{-1}a_{k+1}^{i_{k+1}}$. Note that $b_i \neq b_j$ for every $i \neq j$. Thus there exists a vertex M_2 which is adjacent to infinitely many vertices $\langle b_i \rangle$. The equality $G = \langle b_k \rangle M_2$, $k \geq 1$ implies that there exists j_k such that $b_1^{-1}b_k^{j_k} \in M_2$. Define $c_k = b_1^{-1}b_{k+1}^{j_{k+1}}$. It is not hard to check that $c_i \neq c_j$, for every $i \neq j$ and moreover $c_k \neq a_i$, for each k, i. But every vertex $\langle c_k \rangle$ is not adjacent to M_1 and M_2. Thus there exists a vertex M_3 which is adjacent to infinitely many $\langle c_k \rangle$. Repeating this procedure shows that $\deg(M) = \infty$, a contradiction. So the claim is proved and since $[M:M \cap G'] < \infty$, M is finite and so G is finitely generated. Now by Theorem 4.5 the proof is complete. ■

Now we propose the following conjecture.

Conjecture *Let G be a solvable group and* $\delta(\Gamma(G)) \geq 1$. *If* $\deg(M) < \infty$ *for a maximal subgroup M, then G is a finite group.*

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Remark 4.7 Let G be a simple group. If there exists a vertex H in $\Gamma(G)$ such that $1 \le \deg(H) < \infty$, then G is finite. To see this, let K be a subgroup such that $G = HK$. By [19, Exercise 7, p. 29], $HK^x = G$ for every $x \in G$ and so $[G:N_G(K)] < \infty$. Since G is simple, we conclude that G is a finite group.

Theorem 4.8 *Let G be a finitely generated torsion group and $\delta(\Gamma(G)) \ge 1$. Then for every $g \in G$,* $\deg(\langle g \rangle) < \infty$.

Proof Assume to the contrary that $\langle g \rangle$ is a vertex of infinite degree. If K is adjacent to $\langle g \rangle$, then $[G:K] < \infty$. By [19, Exercise 3, p. 66], G contains finitely many subgroups of index at most $[G:K]$, a contradiction. ∎

5 Co-maximal Graphs of Linear Groups

Let F be a field and n be a positive integer. In this section, we investigate the degrees of vertices of the co-maximal graph of $\mathrm{GL}_n(F)$. For instance, we show that if K is a proper subgroup of $\mathrm{GL}_n(F)$, where F is an algebraically closed field, then $\deg(K) = 0$ or $\deg(K) = \infty$.

Theorem 5.1 *Let F be a field and n be a positive integer. If* $\deg(M)$ *is finite in* $\Gamma(\mathrm{GL}_n(F))$*, for every $M \in Max(\mathrm{GL}_n(F))$, then F is a finite field.*

Proof If $n = 1$, then by Theorem 4.3 G is finite. Thus assume that $n \ge 2$. Let $M \in \mathrm{Max}(\mathrm{GL}_n(F))$. If $\mathrm{SL}_n(F) \subseteq M$, then M is normal and so the result follows from Theorem 4.3. Hence, one may assume that $\mathrm{SL}_n(F) \nsubseteq M$. Assume to the contrary that F is infinite. It follows from [5, Theorem 1] that $F^\times = F \setminus \{0\}$ is not finitely generated. Thus we have the following chain: $\langle x_1 \rangle \subsetneq \langle x_1, x_2 \rangle \subsetneq \cdots$, where $x_i \in F^\times$, for $i \ge 1$. Let $N_i = \langle \left[\begin{smallmatrix} I & 0 \\ 0 & x_i \end{smallmatrix}\right] \rangle \mathrm{SL}_n(F)$. It is clear that $N_i \ne N_j$ for all $i \ne j$, a contradiction. ∎

Now we show that if F is an algebraically closed field, then each vertex in $\Gamma(\mathrm{GL}_n(F))$ is either isolated or has infinite degree.

Theorem 5.2 *Let F be an algebraically closed field, n be a positive integer and $K \in V(\Gamma(\mathrm{GL}_n(F)))$. Then either* $\deg(K) = 0$ *or* $\deg(K) = \infty$.

Proof Assume that $1 \le \deg(K) < \infty$ and H is a vertex adjacent to K. By [19, Exercise 7, p. 29], $KH^x = \mathrm{GL}_n(F)$ for every $x \in \mathrm{GL}_n(F)$ and so $[\mathrm{GL}_n(F):N_{\mathrm{GL}_n(F)}(H)] < \infty$. So there exists a normal subgroup $N \subseteq N_{\mathrm{GL}_n(F)}(H)$ such that $[\mathrm{GL}_n(F):N] < \infty$. Since N is a normal subgroup, we conclude that either N is central or $\mathrm{SL}_n(F) \subseteq N$. Clearly, N is not central and so $\mathrm{SL}_n(F) < N$. Thus $N_{\mathrm{GL}_n(F)}(H)$ contains $\mathrm{SL}_n(F)$ and so $N_{\mathrm{GL}_n(F)}(H)$ is normal. By [4, Theorem 11], H is a normal subgroup of $\mathrm{GL}_n(F)$. Therefore, either H is central or $\mathrm{SL}_n(F) \subseteq H$. Now we divide the proof into the two following cases.

Case 1: The subgroup H is central. Since F is infinite, it follows from [5, Theorem 1] that $F^\times$ is not finitely generated. Thus we have following chain: $\langle x_1 \rangle \subsetneq \langle x_1, x_2 \rangle \subsetneq \cdots$, where $x_i \in F$ for $i \ge 1$. Let $L_i = \langle \left[\begin{smallmatrix} I & 0 \\ 0 & x_i \end{smallmatrix}\right] \rangle H$. Then each L_i is adjacent to K. Since

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$\deg(K) < \infty$, we have $x_i = x_j^t$ for some i, j, t, which is impossible. Therefore, Case 1 does not occur.

Case 2: The subgroup H contains $\mathrm{SL}_n(F)$. We claim that $\det(H) \neq F^\times$. Assume that $\det(H) = F^\times$. Let B be an arbitrary element of $\mathrm{GL}_n(F)$ and $\det(B) = b$. We know that there exists $A \in H$ such that $\det(A) = b$. Hence $BA^{-1} \in \mathrm{SL}_n(F)$ and so $B \in H$. This implies that $H = \mathrm{GL}_n(F)$, a contradiction, so the claim is proved. Since F is an algebraically closed field, we conclude that the index of $\det(H)$ in $F^\times$ is infinite and so $[\mathrm{GL}_n(F) : H]$ is infinite. Hence $\deg(K)$ is infinite, a contradiction. ■

We close this section with the following result.

Theorem 5.3 *Let D be a division ring, n be a positive integer, and* $\deg(\mathrm{SL}_n(D)) < \infty$ *in* $\Gamma(\mathrm{GL}_n(D))$. *Then D is a finite field.*

Proof By Wedderburn's theorem (see [17, 14.1.4]), it is enough to show that D is finite. Assume to the contrary that D is infinite. Note that every element of $\mathrm{GL}_n(D)$ is a product of a diagonal matrix and an element of $\mathrm{SL}_n(D)$. So $1 \leq \deg(\mathrm{SL}_n(D))$. Let L be a subgroup such that $\mathrm{SL}_n(D)L = \mathrm{GL}_n(D)$. Since $\deg(\mathrm{SL}_n(D)) < \infty$, one can suppose that L is a maximal subgroup. We consider two cases.

Case 1: The subgroup L is central. Since $\deg(\mathrm{SL}_n(D)) < \infty$ and L is a normal subgroup, $[\mathrm{GL}_n(D) : L] < \infty$, a contradiction.

Case 2: The subgroup L is not central. One may assume that L is not normal. It follows from [19, Exercise 7, p. 29], that$[\mathrm{GL}_n(D) : L] < \infty$. So $\mathrm{GL}_n(D)$ contains a normal subgroup of finite index, say $N < L$. Therefore, either N is central or contains $\mathrm{SL}_n(D)$. But each case leads to a contradiction, and the proof is complete. ■

6 Some Further Results

A natural question can be raised: for a given group G, when is $\Gamma(G)$ a tree? In this section we completely answer this question for finite groups. Also, we characterize all finitely-generated abelian groups whose co-maximal graphs are forests.

Lemma 6.1 *Let G be a nilpotent group. Then $\Gamma(G)$ is a tree if and only if $G \cong \mathbb{Z}_{pq}$, where p and q are two prime numbers.*

Proof Let $\Gamma(G)$ be a tree. If $\Gamma(G)$ has one vertex, then G is $\mathbb{Z}_{p^2}$. So we can assume that $\Gamma(G)$ has at least two vertices. By Corollary 2.1, we have $\Phi(G) = 1$. Since $\Gamma(G)$ is a tree, G has at least two maximal subgroups. It follows from [12, Lemma 7.4] that every maximal subgroup is normal. Now if G has more than two maximal subgroups, then $\Gamma(G)$ has a cycle, which is impossible. Suppose that $\mathrm{Max}(G) = \{M_1, M_2\}$. The equality $\Phi(G) = 1$ implies that $M_1 \cap M_2 = 1$. So G is a subgroup of $\mathbb{Z}_p \oplus \mathbb{Z}_q$, where p and q are prime numbers. Clearly, $G \ncong \mathbb{Z}_p$ and $G \ncong \mathbb{Z}_q$. Also, if $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$, then $\Gamma(G)$ is not a tree and so $G \cong \mathbb{Z}_{pq}$, where p and q are distinct primes. The converse is clear. ■

Using Lemma 6.1, we have the following theorem.

Theorem 6.2 *Let G be a finite group. Then $\Gamma(G)$ is a tree if and only if*

$$G \in \{\mathbb{Z}_{pq}, \mathbb{Z}_q \rtimes \mathbb{Z}_p\},$$

where p and q are prime numbers.

Proof If $\Gamma(G)$ has one vertex, then G is $\mathbb{Z}_{p^2}$. So we can assume that $\Gamma(G)$ has at least two vertices. Also, it follows from Lemma 6.1 that G is not a p-group. Let p be the smallest prime number which divides $|G|$ and $o(g) = p$, for some $g \in G$. Assume that M is a maximal subgroup of G which is adjacent to $\langle g \rangle$. Since $|G| = p|M|$, by [19, Exercise 3 p. 34] we may suppose that M is normal. We claim that $M \cap M_i = 1$ for every $M \neq M_i \in \mathrm{Max}(G)$. To the contrary, suppose that $M \cap M_i \neq 1$ for some $M \neq M_i \in \mathrm{Max}(G)$. Since $\Gamma(G)$ is connected, there exists a subgroup L such that $L(M \cap M_i) = G$. So we have the cycle $L—M—M_i—L$, a contradiction. Hence the claim is proved. We show that $|G| = pq$, where p and q are distinct primes.

Since G is not a p-group, G contains an element g' of a prime order $q \neq p$. The connectivity of $\Gamma(G)$ implies that there exists a maximal subgroup M_j such that $G = \langle g' \rangle M_j$. Hence G contains a maximal subgroup M_j such that $[G{:}M_j] = q$, where $q \neq p$ is a prime and $MM_j = G$. Thus $|G| = pq$ and by [12, Proposition 6.1] we have $G \cong \mathbb{Z}_{pq}$ or $G \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p$. The converse is clear. ■

Theorem 6.3 *Let G be a finitely generated abelian group. Then $\Gamma(G)$ is a forest if and only if $G \cong \mathbb{Z}_{p^m q}$, where p and q are two primes and $m \geq 0$.*

Proof Suppose that $\Gamma(G)$ is a forest. So we conclude that G has at most two maximal subgroups. If G has exactly one maximal subgroup, then $G \cong \mathbb{Z}_{p^k}$, for some positive integer k. Now assume that G has two maximal subgroups, say M_1, M_2. So $G \cong \mathbb{Z}_{p^m q^n}$, where p and q are two distinct prime numbers. Suppose that $m \geq 1$. Then we show that $n = 1$. Assume to the contrary that $n > 1$. With no loss of generality, suppose that the order of M_1 is $p^{m-1}q^n$ and the order of M_2 is $p^m q^{n-1}$. Let H_1 and H_2 be the subgroup of order q^n and p^m, respectively. Since $H_1 \cap H_2 = 0$, we deduce that H_1 and H_2 are two adjacent vertices in $\Gamma(G)$. So $\Gamma(G)$ has the cycle $H_1—M_2—M_1—H_2—H_1$, a contradiction. Conversely, if $G \cong \mathbb{Z}_{p^m}$, then by Remark 2.9 (ii) we are done. Thus suppose that $G \cong \mathbb{Z}_{p^m q}$, where p and q are two distinct prime numbers and $m \geq 1$. We show that $\Gamma(G)$ is the union of $K_{1,m}$ and $m-1$ copies of K_1. Let H_i be the subgroup of order p^i for $i = 1, \dots, m$ and K_i be the subgroup of order $p^i q$ for $i = 0, \dots, m-1$. Note that H_m is a maximal subgroup which is adjacent to each K_i for $i = 0, \dots, m-1$. Also, each H_i is an isolated vertex for $i = 1, \dots, m-1$. This completes the proof. ■

In the following theorem, we find the clique number and the chromatic number of a co-maximal graph of a finitely generated nilpotent group.

Theorem 6.4 *Let G be a finitely generated nilpotent group. Then*

$$\omega(\Gamma(G)) = \chi(\Gamma(G)) = |\mathrm{Max}(G)|.$$

Proof Since every maximal subgroup is normal (see [12, Lemma 7.4]), the set of maximal subgroups of G forms a clique. We color maximal subgroups with different colors. Let H be a vertex of $\Gamma(G)$. Since H is contained in a maximal subgroup M, we assign the color of M to H. If H is contained in more than one subgroup, choose one of them. Now we show that this coloring is proper. Let K and L be two adjacent vertices. Since $KL = G$, we conclude that K and L are contained in different maximal subgroups and so they have different colors, as desired. ■

By a similar argument to the proof of Theorem 6.4, one can show that $\Gamma(G)$ is a perfect graph.

We close this paper with the following question: is it true that for every group G, $\omega(\Gamma(G)) = \chi(\Gamma(G))$?

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Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran
e-mail: s_akbari@sharif.edu babak.math@gmail.com

Department of Basic Sciences, Jundi-Shapur University of Technology, Dezful, Iran
e-mail: r.nikandish@ipm.ir