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WIENER TAUBERIAN THEOREMS FOR CERTAIN BANACH ALGEBRAS ON REAL RANK ONE SEMISIMPLE LIE GROUPS

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Abstract

We prove Wiener Tauberian theorem type results for various spaces of radial functions, which are Banach algebras on a real-rank-one semisimple Lie group G. These are natural generalizations of the Wiener Tauberian theorem for the commutative Banach algebra of the integrable radial functions on G.

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1. Introduction

The celebrated Wiener Tauberian theorem states that given a function $f \in L^1(\mathbb{R})$, the span of translations $f_a(x) = f(x - a)\,(a \in \mathbb{R})$, or the span of $\{f * g : g \in L^1(\mathbb{R})\}$ is dense in $L^1(\mathbb{R})$ if and only if the Fourier transform of the function f has no real zeros. This theorem has been extended to locally compact abelian groups. In 1955, Ehrenpreis and Mautner observed that the exact analogue of the theorem above fails for the commutative algebra of the integrable K-biinvariant functions on the group $G = \mathrm{SL}(2, \mathbb{R})$, where $K = \mathrm{SO}(2)$ is a maximal compact subgroup of G. Moreover, the Wiener Tauberian theorem does not hold for any noncompact connected semisimple Lie group [8, 9]. Nonetheless, the authors (in [8]) realized that in addition to the nonvanishing condition of the Fourier transforms, a condition on the rate of decay of Fourier transforms at infinity is also necessary. Generally, when f is a K-biinvariant integrable function on G, its Fourier transform is well defined on the strip $S_1 = \{\lambda \in \mathbb{C} : |\mathrm{Im}\lambda| \leq 1\}$. However, for technical reasons, it was necessary for the authors to impose various smoothness conditions and nonvanishing conditions of the Fourier transforms on the *extended strip* $S_{1,\delta} := \{\lambda \in \mathbb{C} : |\mathrm{Im}\lambda| \leq 1 + \delta\}$ for $\delta > 0$. Their theorem is the following.

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THEOREM 1.1 [8, Theorem 6]. *Let f be a K-biinvariant integrable function on $G = \mathrm{SL}(2,\mathbb{R})$ such that:*

(1) *the spherical transform $\widehat{f}$ is a continuous function on the extended strip $S_{1,\delta}$ for $\delta > 0$ and holomorphic on the interior of $S_{1,\delta}$;*
(2) $\lim_{|\lambda|\to\infty} \widehat{f}(\lambda) = 0$ *in $S_{1,\delta}$;*
(3) *$\widehat{f}$ does not vanish on the extended strip $S_{1,\delta}$ and*
(4) $\limsup_{|t|\to\infty} |\widehat{f}(t)|e^{Ke^{|t|}} > 0$ *for all $K > 0$.*

Then the ideal generated by f in $L^1(G//K)$ is dense in $L^1(G//K)$.

Using the nonvanishing condition of the Fourier transforms on the extended strip, the result has been generalized to the full group $\mathrm{SL}(2,\mathbb{R})$ (see [23]) and to the rank-one symmetric spaces (see [3, 24, 25]). For the Wiener Tauberian theorem on symmetric spaces of arbitrary rank, we refer to [18, 19].

In 1995, Ben Natan *et al.* (in [2]) proved an analogue of the Wiener Tauberian theorem in $L^1(\mathrm{SL}(2,\mathbb{R})//\mathrm{SO}(2))$ without any superfluous smoothness conditions or nonvanishing conditions in the extended strip. The main ingredient of their proof is the resolvent transform method developed by Carleman [4] and Domar [7]. In [21], the authors extended the result of [2] to a real-rank-one semisimple Lie group in the K-biinvariant setting.

In 2006, Dahlner (in [6]) gave a qualitative generalization of the result of Ben Natan *et al.* [2] to $L^1(\mathrm{SL}(2,\mathbb{R})//\mathrm{SO}(2),\omega)$ – the convolution algebra of SO(2)-biinvariant functions on $\mathrm{SL}(2,\mathbb{R})$ that are integrable with respect to certain weights ω, where the weight function ω behaves like a Legendre function of the first kind. Our aim is to extend this result to any connected noncompact semisimple real-rank-one Lie group G with finite centre. More precisely, extending the result in [6] to all rank-one cases and enlarging the class of weight functions at the same time, we show that the Wiener Tauberian theorem holds for weighted spaces of K-biinvariant integral functions on the group G, where K is a maximal compact subgroup of G.

For $\lambda \in \mathbb{C}$, let ϕ_λ denote the Harish-Chandra spherical functions on G. We define $\mathbf{S}_\alpha = \{\lambda \in \mathbb{C} : |\mathrm{Im}\lambda| \le \alpha\}$ for $\alpha > 0, r \ge 0$, and

$$\omega_{\alpha,r}(x) = \phi_{i\alpha}(x)(1 + x^+)^r \quad \text{for all } x \in G.$$

Then $L^1(G//K,\omega_{\alpha,r})$ are Banach algebras under convolution, and on these algebras, we prove the following analogue of Wiener Tauberian theorem.

THEOREM 1.2. *Suppose $\{f_\beta : \beta \in \Lambda\}$ is a collection of functions in $L^1(G//K,\omega_{\alpha,r})$ for fixed α and r, such that $\{\widehat{f_\beta} : \beta \in \Lambda\}$ have no common zero in $\mathbf{S}_\alpha$ and $\inf_{\beta\in\Lambda} \delta_\infty^{\alpha\pm}(\widehat{f_\beta}) = 0$, where*

$$\delta_\infty^{\alpha\pm}(\widehat{f}) := \limsup_{t\to\infty} e^{-\pi/2\alpha t} \log|\widehat{f}(\pm t)|.$$

Then the ideal generated by $\{f_\beta : \beta \in \Lambda\}$ in $L^1(G//K,\omega_{\alpha,r})$ is dense in $L^1(G//K,\omega_{\alpha,r})$.

Next, let G be a connected noncompact semisimple real-rank-one Lie group with a finite centre. Then by the *Kunze–Stein phenomenon* (see [5], [20, Remark 6.11]), the Lorentz space $L^{p,1}(G)$ is a Banach algebra for $1 \le p < 2$. Hence, in particular, $L^{p,1}(G//K)$ is a commutative Banach algebra. It is of interest to know whether the Wiener Tauberian theorem holds for the spaces above. The author in [22] answered this affirmatively by proving an analogue of the Wiener Tauberian theorem for $L^{p,1}(\mathrm{SL}(2,\mathbb{R}))$ $(1 \le p < 2)$. Our next result is a Wiener Tauberian theorem for $L^{p,1}(G//K)$ $(1 \le p < 2)$, where G is a complex semisimple Lie group of real rank one; that is, $G = \mathrm{SL}(2,\mathbb{C})$.

THEOREM 1.3. *Let $1 \le p < 2$ and $\gamma_p = (2/p - 1)$. Suppose $\{f_\beta : \beta \in \Lambda\}$ is a subset of $L^{p,1}(G//K)$ such that the collection $\{\widehat{f_\beta} : \beta \in \Lambda\}$ has no common zero in $\mathbf{S}_{\gamma_p}$ and $\inf_{\beta\in\Lambda} \delta_\infty^{\gamma_p\pm}(\widehat{f}) = 0$. Then the ideal generated by $\{f_\beta : \beta \in \Lambda\}$ in $L^{p,1}(G//K)$ is dense in $L^{p,1}(G//K)$.*

1.1. Overview of the proof. We mention that to prove our main results, we follow the approach in [2, 21], which uses the resolvent transform method. The outline of the proof of Theorem 1.2 is as follows. We first determine the maximal ideal space of the Banach algebra $L^1(G//K, \omega_{\alpha,r})$. Then the most crucial step in the proof of Theorem 1.2 is to construct a family of K-biinvariant eigenfunctions of the Laplace–Beltrami operator L that spans a dense subspace of $L^1(G//K, \omega_{\alpha,r})$. These eigenfunctions we denote by b_λ for $\lambda \in \mathbb{C}_+ = \{\lambda \in \mathbb{C} : \mathrm{Im}\lambda > 0\}$. We show for $\mathrm{Im}\lambda > \alpha$ that $b_\lambda \in L^1(G//K, \omega_{\alpha,r})$ and the spherical transform of b_λ is

$$\widehat{b_\lambda}(z) = \frac{1}{z^2 - \lambda^2} \quad \text{for all } z \in \mathbf{S}_\alpha.$$

Using the spherical transform of b_λ, we show the collection $\{b_\lambda : \mathrm{Im}\lambda > \alpha\}$ spans a dense subspace of $L^1(G//K, \omega_{\alpha,r})$. Suppose I is the ideal generated by the functions $\{f_\beta : \beta \in \Lambda\}$ in $L^1(G//K, \omega_{\alpha,r})$. Then for each $g \in L^\infty(G//K, 1/\omega_{\alpha,r})$ that annihilates I, we define its resolvent transform $\mathcal{R}[g]$ by

$$\mathcal{R}[g](\lambda) = \langle b_\lambda, g\rangle, \quad \mathrm{Im}\lambda > \alpha.$$

For a fixed $\lambda_0 \in \mathbb{C}$ with $\mathrm{Im}\lambda_0 > \alpha$, using Banach algebra theory, we show that for $\lambda \in \mathbb{C}$,

$$B_\lambda = (\delta - (\lambda^2 - \lambda_0^2)b_{\lambda_0} + I)^{-1} * (b_{\lambda_0} + I)$$

is a $L^1(G//K, \omega_{\alpha,r})/I$-valued even entire function. Again, crucially using the spherical transform of b_λ, we show

$$B_\lambda = b_\lambda \quad \text{for } \mathrm{Im}\lambda > \alpha.$$

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This implies the formula

$$\mathcal{R}[g](\lambda) = \langle B_\lambda, g\rangle \quad \text{for } \lambda \in \mathbb{C}$$

extends $\mathcal{R}[g]$ analytically to the entire complex plane. Next, we find the representatives of the cosets B_λ for $0 < \mathrm{Im}\lambda < \alpha$ to get the explicit formula of $\mathcal{R}[g]$. With this intent, we show that for every $f \in L^1(G//K, \omega_{\alpha,r})$ and $0 < \mathrm{Im}\lambda < \alpha$, there is a function $T_\lambda f$ in $L^1(G//K, \omega_{\alpha,r})$ such that

$$\widehat{T_\lambda f}(z) = \frac{\widehat{f}(\lambda) - \widehat{f}(z)}{z^2 - \lambda^2} \quad \text{for all } z \in \mathbf{S}_\alpha \backslash \{\pm\lambda\}.$$

Then we show, for $f \in I$ and $\widehat{f}(\lambda) \neq 0$, $T_\lambda f / \widehat{f}(\lambda)$ is a representative of B_λ. Since the spherical transforms of the elements of I have no common zero in $\mathbf{S}_\alpha$, such a representation will always exist. Later, we use the expressions of $\mathcal{R}[g]$ to find estimates for the growth of $\mathcal{R}[g](\lambda)$ outside and inside the boundary of $\mathbf{S}_\alpha$. We also have that $\mathcal{R}[g](\lambda)$ vanishes at infinity from the estimate of $\|b_\lambda\|_{L^1(\omega_{\alpha,r})}$.

Finally, using a log–log-type theorem, we show $\mathcal{R}[g]$ is the zero polynomial. So by the denseness of the span of $\{b_\lambda \in \mathbb{C} : \mathrm{Im}\lambda > \alpha\}$, we conclude $g = 0$.

We follow a similar strategy to prove Theorem 1.3.

This article is organized as follows. We introduce some basic notation and well-known results in Section 2. Then in Section 3, we discuss some weighted $L^1(G//K)$ spaces for which we prove the Wiener Tauberian theorem (Theorem 1.2). Finally, in Section 4, we gather some features specific to the complex semisimple Lie group and prove Theorem 1.3.

2. Preliminaries

2.1. Generalities. In this article, most of our notation is standard, which can be found in [6]. We will denote C as a constant, and its value can change from one line to another. For any two positive expressions f_1 and f_2, $f_1 \asymp f_2$ stands for that there are positive constants C_1, C_2 such that $C_1 f_1 \leq f_2 \leq C_2 f_1$. For $z \in \mathbb{C}$, we use $\mathrm{Re}z$ and $\mathrm{Im}z$ to denote the real and imaginary parts of z, respectively.

2.2. Lorentz spaces. Let (X, m) be a σ-finite measure space. For $f : X \to \mathbb{C}$ a measurable function on X, the distribution function d_f defined on $[0, \infty)$ is given by $d_f(\alpha) = m(\{x \in X : |f(x)| > \alpha\})$. Define for $p \in [1, \infty), q \in [1, \infty]$

$$\|f\|_{p,q} = \begin{cases} \left(\dfrac{q}{p} \displaystyle\int_0^\infty [f^*(\alpha)\alpha^{1/p}]^q \dfrac{d\alpha}{\alpha}\right)^{1/q} & \text{when } q < \infty, \\ \sup_{\alpha>0} \alpha^{1/p} f^*(\alpha) & \text{when } q = \infty, \end{cases} \tag{2-1}$$

where $f^*(s) = \inf\{s > 0 : d_f(\alpha) \leq s\}$ is the *nonincreasing rearrangement* of f (see [13, page 45]). The Lorentz spaces $L^{p,q}(X)$ consist of all measurable functions f for which

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$\|f\|_{p,q}$ is finite. For $p, q \in [1, \infty)$, the following identity gives an alternative expression of $\|\cdot\|_{p,q}$ (see [13, Proposition 1.4.9]),

$$p^{1/q}\Big(\int_0^\infty [d_f(\alpha)^{1/p}s]^q \frac{ds}{s}\Big)^{1/q} = \Big(\frac{q}{p}\int_0^\infty [f^*(\alpha)\alpha^{1/p}]^q \frac{d\alpha}{\alpha}\Big)^{1/q}.$$

We need the following lemma.

LEMMA 2.1. *Suppose $1 \le p < q$ and $r \in (p, q)$. Then for all $f \in L^p(X) \cap L^q(X)$, there exists a positive constant C independent of f such that*

$$\|f\|_{r,1} \le C(\|f\|_p + \|f\|_q). \tag{2-2}$$

PROOF. Suppose $f \in L^p(X) \cap L^q(X)$. From Equation (2-1),

$$\|f\|_{r,1} = \frac{1}{r}\int_0^\infty f^*(\alpha)\alpha^{1/(r-1)}\,d\alpha = \frac{1}{r}\int_0^1 f^*(\alpha)\alpha^{1/(r-1)}\,d\alpha + \frac{1}{r}\int_1^\infty f^*(\alpha)\alpha^{1/(r-1)}\,d\alpha.$$

Next, by Hölder's inequality and $1/q < 1/r < 1/p$, the lemma will follow. □

2.3. Result from complex analysis. Now we borrow a result from complex analysis, which is a consequence of a log–log-type theorem. For any function F on $\mathbb{R}$ and $\alpha > 0$, we let

$$\delta_\infty^{\alpha+}(F) = -\limsup_{t\to\infty} e^{-\pi/2\alpha t}\log|F(t)| \quad \text{and} \quad \delta_\infty^{\alpha-}(F) = -\limsup_{t\to\infty} e^{-\pi/2\alpha t}\log|F(-t)|.$$

THEOREM 2.2. *Let $M : (0, \infty) \to (e, \infty)$ be a continuously differentiable decreasing function with*

$$\lim_{t\to 0^+} t\log\log M(t) < \infty, \qquad \int_0^\infty \log\log M(t)dt < \infty.$$

Let Λ be a collection of bounded holomorphic functions on $\mathbf{S}_\alpha^\circ$ such that

$$\inf_{F\in\Lambda}\delta_\infty^{\alpha+}(F) = \inf_{F\in\Lambda}\delta_\infty^{\alpha-}(F) = 0.$$

Suppose H is a function that satisfies the following estimates for some nonnegative integer N:

$$\begin{aligned}|H(z)| &\le (1+|z|)^N M(d(z, \partial\mathbf{S}_\alpha)), \quad z \in \mathbb{C}\setminus\mathbf{S}_\alpha,\\ |F(z)H(z)| &\le (1+|z|)^N M(d(z, \partial\mathbf{S}_\alpha)), \quad z \in \mathbf{S}_\alpha^\circ \quad \textit{for all } F \in \Lambda.\end{aligned}$$

(1) *If, in addition, H is a holomorphic function on $\mathbf{S}_\alpha\setminus\{\pm\alpha\}$, then H is dominated by a polynomial outside a bounded neighbourhood of $\{\pm\alpha\}$.*
(2) *If H is an entire function, then it is a polynomial.*

PROOF. Proof of the theorem above follows as in [21, Theorem 6.3]. □

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2.4. Real variable theory on semisimple Lie groups of rank one. Let G be a noncompact connected semisimple real-rank-one Lie group with finite centre, with Lie algebra $\mathfrak{g}$. Let θ be a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the associated Cartan decomposition. Let $K = \exp \mathfrak{k}$ be a maximal compact subgroup of G. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Since the group G is of real rank one, $\dim \mathfrak{a} = 1$. Let Σ be the set of nonzero roots of the pair $(\mathfrak{g}, \mathfrak{a})$, and let W be the associated Weyl group. For the rank-one case, it is well known that either $\Sigma = \{-\alpha, \alpha\}$ or $\{-2\alpha, -\alpha, \alpha, 2\alpha\}$, where α is a positive root and the Weyl group W associated to Σ is {-Id, Id}, where Id is the identity operator. Let $\mathfrak{a}^+ = \{H \in \mathfrak{a} : \alpha(H) > 0\}$ be a positive Weyl chamber, and let Σ^+ be the corresponding set of positive roots. In our case, $\Sigma^+ = \{\alpha\}$ or $\{\alpha, 2\alpha\}$. For any root $\beta \in \Sigma$, let $\mathfrak{g}_\beta$ be the root space associated to β. Let $\mathfrak{n} = \sum_{\beta\in\Sigma^+} \mathfrak{g}_\beta$ and $N = \exp \mathfrak{n}$. Then the group G has an Iwasawa decomposition $G = K(\exp \mathfrak{a})N$ and a Cartan decomposition $G = K(\exp \overline{\mathfrak{a}^+})K$. These decompositions are unique. For each $g \in G$, we denote $H(g) \in \mathfrak{a}$ and $g^+ \in \overline{\mathfrak{a}^+}$ as the unique elements such that

$$g = k \exp H(g) n, \quad k \in K, n \in N,$$

and

$$g = k_1 \exp(g^+) k_2, \quad k_1, k_2 \in K. \tag{2-3}$$

Let H_0 be the unique element in $\mathfrak{a}$ such that $\alpha(H_0) = 1$, and through this, we identify $\mathfrak{a}$ with $\mathbb{R}$ as $t \leftrightarrow tH_0$ and $\mathfrak{a}_+ = \{H \in \mathfrak{a} : \alpha(H) > 0\}$ is identified with the set of positive real numbers. We also identify $\mathfrak{a}^*$ and its complexification $\mathfrak{a}^*_{\mathbb{C}}$ with $\mathbb{R}$ and $\mathbb{C}$, respectively, by $t \leftrightarrow t\alpha$ and $z \leftrightarrow z\alpha$, $t \in \mathbb{R}$, $z \in \mathbb{C}$. Let $A = \exp \mathfrak{a} = \{a_t := \exp(tH_0) : t \in \mathbb{R}\}$ and $A^+ = \{a_t : t > 0\}$. Let $m_1 = \dim \mathfrak{g}_\alpha$ and $m_2 = \dim \mathfrak{g}_{2\alpha}$, where $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{2\alpha}$ are the root spaces corresponding to α and 2α. Let $\rho = \frac{1}{2}(m_1 + 2m_2)\alpha$ denote the half sum of the positive roots. By abuse of notation, we denote $\rho(H_0) = \frac{1}{2}(m_1 + 2m_2)$ by ρ.

Let dg, dn, dk and dm be the Haar measures of G, N, K and M, respectively, where $\int_K dk = 1$ and $\int_m dm = 1$. We have the following integral formula corresponding to the Cartan decomposition, which holds for any integrable function f:

$$\int_G f(g)dg = \int_K \int_{\mathbb{R}^+} \int_K f(k_1 a_t k_2)\Delta(t)\, dk_1\, dt\, dk_2,$$

where $\Delta(t) = (2\sinh t)^{m_1+m_2}(2\cosh t)^{m_2}$. A function f is called K-biinvariant if

$$f(k_1 x k_2) = f(x) \quad \text{for all } x \in G, k_1, k_2 \in K. \tag{2-4}$$

For a class of functions $\mathcal{F}$ on G, we denote the corresponding subclass of K-biinvariant functions by $\mathcal{F}(G//K)$.

2.5. Spherical function. Let $\mathbb{D}(G/K)$ be the algebra of G-invariant differential operators on G/K. The elementary spherical functions ϕ are C^∞ functions and are joint eigenfunctions of all $D \in \mathbb{D}(G/K)$ for some complex eigenvalue $\lambda(D)$. That is,

$$D\phi = \lambda(D)\phi, \quad D \in \mathbb{D}(G/K).$$

They are parametrized by $\lambda \in \mathbb{C}$. The algebra $\mathbb{D}(G/K)$ is generated by the Laplace–Beltrami operator L. Then we have, for all $\lambda \in \mathbb{C}$, that ϕ_λ is a C^∞ solution of

$$L\phi = -(\lambda^2 + \rho^2)\phi. \tag{2-5}$$

The A-radial part of the Laplace–Beltrami operator is given by

$$L_A f(a_t) := \frac{d^2}{dt^2} f(a_t) + ((m_1 + m_2)\coth t + m_2 \tanh t)\frac{d}{dt} f(a_t), \quad t > 0.$$

For $\lambda \neq -i, -2i, \ldots$, we have another solution Φ_λ of Equation (2-5) on $(0, \infty)$ given by (see [21, Equation (2.7)]),

$$\Phi_\lambda(a_t) = (2\cosh t)^{i\lambda-\rho}\, {}_2F_1\Big(\frac{\rho - i\lambda}{2}, \frac{m_1+2}{4} - \frac{i\lambda}{2}; 1 - i\lambda; \cosh^{-2} t\Big),$$

where ${}_2F_1$ is the Gauss hypergeometric function.

The function Φ_λ has a series representation, called the *Harish-Chandra series* for $t > 0$, and has a singularity at $t = 0$. Using the Cartan decomposition, we extend Φ_λ as a K-biinvariant function on $G/K \setminus \{eK\}$. Therefore, Φ_λ is a solution of Equation (2-5) on $G/K \setminus \{eK\}$ and we also have for $t \to \infty$,

$$\Phi_\lambda(a_t) = e^{(i\lambda-\rho)t}(1 + O(1)). \tag{2-6}$$

For $\lambda \in \mathbb{C} \setminus i\mathbb{Z}$, Φ_λ and $\Phi_{-\lambda}$ are two linearly independent solutions. Therefore, for $\lambda \in \mathbb{C} \setminus i\mathbb{Z}$, ϕ_λ is a linear combination of both Φ_λ and $\Phi_{-\lambda}$; that is,

$$\phi_\lambda = c(\lambda)\Phi_\lambda + c(-\lambda)\Phi_{-\lambda},$$

where $c(\lambda)$ is the Harish-Chandra c-function given by

$$c(\lambda) = \frac{2^{\rho-i\lambda}\Gamma(\frac{m_1+m_2+1}{2})\Gamma(i\lambda)}{\Gamma(\frac{\rho+i\lambda}{2})\Gamma(\frac{m_1+2}{4} + \frac{i\lambda}{2})}.$$

We have the following asymptotic estimate of ϕ_λ (see [14]) for $\mathrm{Im}\lambda < 0$ and $t \to \infty$,

$$\phi_\lambda(a_t) = c(\lambda)e^{(i\lambda-\rho)t}(1 + O(1)). \tag{2-7}$$

The c-function has neither a zero nor a pole in the region $\mathrm{Im}\lambda < 0$ (see [16, Theorem 6.4, Ch. IV]); so it follows that for any fixed $\lambda \in \mathbb{C}$ with $\mathrm{Im}\lambda < 0$,

$$|\phi_\lambda(a_t)| \asymp e^{-(\mathrm{Im}\lambda+\rho)t}. \tag{2-8}$$

For any $\lambda \in \mathbb{C}$, the elementary spherical function ϕ_λ has the following integral representation:

$$\phi_\lambda(x) = \int_K e^{-(i\lambda+\rho)H(xk)}\, dk \quad \text{for all } x \in G. \tag{2-9}$$

We now list down some well-known properties of the elementary spherical functions which are important for us [11, Proposition 3.1.4 and Ch. 4, Section 4.6], [17, Lemma 1.18, page 221].

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(1) $\phi_\lambda(g)$ is K-biinvariant in $g \in G$, $\phi_\lambda = \phi_{-\lambda}$, $\phi_\lambda(g) = \phi_\lambda(g^{-1})$.
(2) $\phi_\lambda(g)$ is C^∞ in $g \in G$ and holomorphic in $\lambda \in \mathbb{C}$.
(3) The following inequality holds:

$$e^{-\rho t} \le \phi_0(a_t) \le (1 + |t|)\, e^{-\rho t}, \quad t \ge 0.$$

(4) $|\phi_\lambda(x)| \le 1$ for all $x \in G$ if and only if $\lambda \in \mathbf{S}_\rho = \{\lambda \in \mathbb{C} : |\mathrm{Im}\lambda| \le \rho\}$.

We also have the following proposition from [15, Ch. IV, Proposition 2.2].

PROPOSITION 2.3. *Let f be a complex-valued continuous function on G, not identically* 0. *Then f is a spherical function if and only if*

$$\int_K f(xky)\, dk = f(x)f(y)$$

for all $x, y \in G$.

The following proposition from [20, Proposition 2.1] will be useful.

LEMMA 2.4. *The elementary spherical function ϕ_λ satisfies the following properties.*

(1) *For any $\lambda_1, \lambda_2 \in \mathbb{C}$ with $|\mathrm{Im}\lambda_1| > |\mathrm{Im}\lambda_2| > 0$ and for any $r \ge 0$, $|\phi_{\lambda_2}(x)|(1 + x^+)^r \le C|\phi_{\lambda_1}(x)|$ for all $x \in G$, where C is a constant depending on λ_1, λ_2 and r.*
(2) *For $1 \le p < 2$, $\phi_\lambda \in L^{p',\infty}(G//K)$ if and only if $\lambda \in \mathbf{S}_{\gamma_p}$.*

2.6. Spherical Fourier transform. The spherical transform $\widehat{f}$ of a suitable K-biinvariant function f is defined by the formula

$$\widehat{f}(\lambda) = \int_G f(x)\phi_\lambda(x^{-1})\, dx.$$

It is well known that if $f \in L^1(G//K)$, then $\widehat{f}$ is analytic on $\mathbf{S}_\rho^\circ$, continuous on $\mathbf{S}_\rho$ and $|\widehat{f}(\lambda)| \to 0$ as $|\lambda| \to \infty$ in $\mathbf{S}_\rho$.

Let $C_c^\infty(G//K)$ be the set of all C^∞ compactly supported K-biinvariant functions on G. Also let $PW(\mathbb{C})$ be the set of all entire functions $h : \mathbb{C} \to \mathbb{C}$ such that h is of exponential type T for some $T > 0$, that is, for each $N \in \mathbb{N}$,

$$\sup_{\lambda \in \mathbb{C}} (1 + |\lambda|)^N |h(\lambda)| e^{-T|\mathrm{Im}\lambda|} < \infty$$

and let $PW(\mathbb{C})_e$ be the set of all even functions in $PW(\mathbb{C})$. Then we have the following Paley–Wiener theorem.

THEOREM 2.5 [1, Theorem 2]. *The function $f \mapsto \widehat{f}$ is a topological isomorphism between $C_c^\infty(G//K)$ and $PW(\mathbb{C})_e$.*

3. Wiener Tauberian theorem on weighted spaces

Let G be a connected, noncompact, real-rank-one semisimple Lie group with finite centre, and K be a maximal compact subgroup of G. For fixed $\alpha > 0$ and $r \ge 0$,

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we define

$$\omega_{\alpha,r}(x) = \phi_{i\alpha}(x)(1 + x^+)^r$$

for all $x \in G$, where we recall x^+ from Equation (2-3). For α and r as above, we define the weighted L^1-spaces as

$$L^1(G//K, \omega_{\alpha,r}) := \{f : G \to \mathbb{C} : f \text{ is measurable and } K\text{-biinvariant with } \|f\|_{L^1_{\omega_{\alpha,r}}} < \infty\},$$

where

$$\|f\|_{L^1_{\omega_{\alpha,r}}} = \int_G |f(x)|\, \omega_{\alpha,r}(x)\, dx.$$

From the inequality in [11, Proposition 4.6.11], we have

$$(1 + y^+)/(1 + x^+) \le (1 + (xy)^+) \le (1 + x^+)(1 + y^+).$$

Then it follows that $L^1(G//K, \omega_{\alpha,r})$ is a Banach algebra.

We note that $L^1(G//K, \omega_{\rho,0}) = L^1(G//K)$, and for convenience henceforth, we write $L^1(\omega_{\alpha,r})$ for $L^1(G//K, \omega_{\alpha,r})$.

First, we determine the maximal ideal space of $L^1(\omega_{\alpha,r})$. Let $\Lambda : L^1(\omega_{\alpha,r}) \to \mathbb{C}$ be a nonzero algebra homomorphism. Then by the Riesz representation theorem, there exists a function $g_\Lambda \in L^\infty(G//K, 1/\omega_{\alpha,r})$, such that

$$\Lambda(f) = \int_G f(x) g_\Lambda(x)\, dx$$

for all $f \in L^1(w_{\alpha,r})$. Since

$$\Lambda(f_1 * f_2) = \Lambda(f_1)\Lambda(f_2) \quad \text{for all } f_1, f_2 \in L^1(\omega_{\alpha,r}),$$

we get that

$$\int_K g_\Lambda(xky)\, dk = g_\Lambda(x) g_\Lambda(y)$$

for all $x, y \in G$. Thus, from Proposition 2.3, we have $g_\Lambda = \phi_\lambda$ for some $\lambda \in \mathbb{C}$. Therefore, the maximal ideal space of $L^1(\omega_{\alpha,r})$ is

$$\Sigma_{\omega_{\alpha,r}} = \Big\{\lambda \in \mathbb{C} : \sup_{x \in G} |\phi_\lambda(x)|/w_{\alpha,r}(x) < \infty\Big\}.$$

Then it follows that for $f \in L^1(\omega_{\alpha,r})$, its spherical Fourier transform

$$\widehat{f}(\lambda) = \int_G f(x)\phi_\lambda(x)\, dx$$

exists for $\lambda \in \Sigma_{\omega_{\alpha,r}}$. Moreover, $\widehat{f}$ is analytic on $\Sigma^\circ_{\omega_{\alpha,r}}$ and continuous on $\Sigma_{\omega_{\alpha,r}}$.

In the following lemma, we determine $\Sigma_{\omega_{\alpha,r}}$ explicitly using the asymptotic estimate in Equation (2-7) of $\phi_\lambda(a_t)$ and show that $\Sigma_{\omega_{\alpha,r}}$ is independent of r. We recall here that $\mathbf{S}_\alpha = \{\lambda \in \mathbb{C} : |\mathrm{Im}\lambda| \le \alpha\}$.

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LEMMA 3.1. *Suppose $\alpha > 0$, then $\Sigma_{\omega_{\alpha,r}} = \mathbf{S}_\alpha$ for any $r \geq 0$.*

PROOF. Since $\phi_\lambda(x) = \phi_{-\lambda}(x)$ and $|\phi_\lambda(x)| \leq \phi_{i\mathrm{Im}\lambda}(x)$ for all $x \in G$ (see Equation (2-9)), it follows that $\mathbf{S}_\alpha \subset \Sigma_{\omega_{\alpha,r}}$. Now to prove $\mathbf{S}_\alpha = \Sigma_{\omega_{\alpha,r}}$, let $\lambda \in \mathbb{C}$ with $\mathrm{Im}\lambda > \alpha$. Then for $x = k_1 a_t k_2$,

$$\frac{|\phi_\lambda(x)|}{\phi_{i\alpha}(x)(1+x^+)^r} = \frac{|\phi_\lambda(a_t)|}{\phi_{i\alpha}(a_t)(1+t)^r} \asymp \frac{e^{(\mathrm{Im}\lambda-\alpha)t}}{(1+t)^r},$$

which goes to infinity as $t \to \infty$. Therefore, $\lambda \notin \Sigma_{\omega_{\alpha,r}}$ and since $\phi_\lambda = \phi_{-\lambda}$, we conclude $\mathbf{S}_\alpha = \Sigma_{\omega_{\alpha,r}}$. □

3.1. A dense subspace of $L^1(w_{\alpha,r})$. We now construct a dense subset of $L^1(\omega_{\alpha,r})$, which plays a crucial role towards proving Theorem 1.2. This collection is a suitable scalar multiple of Φ_λ. For $\lambda \in \mathbb{C}_+ := \{\lambda \in \mathbb{C} : \mathrm{Im}\lambda > 0\}$, we define

$$b_\lambda(a_t) := \frac{i}{2\lambda c(-\lambda)}\Phi_\lambda(a_t) \quad \text{for } t > 0. \tag{3-1}$$

We extend b_λ to a K-biinvariant function on $G\backslash K$, using the Cartan decomposition

$$b_\lambda(k_1 a_t k_2) = b_\lambda(a_t) \quad \text{for all } t > 0. \tag{3-2}$$

Hence, b_λ is also a solution of Equation (2-5) on $G/K \setminus \{eK\}$. Next, using the asymptotic estimates of $\Phi_\lambda(a_t)$ (see Equation (2-6)) and $\phi_{i\alpha}(a_t)$ (see Equation (2-7)) near $t = \infty$, we observe that if $\mathrm{Im}\lambda \leq \alpha$, then the functions b_λ do not belong to $L^1(\omega_{\alpha,r})$. In the following lemma, we show for $\lambda \in \mathbb{C}_+$ with $\mathrm{Im}\lambda > \alpha$, $b_\lambda \in L^1(\omega_{\alpha,r})$, and along with this, the lemma also gives us a dense subspace of $L^1(\omega_{\alpha,r})$.

LEMMA 3.2. *The functions $\{b_\lambda : \mathrm{Im}\lambda > \alpha\}$ span a dense subspace of $L^1(\omega_{\alpha,r})$.*

We prove this lemma step by step. First, we show that for $\mathrm{Im}\lambda > \alpha$, b_λ belongs to $L^1(\omega_{\alpha,r})$. To show this, we borrow the estimates of $b_\lambda(a_t)$ from [21, Lemma 3.1] near $t = 0$ and away from zero. Later in Lemma 3.5, we find the spherical transforms of b_λ, which are essential to prove that the collection $\{b_\lambda : \mathrm{Im}\lambda > \alpha\}$ spans a dense subspace of $L^1(\omega_{\alpha,r})$.

LEMMA 3.3 [21, Lemma 3.1]. *Let $\lambda \in \mathbb{C}_+$. Then $b_\lambda(a_t)$ satisfies the following estimates near $t = 0$ and ∞.*
(a) There is a positive constant C such that for all $t \in (0, 1/2]$,

$$|b_\lambda(a_t)| \leq \begin{cases} C(1+|\lambda|)^N t^{-(m_1+m_2-1)} & \text{if } m_1 + m_2 > 1, \\ C \log \dfrac{1}{t} & \text{if } m_1 + m_2 = 1. \end{cases}$$

(b) There is a positive constant C and a natural number M such that for all $t \in [1/2, \infty]$,

$$|b_\lambda(a_t)| \leq C(1+|\lambda|)^M e^{-(\mathrm{Im}\lambda+\rho)t}.$$

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LEMMA 3.4. *If* $\mathrm{Im}\lambda > \alpha$, *then* $b_\lambda \in L^1(G//K, \omega_{\alpha,r})$. *Moreover, there exist* $n, N_0 \in \mathbb{N}$ *and a constant* $C > 0$ *(independent of* λ*) such that the following estimate holds:*

$$\|b_\lambda\|_{L^1_{\omega_{\alpha,r}}} \leq C\frac{(1+|\lambda|)^{N_0}}{(\mathrm{Im}\lambda-\alpha)^{n+1}}.$$

PROOF. Let $\lambda \in \mathbb{C}_+$ with $\mathrm{Im}\lambda > \alpha$ and $n = [r]+1$. Then using the estimates of $b_\lambda(a_t)$ and Equation (2-8) of $\phi_\alpha(a_t)$,

$$\begin{aligned}
\|b_\lambda\|_{L^1_{\omega_{\alpha,r}}} &\leq \int_0^\infty |b_\lambda(a_t)|\phi_{i\alpha}(a_t)(1+t)^n\Delta(t)\,dt \\
&\leq C(1+|\lambda|)^{\max\{N,M\}}\Big(\int_0^{1/2} t\,\phi_{i\alpha}(a_t)(1+t)^n dt + \int_{1/2}^\infty e^{(\alpha-\mathrm{Im}\lambda)t}(1+t)^n\,dt\Big) \\
&\leq C(1+|\lambda|)^{\max\{N,M\}}\Big(1+\frac{1}{(\mathrm{Im}\lambda-\alpha)^{n+1}}\Big) \\
&\leq \frac{C(1+|\lambda|)^{N_0}}{(\mathrm{Im}\lambda-\alpha)^{n+1}}.
\end{aligned}$$
□

LEMMA 3.5. *Let* $\lambda \in \mathbb{C}_+$ *with* $\mathrm{Im}\lambda > \alpha$. *Then we have*

$$\widehat{b_\lambda}(z) = \frac{1}{z^2-\lambda^2} \quad \textit{for all } z \in \mathbf{S}_\alpha.$$

PROOF. Suppose $\lambda \in \mathbb{C}_+$ with $\mathrm{Im}\lambda > \alpha$. Then from [21, Lemma 3.4], we get $\widehat{b_\lambda}(\xi) = 1/(\xi^2-\lambda^2)$ for all $\xi \in \mathbb{R}$. From Lemma 3.4, we have $b_\lambda \in L^1(\omega_{\alpha,r})$. Hence, $\widehat{b_\lambda}$ is a well-defined continuous function on the strip $\mathbf{S}_\alpha$ and holomorphic on $\mathbf{S}_\alpha^\circ$. Therefore, by analytic continuation, the lemma follows. □

PROOF OF LEMMA 3.2. We will show that $\overline{\mathrm{span}\{b_\lambda : \mathrm{Im}\lambda > \alpha\}}$ contains $C_c^\infty(G//K)$ and since $C_c^\infty(G//K)$ is dense in $L^1(\omega_{\alpha,r})$, the lemma will follow. Let $f \in C_c^\infty(G//K)$, then $\widehat{f}$ is entire (see Theorem 2.5). Applying Cauchy's integral formula for $\widehat{f}$,

$$\widehat{f}(w) = \frac{1}{2\pi i}\int_{\zeta_R}\frac{\widehat{f}(z)}{z-w}\,dz \quad \text{for all } w \in \mathbf{S}_\alpha, \tag{3-3}$$

where ζ_R is the contour consisting of a rectangle with vertices $R+i(\alpha+1)$, $-R+i(\alpha+1)$, $-R-i(\alpha+1)$, $R-i(\alpha+1)$ (R is sufficiently large) and the positive counterclockwise orientation. From Theorem 2.5, we get the integrals on the vertical sides of ζ_R to go to 0, as $R \to \infty$. Therefore, Equation (3-3) gives

$$\widehat{f}(w) = \frac{1}{2\pi i}\int_A\frac{\widehat{f}(z)}{z-w}\,dz + \frac{1}{2\pi i}\int_B\frac{\widehat{f}(z)}{z-w}\,dz \quad \text{for } w \in \mathbf{S}_\alpha,$$

where $A = \mathbb{R}+i(\alpha+1)$ and $B = \mathbb{R}-i(\alpha+1)$. We know $\widehat{f}(z)$ is an even function, and so by the change of variable $z \to -z$ in the second integral, we get for all $w \in \mathbf{S}_\alpha$,

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$$\widehat{f}(w) = \frac{1}{2\pi i}\int_A \frac{2z\widehat{f}(z)}{z^2 - w^2}\,dz.$$

Now for $z \in A$, $\mathrm{Im} z > \alpha$, so (by Lemma 3.4) b_z is in $L^1(\omega_{\alpha,r})$. Also, from Lemma 3.5, we can write

$$\widehat{f}(w) = \frac{1}{2\pi i}\int_A 2z\widehat{f}(z)\widehat{b_z}(w)\,dz. \tag{3-4}$$

From the estimate of $\|b_z\|_{\omega_{\alpha,r}}$ and the decay condition of $\widehat{f}$, it follows that the $L^1(\omega_{\alpha,r})$ integral

$$\frac{1}{2\pi i}\int_A 2z\widehat{f}(z)b_z(\cdot)\,dz$$

converges. Equation (3-4) shows that it must converge to f. Since the Riemann sums of the integral are nothing but finite linear combinations of b_λ, we conclude that f is in the closed subspace spanned by $\{b_\lambda : \mathrm{Im}\lambda > \alpha\}$. Hence, the lemma follows. □

We need to prove $\|b_\lambda\|_{L^1_{\omega_{\alpha,r}}} \to 0$ as $\lambda \to \infty$ along the positive imaginary axis. For the $r = 0$ case, Dahlner [6] proved it by using $\|b_{i\xi}\|_{L^1} = \widehat{b_{i\xi}}(i\alpha)$ for large ξ. However, in general, for $r > 0$, this is not true.

Lemma 3.6. *We have $\|b_\lambda\|_{L^1_{\omega_{\alpha,r}}} \to 0$ as $\lambda \to \infty$ along the positive imaginary axis.*

Proof. Let us take $\lambda = i\xi$, where $\xi > 0$ is very large. Then from Equation (3-1), we have $b_{i\xi}(a_t)$ is positive, and so

$$\begin{aligned}\|b_{i\xi}\|_{L^1_{\omega_{\alpha,r}}} &= \int_0^\infty b_{i\xi}(a_t)\phi_{i\alpha}(a_t)(1+t)^r\Delta(t)\,dt\\ &\le C\int_0^\infty b_{i\xi}(a_t)\phi_{2i\alpha}(a_t)\Delta(t)\,dt \quad \text{(using Proposition 2.4)}\\ &= C\frac{1}{\xi^2 - 4\alpha^2}.\end{aligned}$$

The lemma follows by sending ξ to infinity. □

3.2. Resolvent transform. Let $L^1_\delta(\omega_{\alpha,r})$ be the unitization of $L^1(\omega_{\alpha,r})$, where δ is the K-biinvariant distribution on G, defined by $\delta(\phi) = \phi(e)$ for all $\phi \in C_c^\infty(G//K)$. The maximal ideal space of $L^1_\delta(\omega_{\alpha,r})$ is $\{L_z : z \in \mathbf{S}_\alpha \cup \{\infty\}\}$, where the complex homomorphisms L_z on $L^1_\delta(\omega_{\alpha,r})$ are defined by the following:

$$L_z(f) = \widehat{f}(z), \quad z \in \mathbf{S}_\alpha, \quad L_\infty(f) = \begin{cases} 1 & \text{if } f = \delta \\ 0 & \text{if } f \in L^1(\omega_{\alpha,r})\end{cases} \quad \text{for all } f \in L^1_\delta(\omega_{\alpha,r}).$$

Let I denote the closed ideal from the hypothesis of Theorem 1.2. Then the spherical transform of the functions in I does not have any common zero in $\mathbf{S}_\alpha$. Therefore, the

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maximal ideal space of the quotient algebra $L^1_\delta(\omega_{\alpha,r})/I$ is the complex homomorphism $\tilde{L}_\infty$ defined by

$$\tilde{L}_\infty(f) = \widehat{f}(\infty) \quad \text{for all } f \in L^1_\delta(\omega_{\alpha,r})/I.$$

It also follows that an element $f + I \in L^1_\delta(\omega_{\alpha,r})/I$ is invertible if and only if $\widehat{f}(\infty) \neq 0$.

Now suppose λ_0 is a fixed complex number with $\mathrm{Im}\lambda_0 > \alpha$. So by Lemma 3.4, $b_{\lambda_0} \in L^1(\omega_{\alpha,r})$, and for $\lambda \in \mathbb{C}$, the function $\lambda \to \widehat{\delta} - (\lambda^2 - \lambda_0^2)\widehat{b_{\lambda_0}}$ does not vanish at ∞. Hence, $\delta - (\lambda^2 - \lambda_0^2)b_{\lambda_0} + I$ is invertible in the quotient algebra $L^1_\delta(\omega_{\alpha,r})/I$ for all $\lambda \in \mathbb{C}$. We put

$$B_\lambda := (\delta - (\lambda^2 - \lambda_0^2)b_{\lambda_0} + I)^{-1} * (b_{\lambda_0} + I). \tag{3-5}$$

For each $g \in L^\infty(G//K, 1/\omega_{\alpha,r})$ that annihilates the closed ideal I, we associate its resolvent transform

$$\mathcal{R}[g](\lambda) = \langle B_\lambda, g\rangle \quad \text{for } \lambda \in \mathbb{C}.$$

3.3. Properties of $\mathcal{R}[g](\lambda)$. We first find the representative of B_λ in $L^1(G//K, \omega_{\alpha,r})$ that gives the explicit expression of $\mathcal{R}[g](\lambda)$. Let $\lambda \in \mathbb{C}$ be such that $0 < \mathrm{Im}\lambda < \alpha$ and $f \in L^1(\omega_{\alpha,r})$. Then we define

$$T_\lambda f(a_t) = b_\lambda(a_t)\int_t^\infty f(a_s)\phi_\lambda(a_s)\Delta(s)\,ds - \phi_\lambda(a_t)\int_t^\infty f(a_s)b_\lambda(a_s)\Delta(s)\,ds, \quad t > 0, \tag{3-6}$$

and extend it to a K-biinvariant function on $G \setminus K$.

We first show that for a given $f \in L^1(\omega_{\alpha,r})$, $T_\lambda f \in L^1(\omega_{\alpha,r})$, and find a good quantitative bound of $\|T_\lambda f\|_{L^1_{\omega_{\alpha,r}}}$ (in the next lemma). Then (in Lemma 3.9), we find its spherical Fourier transform. This is essential to finding the representative of B_λ in terms of $T_\lambda f$ for $0 < \mathrm{Im}\lambda < \alpha$.

Lemma 3.7. *Let $0 < \mathrm{Im}\lambda < \alpha$ and $f \in L^1(\omega_{\alpha,r})$. Then, $T_\lambda f \in L^1(\omega_{\alpha,r})$. Moreover, if $\lambda \notin B_\alpha(0)$,*

$$\|T_\lambda f\|_{L^1_{\omega_{\alpha,r}}} \leq C\|f\|_{L^1_{\omega_{\alpha,r}}}(1 + |\lambda|)^L d(\lambda, \partial\mathbf{S}_\alpha)^{-1}$$

for some nonnegative integer L, where $d(\lambda, \partial\mathbf{S}_\alpha)$ denotes the Euclidean distance of λ from the boundary $\partial\mathbf{S}_\alpha$ of the strip $\mathbf{S}_\alpha$ and $B_\alpha(0)$ denotes the closed ball in $\mathbb{C}$ of radius α centred at zero.

To find the estimate of $\|T_\lambda f\|_{L^1(\omega_{\alpha,r})}$, we need the following property of spherical functions.

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LEMMA 3.8. *Suppose* $0 < \nu < \alpha$, *then we have* $\phi_{i\nu}(x) \leq \phi_{i\alpha}(x)$ *for all* $x \in G$.

PROOF. By the Cartan decomposition, it is enough to show that $\phi_{i\nu}(a_t) \leq \phi_{i\alpha}(a_t)$ for all $t > 0$, when $0 < \nu < \alpha$. For any two smooth functions f and g on $(0, \infty)$, we define

$$[f, g](t) := (f'(t)g(t) - f(t)g'(t))\, \Delta(t), \quad t > 0.$$

An easy calculation shows that

$$[f, g]'(t) = (Lf \cdot g - f \cdot Lg)(t)\Delta(t), \tag{3-7}$$

where we recall L is the Laplace–Beltrami operator on G/K and

$$\Delta(t) = (2 \sinh t)^{m_1+m_2}(2 \cosh t)^{m_2}.$$

Now putting $f = \phi_{i\nu}$ and $g = \phi_{i\alpha}$ in Equation (3-7),

$$\begin{aligned}[\phi_{i\nu}, \phi_{i\alpha}]'(a_t) &= ((\nu^2 - \rho^2)\phi_{i\nu}\phi_{i\alpha} - \phi_{i\nu}\phi_{i\alpha}(\alpha^2 - \rho^2))\Delta(t) \\ &= (\nu^2 - \alpha^2)\phi_{i\nu}\phi_{i\alpha}\, \Delta(t).\end{aligned}$$

Hence, $[\phi_{i\nu}, \phi_{i\alpha}](a_t) \leq 0$ for all $t > 0$. So we have

$$\left(\frac{\phi_{i\nu}}{\phi_{i\alpha}}\right)'(a_t) = \left(\frac{\phi_{i\alpha}(a_t)\phi'_{i\nu}(a_t) - \phi_{i\nu}(a_t)\phi'_{i\alpha}(a_t)}{\phi^2_{i\alpha}(a_t)}\right) \leq 0.$$

Therefore, $\phi_{i\nu}(a_t) \leq \phi_{i\alpha}(a_t)$ for all $t > 0$. □

PROOF OF LEMMA 3.7. We have

$$\|T_\lambda f\|_{L^1_{\omega_{\alpha,r}}} = \int_0^\infty |T_\lambda f(a_t)|\phi_{i\alpha}(a_t)\Delta(t)(1 + t)^r\, dt.$$

Let

$$r_\lambda(t) = \begin{cases} C(1 + |\lambda|)^N t^{(1-m_1-m_2)} & \text{if } m_1 + m_2 > 1, \\ C \log \dfrac{1}{t} & \text{if } m_1 + m_2 = 1. \end{cases}$$

We observe the following properties of $r_\lambda(t)$: :

(i) r_λ is a decreasing function;
(ii) $\int_0^{1/2} r_\lambda(t)\Delta(t)\, dt \leq C(1 + |\lambda|)^N$;
(iii) $|b_\lambda(a_t)| \leq r_\lambda(t), \; t \in (0, 1/2]$.

We write

$$\int_0^\infty |T_\lambda f(a_t)|\phi_{i\alpha}(a_t)\Delta(t)(1 + t)^r\, dt \leq I_1 + I_2 + I_3 + I_4,$$

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where

$$I_1 = \int_0^{1/2} |b_\lambda(a_t)|\phi_{i\alpha}(a_t)\Delta(t)(1+t)^r\Big(\int_t^\infty |f(a_s)||\phi_\lambda(a_s)|\Delta(s)\,ds\Big)\,dt,$$
$$I_2 = \int_0^{1/2} |\phi_\lambda(a_t)|\phi_{i\alpha}(a_t)\Delta(t)(1+t)^r\Big(\int_t^\infty |f(a_s)||b_\lambda(a_s)|\Delta(s)\,ds\Big)\,dt,$$
$$I_3 = \int_{1/2}^\infty |b_\lambda(a_t)||\phi_{i\alpha}(a_t)\Delta(t)(1+t)^r\Big(\int_t^\infty |f(a_s)||\phi_\lambda(a_s)|\Delta(s)\,ds\Big)\,dt,$$
$$I_4 = \int_{1/2}^\infty |\phi_\lambda(a_t)|\phi_{i\alpha}(a_t)\Delta(t)(1+t)^r\Big(\int_t^\infty |f(a_s)||b_\lambda(a_s)|\Delta(s)\,ds\Big)\,dt.$$

Then,

$$\begin{aligned} I_1 &\le C\int_0^{1/2} r_\lambda(t)\phi_{i\alpha}(a_t)\Delta(t)(1+t)^r\Big(\int_0^\infty |f(a_s)|\phi_{i\alpha}(a_s)\Delta(s)\,ds\Big)\,dt \quad \text{(using Lemma 3.8)}\\ &\le C(1+|\lambda|)^N\|f\|_{L^1_{\omega\alpha,r}}, \end{aligned}$$

$$\begin{aligned} I_2 &\le \int_0^{1/2} \phi_{i\mathrm{Im}\lambda}(a_t)\phi_{i\alpha}(a_t)\Delta(t)(1+t)^r\Big(\int_t^{1/2} |f(a_s)||b_\lambda(a_s)|\Delta(s)\,ds\Big)\,dt\\ &\quad + \int_0^{1/2} \phi_{i\mathrm{Im}\lambda}(a_t)\phi_{i\alpha}(a_t)\Delta(t)(1+t)^r\Big(\int_{1/2}^\infty |f(a_s)||b_\lambda(a_s)|\Delta(s)\,ds\Big)\,dt.\\ &\le C\int_0^{1/2} r_\lambda(t)\Delta(t)\,dt\int_0^{1/2} |f(a_s)|\phi_{i\alpha}(a_s)\Delta(s)(1+s)^r\Delta(s)\,ds\\ &\quad + C(1+|\lambda|)^M\int_{1/2}^\infty |f(a_s)|e^{(\alpha-\rho)s}\Delta(s)\,ds\\ &\le C(1+|\lambda|)^{\max(M,N)}\Big(\int_0^{1/2} |f(a_s)|\phi_{i\alpha}(a_s)\Delta(s)(1+s)^r\,ds\\ &\quad + \int_{1/2}^\infty |f(a_s)|\phi_{i\alpha}(a_s)\Delta(s)(1+s)^r\,ds\Big) \quad \text{(using Equation (2 − 7))}\\ &\le C(1+|\lambda|)^{\max(M,N)}\|f\|_{L^1_{\omega\alpha,r}}. \end{aligned}$$

Next, using the estimates of $b_\lambda(a_t)$ and $\phi_\lambda(a_t)$ (see Equation (2-7) and Lemma 3.3) and changing the order of integration,

$$\begin{aligned} I_3 &\le C(1+|\lambda|)^M|c(-\lambda)||c(-\alpha)|\int_{1/2}^\infty e^{(\alpha-\mathrm{Im}\lambda)t}(1+t)^r\Big(\int_t^\infty |f(a_s)|e^{(\mathrm{Im}\lambda-\rho)s}\Delta(s)\,ds\Big)\,dt\\ &\le C(1+|\lambda|)^M|c(-\lambda)||c(-\alpha)|\int_{1/2}^\infty |f(a_s)|e^{(\alpha-\mathrm{Im}\lambda)s}\Delta(s)\Big(\int_0^s e^{(\alpha-\mathrm{Im}\lambda)t}(1+t)^r\,dt\Big)\,ds\\ &\le C(1+|\lambda|)^M|c(-\lambda)||c(-\alpha)|\int_{1/2}^\infty |f(a_s)|e^{(\mathrm{Im}\lambda-\rho)s}(1+s)^r\Delta(s)\Big(\frac{e^{(\alpha-\mathrm{Im}\lambda)s}-e^{(\alpha-\mathrm{Im}\lambda)1/2}}{\alpha-\mathrm{Im}\lambda}\Big)\,ds\\ &\le C(1+|\lambda|)^M|c(-\lambda)|\|f\|_{L^1_{\omega\alpha,r}}\,d(\lambda,\partial\mathbf{S}_\alpha)^{-1}. \end{aligned}$$

Simliarly, we can prove

$$I_4 \leq C(1+|\lambda|)^M |c(-\lambda)| \|f\|_{L^1_{\omega_{\alpha,r}}} \, d(\lambda, \partial\mathbf{S}_\alpha)^{-1}.$$

Since

$$c(-\lambda) = \frac{2^{\rho-i\lambda}\Gamma(\frac{m_1+m_2+1}{2})\Gamma(i\lambda)}{\Gamma(\frac{\rho+i\lambda}{2})\Gamma(\frac{m_1+2}{4}+\frac{i\lambda}{2})} = \frac{2^{\rho-i\lambda}\Gamma(\frac{m_1+m_2+1}{2})\Gamma(1+i\lambda)}{i\lambda\,\Gamma(\frac{\rho+i\lambda}{2})\Gamma(\frac{m_1+2}{4}+\frac{i\lambda}{2})},$$

by the polynomial approximation of gamma functions,

$$|c(-\lambda)| \leq \frac{C}{|\lambda|(1+|\lambda|)^{(m_1+m_2-2)/2}}.$$

Now $\lambda \notin B_\alpha(0)$, so we can dominate $|c(-\lambda)|$ by a polynomial. Finally, from the estimates I_1, I_2, I_3 and I_4, the lemma follows. □

We will use the inverse spherical transform to show $T_\lambda f$ is a representative of B_λ for $0 < \mathrm{Im}\lambda < \alpha$. To apply an inverse spherical transform, we need the following lemma.

LEMMA 3.9. *Let* $0 < \mathrm{Im}\lambda < \alpha$ *and* $f \in L^1(G//K, \omega_{\alpha,r})$. *Then,*

$$\widehat{T_\lambda f}(z) = \frac{\widehat{f}(\lambda) - \widehat{f}(z)}{z^2 - \lambda^2} \quad \text{for all } z \in \mathbf{S}_\alpha \setminus \{\pm\lambda\}.$$

PROOF. Using the definition of $T_\lambda f$ in Equation (3-6),

$$\begin{aligned}\widehat{T_\lambda f}(z) &= \int_0^\infty b_\lambda(a_t)\phi_z(a_t)\Big(\int_t^\infty f(a_s)\phi_\lambda(a_s)\Delta(s)\,ds\Big)\Delta(t)\,dt \\ &\quad - \int_0^\infty \phi_\lambda(a_t)\phi_z(a_t)\Big(\int_t^\infty f(a_s)b_\lambda(a_s)\Delta(s)\,ds\Big)\Delta(t)\,dt.\end{aligned}$$

By changing the order of integration,

$$\widehat{T_\lambda f}(z) = \int_0^\infty f(a_s)\Big(\phi_\lambda(a_s)\int_0^s b_\lambda(a_t)\phi_z(a_t)\Delta(t)\,dt - b_\lambda(a_s)\int_0^s \phi_\lambda(a_t)\phi_z(a_t)\Delta(t)\,dt\Big)\Delta(s)\,ds.$$

Putting $f = \Phi_\lambda$, $g = \phi_z$ in Equation (3-7), we get for any $0 < r < s$,

$$\int_r^s \Phi_\lambda(a_t)\phi_z(a_t)\Delta(t)\,dt = \frac{1}{z^2-\lambda^2}([\Phi_\lambda, \phi_z](a_s) - [\Phi_\lambda, \phi_z](a_r)). \tag{3-8}$$

Now sending $r \to 0$ and using the asymptotic behaviour of $\Phi_\lambda(a_t)$ near $t = 0$, we get (see [21, Lemma 8.1])

$$\lim_{r\to 0^+}[\Phi_\lambda, \phi_z](a_r) = 2i\lambda c(-\lambda).$$

Consequently, Equation (3-8) becomes

$$\int_0^s b_\lambda(a_t)\phi_z(a_t)\Delta(t)\,dt = \frac{1}{z^2-\lambda^2}([b_\lambda, \phi_z](a_s) + 1).$$

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Similarly, putting $f = \Phi_\lambda$, $g = \phi_\lambda$ in Equation (3-7), we get $[\phi_\lambda, \Phi_\lambda](\cdot)$ is constant on $(0, \infty)$. Next, using the asymptotic behaviour of $\Delta(t)$ and $\Phi_\lambda(a_t)$ near $t = \infty$, we get (see [21, Lemma 8.1])

$$[\phi_\lambda, \Phi_\lambda](\cdot) = \lim_{t\to\infty}[\phi_\lambda, \Phi_\lambda](a_t) = -2i\lambda c(-\lambda).$$

Next, from the equations above, it follows that

$$\begin{aligned} I(s) &:= \phi_\lambda(a_s)\Delta(s)\int_0^s b_\lambda(a_t)\phi_z(a_t)\Delta(t)\,dt - b_\lambda(a_s)\Delta(s)\int_0^s \phi_\lambda(a_t)\phi_z(a_t)\Delta(t)\,dt \\ &= \frac{1}{z^2-\lambda^2}(\phi_\lambda(a_s)\Delta(s)([b_\lambda, \phi_z](a_s)+1) - b_\lambda(a_s)\Delta(s)[\phi_z, \phi_\lambda](a_s)) \\ &= \frac{1}{z^2-\lambda^2}(\phi_\lambda(a_s) - \phi_z(a_s))\Delta(s) \text{ (here we use the fact } [\phi_\lambda, b_\lambda](\cdot) = 1). \end{aligned}$$

Therefore,

$$\widehat{T_\lambda f}(z) = \frac{\widehat{f}(\lambda) - \widehat{f}(z)}{z^2-\lambda^2} \quad \text{for all } z \in \mathbf{S}_\alpha \setminus \{\pm\lambda\}. \qquad \square$$

We are now equipped with all the tools to find out the explicit formula of $\mathcal{R}[g]$. The relevancy of the functions b_λ and $T_\lambda f$ is made clear in the following lemma, where we summarize the necessary properties of the resolvent transform.

LEMMA 3.10. *Assume that $g \in L^\infty(G//K, 1/\omega_{\alpha,r})$ annihilates the ideal I. Then:*

(a) *$\mathcal{R}[g](\lambda)$ is an even holomorphic function on $\mathbb{C}$;*

(b) *for* $\mathrm{Im}\lambda > \alpha$,

$$\mathcal{R}[g](\lambda) = \langle b_\lambda, g\rangle;$$

(c) *for any function $f \in I$ and* $0 < \mathrm{Im}\lambda < \alpha$,

$$\mathcal{R}[g](\lambda) = \frac{\langle T_\lambda f, g\rangle}{\widehat{f}(\lambda)}$$

provided $\lambda \notin Z(\widehat{f}) := \{z \in \mathbf{S}_\alpha : \widehat{f}(z) = 0\}$;

(d) *for* $|\mathrm{Im}\lambda| > \alpha$,

$$|\mathcal{R}[g](\lambda)| \le C\|g\|_{L^\infty_{1/\omega_{\alpha,r}}}\frac{(1+|\lambda|)^N}{d(\lambda, \partial\mathbf{S}_\alpha)^n};$$

(e) *for* $|\mathrm{Im}\lambda| < \alpha$,

$$|\widehat{f}(\lambda)\mathcal{R}[g](\lambda)| \le C\|f\|_{L^1_{\omega_{\alpha,r}}}\|g\|_{L^\infty_{1/\omega_{\alpha,r}}}\frac{(1+|\lambda|)^N}{d(\lambda, \partial\mathbf{S}_\alpha)},$$

where the constant C is independent of λ and $f \in I$.

PROOF. (a) This follows from the definition of B_λ (see Equation (3-5)).

(b) Suppose λ_0 be a fixed complex number with $\mathrm{Im}\lambda_0 > \alpha$. We recall from Lemma 3.4 that for $\mathrm{Im}\lambda > \alpha$, we have $b_\lambda \in L^1(\omega_{\alpha,r})$. Then from Lemma 3.5, for all $z \in \mathbf{S}_\alpha$,

$$(1 - (\lambda^2 - \lambda_0^2)\widehat{b}_{\lambda_0}(z))\widehat{b_\lambda}(z) = \widehat{b_{\lambda_0}}(z), \quad \text{for all } z \in \mathbf{S}_\alpha.$$

Hence, by the inverse spherical transform,

$$(\delta - (\lambda^2 - \lambda_0^2)b_{\lambda_0}(\cdot)) * b_\lambda(\cdot) = b_{\lambda_0}(\cdot)$$

as $L^1_\delta(\omega_{\alpha,r})$ functions. Therefore, in the quotient algebra $L^1_\delta(\omega_{\alpha,r})/I$,

$$(\delta - (\lambda^2 - \lambda_0^2)b_{\lambda_0} + I) * (b_\lambda + I) = b_{\lambda_0} + I. \tag{3-9}$$

Since $(\delta - (\lambda^2 - \lambda_0^2)b^\lambda_{\lambda_0} + I)$ is invertible in $L^1_\delta(\omega_{\alpha,r})/I$, from the definition of B_λ (see Equation (3-5)) and Equation (3-9), we get $B_\lambda = b_\lambda + I$. Accordingly,

$$\mathcal{R}[g](\lambda) = \langle b_\lambda, g\rangle. \tag{3-10}$$

(c) Let $f \in I$ and $\lambda \in \mathbb{C}$ with $0 < \mathrm{Im}\lambda < \alpha$, then by Lemma 3.7, $T_\lambda f$ is in $L^1(\omega_{\alpha,r})$. Now if λ is such that $\widehat{f}(\lambda) \neq 0$, then, similarly as in the previous case, we have from Lemma 3.9 and the inverse spherical transform,

$$(\delta - (\lambda^2 - \lambda_0^2)b_{\lambda_0}(\cdot)) * \left(\frac{T_\lambda f(\cdot)}{\widehat{f}(\lambda)}\right) = b_{\lambda_0}(\cdot) - \frac{b_{\lambda_0}(\cdot) * f(\cdot)}{\widehat{f}(\lambda)}$$

as $L^1_\delta(\omega_{\alpha,r})$ functions. Since $f \in I$,

$$(\delta - (\lambda^2 - \lambda_0^2)b_{\lambda_0} + I) * \left(\frac{T_\lambda f}{\widehat{f}(\lambda)} + I\right) = b_{\lambda_0} + I.$$

From the definition in Equation (3-5) of B_λ, we get $B_\lambda = T_\lambda f/\widehat{f}(\lambda) + I$. Hence, from the equation above and the fact that the spherical transforms of functions in I do not have a common zero, we get the expression

$$\mathcal{R}[g](\lambda) = \left\langle \frac{T_\lambda f}{\widehat{f}(\lambda)}, g\right\rangle$$

provided $\lambda \notin Z(\widehat{f}) = \{z \in S_\alpha : \widehat{f}(z) = 0\}$.

(d) From item (a), we have $\mathcal{R}[g]$ is an even function, so it is enough to consider the case $\mathrm{Im}\lambda > \alpha$. Now for $\mathrm{Im}\lambda > \alpha$, we have from Equation (3-10) and Lemma 3.4,

$$|\mathcal{R}[g](\lambda)| \le C\|g\|_{L^\infty_{1/\omega_{\alpha,r}}} \frac{(1+|\lambda|)^N}{d(\lambda, \partial S_\alpha)^n} \quad \text{for some } C > 0.$$

(e) From the estimate of $\|T_\lambda f\|_{L^1_{\omega_{\alpha,r}}}$ (Lemma 3.7), we get for $0 < \mathrm{Im}\lambda < \alpha$, $\lambda \notin B_\alpha(0)$,

$$|\widehat{f}(\lambda)\mathcal{R}[g](\lambda)| \le C\|f\|_{L^1_{\omega_{\alpha,r}}}\|g\|_{L^\infty_{1/\omega_{\alpha,r}}} \frac{(1+\lambda)^L}{d(\lambda, \partial S_\alpha)}. \tag{3-11}$$

As $\widehat{f}(\lambda)\mathcal{R}[g](\lambda)$ is an even function on S_α, we get the same estimate for $0 < |\mathrm{Im}\lambda| < \alpha$, $\lambda \notin B_\alpha(0)$. Now from continuity of $\mathcal{R}[g](\lambda)$ and using Hölder's inequality for $\lambda \in B_\alpha(0)$,

$$|\widehat{f}(\lambda)\mathcal{R}[g](\lambda)| \le C\|f\|_{L^1_{\omega_{\alpha,r}}},$$

where C is a constant independent of f and λ. Hence, using continuity of $\mathcal{R}[g](\lambda)$ and $\widehat{f}(\lambda)$ again, we can find a constant $C > 0$ such that for all $\lambda \in \mathbb{C}$ with $0 \le \mathrm{Im}\lambda < \alpha$, Equation (3-11) holds. □

Proof of Theorem 1.2. Since the ideal generated by $\{f_\beta : \beta \in \Lambda\}$ is the same as the ideal generated by the elements $\{f_\beta/\|f_\beta\|_{L^1_{\omega_{\alpha,r}}} : \beta \in \Lambda\}$ and $\delta_\infty^{\alpha\pm}(\widehat{f}) = \delta_\infty^{\alpha+}(\widehat{f}/\|f\|_{L^1_{\omega_{\alpha,r}}})$, we assume that the functions f_β are of unit $L^1(\omega_{\alpha,r})$ norm. Let $g \in L^\infty(G//K, 1/\omega_{\alpha,r})$ annihilate the closed ideal I generated by $\{f_\beta : \beta \in \Lambda\}$. We will show that $g = 0$. Then by an application of the Hahn–Banach theorem, it will follow that $I = L^1(G//K, \omega_{\alpha,r})$. From the hypothesis,

$$\inf_{\beta\in\Lambda} \delta_\infty^{\alpha+}(\widehat{f_\beta}) = \inf_{\beta\in\Lambda} \delta_\infty^{\alpha-}(\widehat{f_\beta}) = 0.$$

By Lemma 3.10, the entire function $\mathcal{R}[g]$ satisfies the following estimates:

$$\begin{aligned} |\mathcal{R}[g](z)| &\le C(1+|z|)(d(z,\partial\mathbf{S}_\alpha))^{-n-1}, \quad z \in \mathbb{C}\backslash\mathbf{S}_\alpha,\\ |\widehat{f_\beta}(z)\mathcal{R}[g](z)| &\le C(1+|z|)(d(z,\partial\mathbf{S}_\alpha))^{-1}, \quad z \in \mathbf{S}_\alpha^0, \end{aligned}$$

for all $\beta \in \Lambda$, where C is a constant and we choose it to be greater than e and we recall $n = [r] + 1$. Let $M : (0,\infty) \to (e,\infty)$ be a continuously differentiable decreasing function such that $M(t) = \frac{C}{t^{n+1}}$ for $0 < t < 1$, and $\int_1^\infty \log\log M(t)\, dt < \infty$. With this definition of M,

$$\begin{aligned} |\mathcal{R}[g](z)| &\le (1+|z|)M(d(z,\partial\mathbf{S}_\alpha)) \quad z \in \mathbb{C}\backslash\mathbf{S}_\alpha,\\ |\widehat{f_\beta}(z)\mathcal{R}[g](z)| &\le (1+|z|)M(d(z,\partial\mathbf{S}_\alpha)) \quad z \in \mathbf{S}_\alpha^0, \quad \text{for all } \beta \in \Lambda. \end{aligned}$$

Therefore, applying Theorem 2.2, we get $\mathcal{R}[g](z)$ is a polynomial. Now from Lemma 3.6, we get $\|b_z\|_{L^1_{\omega_{\alpha,r}}} \to 0$ as z goes to infinity along the positive imaginary axis. However, Lemma 3.10 gives

$$|\mathcal{R}[g](z)| \le \|b_z\|_{L^1_{\omega_{\alpha,r}}} \|g\|_{L^\infty_{1/\omega_{\alpha,r}}} \quad \text{for } \mathrm{Im}z > \alpha,$$

so we get $\mathcal{R}[g](z) \to 0$, when $z \to \infty$ along the positive imaginary axis. Therefore, $\mathcal{R}[g]$ is the zero polynomial. Hence, $\langle b_\lambda, g\rangle = 0$ whenever $\mathrm{Im}\lambda > \alpha$, but the collection $\{b_\lambda : \mathrm{Im}\lambda > \alpha\}$ spans a dense subset of $L^1(G//K, \omega_{\alpha,r})$. So $g = 0$, and the theorem is proved. □

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4. Wiener Tauberian theorem for complex semisimple Lie groups

In this section, we prove an analogue of the Wiener Tauberian theorem for Lorentz spaces $L^{p,1}$, $1 \le p < 2$ of K-biinvariant functions on a noncompact, real-rank-one *complex* semisimple Lie group $G = \mathrm{SL}(2, \mathbb{C})$. It is known that for a noncompact *complex* semisimple Lie group $\alpha \in \Sigma^+$, $m_1 = 2$ and $m_2 = 0$ [16, Theorem 6.14]. As in the real case, we identify ρ with 1. Throughout this section, p will always lie in $[1, 2)$. We also recall $\gamma_p = (2/p - 1)$ and $\mathbf{S}_{\gamma_p} = \{\lambda \in \mathbb{C} : |\mathrm{Im}\lambda| \le \gamma_p\}$.

We have the following integral formulae corresponding to the Cartan decomposition (see [10, Section 3]):

$$\int_G f(g)dg = \int_K \int_{\mathbb{R}^+} \int_K f(k_1 a_t k_2)\, \widetilde{\Delta}(t)\, dk_1\, dt\, dk_2,$$

where $\widetilde{\Delta}(t) = (e^t - e^{-t})^2$, $t \in \mathbb{R}$.

4.1. Spherical functions. The spherical functions on G with respect to K have the following formula (see [12, Equation (2.2)] and [16, page 432, Theorem 5.7]):

$$\phi_\lambda(a_t) = ic(\lambda)\frac{\sin \lambda t}{\sinh t},$$

where

$$c(\lambda) = \frac{2^{(1-i\lambda)}\Gamma(\frac{3}{2})\Gamma(i\lambda)}{\Gamma(1 + \frac{i\lambda}{2})\Gamma(\frac{1+i\lambda}{2})} = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{i\lambda}{2})\Gamma(\frac{1+i\lambda}{2})}{\sqrt{\pi}\,\Gamma(1 + \frac{i\lambda}{2})\Gamma(\frac{1+i\lambda}{2})} = \frac{1}{i\lambda}.$$

The spherical transform $\widehat{f}$ of a function $f \in L^{p,1}(G//K)$ is defined as

$$\widehat{f}(\lambda) = \int_G f(x)\phi_{-\lambda}(x^{-1})\, dx \quad \text{for all } \lambda \in \mathbf{S}_{\gamma_p}.$$

As in the rank-one real semisimple Lie group, similar properties of spherical functions and spherical Fourier transforms hold for the rank-one complex semisimple Lie group. We refer the reader to [10] for further details.

4.2. A dense subspace of $L^{p,1}(G//K)$. Now, to prove Theorem 1.3, we follow a similar strategy as in Theorem 1.2. First, we construct a dense subspace of $L^{p,1}(G//K)$. We define for $\lambda \in \mathbb{C}_+$,

$$b_\lambda(a_t) = \frac{ie^{i\lambda t}}{2\lambda c(-\lambda)J(t)} = \frac{e^{i\lambda t}}{2J(t)} \quad \text{for all } t > 0,$$

where $J(t) = (e^t - e^{-t})$. We extend $b_\lambda(\cdot)$ as a K-biinvariant function in $G/K \setminus \{eK\}$ using the Cartan decomposition of G.

LEMMA 4.1. *The functions $\{b_\lambda : \mathrm{Im}\lambda > \gamma_p\}$ span a dense subspace of $L^{p,1}(G//K)$.*

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To prove the lemma above, we first show that for $\mathrm{Im}\lambda > \gamma_p$, $b_\lambda \in L^{p,1}(G//K)$. After that, we show that the spherical transform of b_λ is $\widehat{b_\lambda}(z) = 1/(z^2 - \lambda^2)$ for all $z \in \mathbf{S}_{\gamma_p}$, and then the proof of the lemma above follows exactly as in Lemma 3.2.

LEMMA 4.2. *Let* $1 \leq p < 2$ *and* $\lambda \in \mathbb{C}_+$ *with* $\mathrm{Im}\lambda > \gamma_p$, *then* $b_\lambda \in L^{p,1}(G//K)$. *Moreover, there exists a constant* $C > 0$ *(independent of* λ*) such that the following estimate holds:*

$$\|b_\lambda\|_{p,1} \leq C\frac{1+|\lambda|}{\mathrm{Im}\lambda - \gamma_p}. \tag{4-1}$$

PROOF. Suppose $\lambda \in \mathbb{C}_+$. Then using an asymptotic estimate of $J(t)$, we get

$$|b_\lambda(a_t)| \leq C(f(t) + g(t)), \tag{4-2}$$

where

$$f(t) = \chi_{(0,1]}(t)t^{-1} \quad \text{and} \quad g(t) = \chi_{[1,\infty)}(t)e^{-(\mathrm{Im}\lambda+1)t} \quad \text{for } t \geq 0.$$

We extend f and g as K biinvariant functions in G similarly as in Equation (3-2). The proof will be completed if we show that $\|f\|_{p,1} + \|g\|_{p,1}$ is dominated by the right-hand side of Equation (4-1). We have, for $\alpha > 0$,

$$d_f(\alpha) = m\{t \in [0,\infty] : |f(t)| > \alpha\} = m\{t \in (0,1] : t^{-1} > \alpha\} = m\Big\{t \in (0,1] : t < \frac{1}{\alpha^{-1}}\Big\},$$

where m is the Haar measure on G in the Cartan decomposition. We observe that for $\alpha < 1$, $d_f(\alpha)$ is constant. For $\alpha > 1$, we have

$$d_f(\alpha) = \int_0^{\alpha^{-1}} \widetilde{\Delta}(t)\,dt = \int_0^{\alpha^{-1}} (e^t - e^{-t})^2\,dt \asymp \int_0^{\alpha^{-1}} t^2\,dt \asymp \alpha^{-3}.$$

Now,

$$\|f\|_{p,1} = p\Big(\int_0^\infty d_f(\alpha)^{1/p} d\alpha\Big) \leq p\Big(C + \int_1^\infty \alpha^{-3/p} d\alpha\Big) = C \text{ (since } p < 2). \tag{4-3}$$

For the function g,

$$d_g(\alpha) = m\{t \in [1/2,\infty) : e^{-(\mathrm{Im}\lambda+1)t} > \alpha\} = m\Big\{t \in [1/2,\infty) : t < \frac{1}{(\mathrm{Im}\lambda+1)}\log\frac{1}{\alpha}\Big\}.$$

Therefore, $d_g(\alpha) = 0$ whenever $\alpha > e^{-(\mathrm{Im}\lambda+1)/2}$ and for the range $0 < \alpha < e^{-(\mathrm{Im}\lambda+1)/2}$,

$$d_g(\alpha) \asymp \int_{1/2}^{1/(\mathrm{Im}\lambda+1)\log 1/\alpha} e^{2t}\,dt \leq \frac{1}{\alpha^{2/(\mathrm{Im}\lambda+1)}}.$$

Then,

$$\|g\|_{p,1} = p\Big(\int_0^\infty d_g(\alpha)^{1/p} d\alpha\Big) \leq p\Big(\int_0^1 \alpha^{-2/p(\mathrm{Im}\lambda+1)} d\alpha\Big) = \frac{1}{1 - \frac{2}{p(\mathrm{Im}\lambda+1)}}. \tag{4-4}$$

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The integral in Equation (4-4) converges since $2/(\mathrm{Im}\lambda + 1)p < 1$ for $\mathrm{Im}\lambda > \gamma_p$. Next, from Equations (4-3) and (4-4),

$$\|b_\lambda\|_{p,1} \le C + \frac{1}{1 - \frac{2}{p(\mathrm{Im}\lambda+1)}} \le C\frac{1 + |\lambda|}{\mathrm{Im}\lambda - \gamma_p}.$$

This completes the proof of the lemma. □

LEMMA 4.3. *For $\lambda \in \mathbb{C}_+$ with $\mathrm{Im}\lambda > \gamma_p$,*

$$\widehat{b_\lambda}(z) = \frac{1}{z^2 - \lambda^2}, \quad \textit{for all } z \in \mathbf{S}_{\gamma_p}.$$

PROOF. Suppose $z \in \mathbf{S}_{\gamma_p}$, then,

$$\begin{aligned}\widehat{b_\lambda}(z) &= \frac{1}{2z}\int_0^\infty \frac{e^{i\lambda t}\sin(zt)}{J(t)\sinh t}(e^t - e^{-t})^2\,dt \\ &= \frac{1}{2iz}\int_0^\infty (e^{i(\lambda+z)t} - e^{i(\lambda-z)t})\,dt \\ &= \frac{1}{-2z}\frac{2z}{(\lambda^2 - z^2)} = \frac{1}{(z^2 - \lambda^2)}.\end{aligned}$$

□

From the pointwise estimate in Equation (4-2) of $b_\lambda(a_t)$, we observe that if $\mathrm{Im}(\lambda)$ is sufficiently large, then $b_\lambda \in L^{p,1}(G//K)$ for any small p. In particular, if $\mathrm{Im}\lambda > \gamma_p + 1$, then $\widehat{b_\lambda}$ exists at the point i and $\widehat{b_\lambda}(i) = 1/-(\lambda^2 + 1)$. Using this along with the estimate of $b_\lambda(a_t)$, we show that for all $p \in [1, 2)$, $\|b_\lambda\|_{p,1} \to 0$ whenever $\lambda \to \infty$ along the positive imaginary axis.

LEMMA 4.4. *We have $\|b_\lambda\|_{p,1} \to 0$ as $\lambda \to \infty$ along the positive imaginary axis.*

PROOF. We are going to show for $\lambda = \zeta + i\xi$, $\|b_\lambda\|_{p,1} \to 0$ as $\xi \to \infty$, for any fixed $\zeta \in \mathbb{R}$. Suppose $\lambda = \zeta + i\xi$ and ξ is a large positive real number and $q \in [1, 2)$. Then, $b_{i\xi}(a_t)$ is positive and for a fixed ζ, $|b_\lambda(a_t)| \le Cb_{i\xi}(a_t)$ for all $t > 0$. Hence, for large $\xi > 0$,

$$\|b_{i\xi}\|_1 = \int_{\mathbb{R}_+} b_{i\xi}(a_t)\widetilde{\Delta}(t)\,dt = \widehat{b_{i\xi}}(i) = \frac{1}{\xi^2 - 1}$$

goes to zero as $\xi \to \infty$. Next, we show that for any $q \in [1, 2)$,

$$\lim_{\xi\to\infty} \|b_{i\xi}\|_q = 0. \tag{4-5}$$

We have

$$\begin{aligned}\|b_{i\xi}\|_q^q &\le C\int_0^1 \left(\frac{1}{t}\right)^q \widetilde{\Delta}(t)\,dt + C\int_1^\infty e^{-(\xi+1)qt+2t}\,dt \\ &\le C + Ce^{1-(\xi+1)q/2}.\end{aligned}$$

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Let $1 < p < q$, then there exists $\theta \in (0,1)$ such that $1/p = (1-\theta) + \theta/p$. By Hölder's inequality,

$$\lim_{\xi\to\infty} \|b_{i\xi}\|_p \le \lim_{\xi\to\infty} \|b_{i\xi}\|_1^{1-\theta} \lim_{\xi\to\infty} \|b_{i\xi}\|_q^{\theta} = 0$$

as $\lim_{\xi\to\infty} \|b_{i\xi}\|_1 = 0$ (see [21, Lemma 3.5]). Since p, q are arbitrary, we get Equation (4-5). Now for $p_1 < p < p_2$, we have from Lemma 2.1,

$$\|b_{i\xi}\|_{p,1} \le C(\|b_{i\xi}\|_{p_1} + \|b_{i\xi}\|_{p_2}).$$

This implies $\|b_\lambda\|_{p,1} \to 0$ as $\lambda \to \infty$ through any vertical line. □

4.3. Resolvent transform. As before, here also we define a resolvent transform associated to each $g \in L^{p',\infty}(G//K)$ that annihilates the closed ideal I generated by the functions $\{f_\beta : \beta \in \Lambda\}$ from the hypothesis of Theorem 1.3. Let $L^{p,1}_\delta(G//K)$ be the unitization of $L^{p,1}(G//K)$ and δ. The maximal ideal space of $L^{p,1}_\delta(G//K)$ is $\{L_z : z \in \mathbf{S}_{\gamma_p} \cup \infty\}$, where L_z are complex homomorphisms on $L^{p,1}_\delta(G//K)$ defined similarly as in Equation (3-2). We have that the collection $\{\widehat{f_\beta} : \beta \in \Lambda\}$ does not have any common zero in $\mathbf{S}_{\gamma_p}$. So by Banach algebra theory, we get that the maximal ideal space of $L^{p,1}_\delta(G//K)/I$ is the complex homomorphism $\tilde{L}_\infty$, defined by

$$\tilde{L}_\infty(f + I) = \widehat{f}(\infty) \quad \text{for all } f \in L^{p,1}_\delta(G//K)/I.$$

For each $g \in L^{p',\infty}(G//K)$ that annihilates the closed ideal I, we associate its resolvent transform

$$\mathcal{R}[g](\lambda) = \langle b_\lambda, g\rangle, \quad \operatorname{Im}\lambda > \gamma_p.$$

Now suppose λ_0 is a fixed complex number with $\operatorname{Im}\lambda_0 > \gamma_p$. Then, using an analogous argument as in Equation (3-5), we define the resolvent transform

$$\mathcal{R}[g](\lambda) = \langle B_\lambda, g\rangle \quad \lambda \in \mathbb{C},$$

which is analytic on the entire complex plane, where

$$B_\lambda = (\delta - (\lambda^2 - \lambda_0^2) b_{\lambda_0} + I)^{-1} * (b_{\lambda_0} + I).$$

Our next objective is to find an explicit formula for $\mathcal{R}[g](\lambda)$ by acquiring representatives of B_λ in $L^{p,1}(G//K)/I$. We will show, for $\operatorname{Im}\lambda > \gamma_p$, a representative of B_λ is b_λ, which is in $L^{p,1}_\delta(G//K)$. Before that, we find a representative of B_λ in $L^{p,1}_\delta(G//K)/I$ for $0 < \operatorname{Im}\lambda < \gamma_p$. Suppose $\lambda \in \mathbb{C}$ with $0 < \operatorname{Im}\lambda < \gamma_p$ and $f \in L^{p,1}(G//K)$. Then for all $t > 0$, we define

$$T_\lambda f(a_t) = b_\lambda(a_t) \int_t^\infty f(a_s)\phi_\lambda(a_s)\widetilde{\Delta}(s)\, ds - \phi_\lambda(a_t) \int_t^\infty f(a_s) b_\lambda(a_s)\widetilde{\Delta}(s)\, ds, \tag{4-6}$$

and extend it as a K biinvariant function on G. Next, using the estimate of $b_\lambda(a_t)$, we find a quantitative estimate of $\|T_\lambda f\|_{p,1}$ in the following lemma.

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LEMMA 4.5. *For $\lambda \in \mathbb{C}_+$ with $0 < \operatorname{Im}\lambda < \gamma_p$, we have $T_\lambda f \in L^{p,1}(G//K)$ and moreover for $\lambda \notin B_{\gamma_p}(0)$, its $L^{p,1}(G//K)$ norm satisfies $\|T_\lambda f\|_{p,1} \leq C(1+|\lambda|)/d(\lambda, \partial \mathbf{S}_{\gamma_p})$.*

PROOF. Suppose $h \in L^{p',\infty}(G//K)$ with $\|h\|_{p',\infty} \leq 1$, then from Equation (4-6),

$$\begin{aligned}\Big|\int_G T_\lambda f(x)h(x)\,dx\Big| &= \Big|\int_0^\infty T_\lambda f(a_t)h(a_t)\widetilde{\Delta}(t)\,dt\Big| \\ &= \Big|\int_0^\infty \Big(b_\lambda(a_t)\int_t^\infty f(a_s)\phi_\lambda(a_s)\widetilde{\Delta}(s)\,ds \\ &\qquad - \phi_\lambda(a_t)\int_t^\infty f(a_s)b_\lambda(a_s)\widetilde{\Delta}(s)\,ds\Big)h(t)\widetilde{\Delta}(t)\,dt\Big|.\end{aligned}$$

Now we divide the integral into four parts to use the estimates of $b_\lambda(a_t)$:

$$\Big|\int_G T_\lambda f(x)h(x)\,dx\Big| \leq I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned} I_1 &= \int_0^1 |b_\lambda(a_t)||h(a_t)|\Big(\int_t^\infty |f(a_s)||\phi_\lambda(a_s)|\widetilde{\Delta}(s)\,ds\Big)\widetilde{\Delta}(t)\,dt, \\ I_2 &= \int_0^1 |\phi_\lambda(a_t)||h(a_t)|\Big(\int_t^\infty |f(a_s)||b_\lambda(a_s)|\widetilde{\Delta}(s)\,ds\Big)\widetilde{\Delta}(t)\,dt, \\ I_3 &= \int_1^\infty |b_\lambda(a_t)||h(a_t)|\Big(\int_t^\infty |f(a_s)||\phi_\lambda(a_s)|\widetilde{\Delta}(s)\,ds\Big)\widetilde{\Delta}(t)\,dt \end{aligned}$$

and

$$I_4 = \int_1^\infty |\phi_\lambda(a_t)||h(a_t)|\Big(\int_t^\infty |f(a_s)||b_\lambda(a_s)|\widetilde{\Delta}(s)\,ds\Big)\widetilde{\Delta}(t)\,dt.$$

Then, using Hölder's inequality and our estimates of b_λ,

$$\begin{aligned} I_1 &\leq \int_0^1 |b_\lambda(a_t)||h(a_t)|\Big(\int_t^\infty |f(a_s)||\phi_\lambda(a_s)|\widetilde{\Delta}(s)\,ds\Big)\widetilde{\Delta}(t)\,dt \\ &\leq C\|f\|_{p,1}\|\phi_\lambda\|_{p',\infty}\|h\|_{p',\infty} \\ I_2 &\leq \int_0^1 |\phi_\lambda(a_t)||h(a_t)|\Big(\int_t^\infty |f(a_s)||b_\lambda(a_s)|\widetilde{\Delta}(s)\,ds\Big)\widetilde{\Delta}(t)\,dt \\ &\leq C\int_0^{1/2} |\phi_\lambda(a_t)||h(a_t)|\|f\|_{p,1}(C + \|\chi_{[1,\infty)}(s)e^{-(\operatorname{Im}\lambda+1)s}\|_{p',\infty})\,dt \\ &\leq C\|f\|_{p,1}\|h\|_{p',\infty}. \end{aligned}$$

Since λ is inside the strip $\mathbf{S}_{\gamma_p}$, that is, $0 < \operatorname{Im}\lambda < \gamma_p$, by a similar calculation as in Equation (4-2), we can show $\|\phi_\lambda\|_{p',\infty}$ and $\|\chi_{[1,\infty)}(s)b_\lambda(a_s)\|_{p',\infty}$ is bounded by a constant independent of λ.

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Before we estimate I_3 and I_4, we need to find an $L^{p,1}$-norm estimate of the following K biinvariant function on G, defined by

$$g(t) = e^{(\mathrm{Im}\lambda-1)t}\chi_{[1,s]}(t), \quad \text{where } s \geq 1 \text{ is fixed.}$$

Then the distribution function is

$$d_g(\alpha) = m\Big\{t \in [1,s] : t < \frac{1}{(1-\mathrm{Im}\lambda)}\log\frac{1}{\alpha}\Big\}.$$

Recall m is the Haar measure on G in the Cartan decomposition. Now we observe, unless $0 < \alpha < e^{(\mathrm{Im}\lambda-1)}$, $d_g(\alpha) = 0$. Furthermore, when $0 < \alpha < e^{(\mathrm{Im}\lambda-1)s}$,

$$d_g(\alpha) \asymp \int_1^s e^{2t}\,dt = \frac{e^{2s}-e^2}{2}$$

and for $e^{(\mathrm{Im}\lambda-1)s} < \alpha < e^{(\mathrm{Im}\lambda-1)}$, we get

$$d_g(\alpha) \asymp \int_1^{(\log 1/\alpha)/(1-\mathrm{Im}\lambda)} e^{2t}\,dt = \frac{1}{2\alpha^{2/(1-\mathrm{Im}\lambda)}} - \frac{e^2}{2}.$$

Therefore,

$$\begin{aligned}\|g\|_{p,1} = p\int_0^\infty d_g(\alpha)^{1/p}d\alpha &= p\int_0^{e^{(\mathrm{Im}\lambda-1)s}} d_g(\alpha)^{1/p}d\alpha + p\int_{e^{(\mathrm{Im}\lambda-1)s}}^{e^{(\mathrm{Im}\lambda-1)}} d_g(\alpha)^{1/p}d\alpha\\ &\leq Ce^{(\mathrm{Im}\lambda+2/p-1)s} + \frac{e^{\mathrm{Im}\lambda-1+2/p}-e^{(\mathrm{Im}\lambda-1+2/p)s}}{1-\frac{2}{p(-\mathrm{Im}\lambda)}}\\ &\leq C(1+|\lambda|)\frac{e^{(\mathrm{Im}\lambda+2/p-1)s}}{(\frac{2}{p}-1)1+\mathrm{Im}\lambda} \leq C(1+|\lambda|)\frac{e^{(2/p-1+\mathrm{Im}\lambda)s}}{(\frac{2}{p}-1)-\mathrm{Im}\lambda}.\end{aligned}$$

Similarly, if we take $\tilde{g}(t) = e^{-(\mathrm{Im}\lambda+1)t}\chi_{[1,s]}(t)$, for a fixed $s \geq 1$,

$$\|\tilde{g}\|_{p,1} \leq C(1+|\lambda|)\frac{e^{(2/p-1-\mathrm{Im}\lambda)s}}{(\frac{2}{p}-1)-\mathrm{Im}\lambda}. \tag{4-7}$$

Now, using the estimates of b_λ, ϕ_λ and changing the order of integration,

$$\begin{aligned}I_3 &\leq \frac{C}{|\lambda|}\int_1^\infty e^{-(\mathrm{Im}\lambda+1)t}|h(t)|\Big(\int_t^\infty |f(a_s)|e^{(\mathrm{Im}\lambda-1)s}\widetilde{\Delta}(s)\,ds\Big)\widetilde{\Delta}(t)\,dt\\ &\leq \frac{C}{|\lambda|}\int_1^\infty |f(a_s)|e^{(\mathrm{Im}\lambda-1)s}\Big(\int_1^s e^{-(\mathrm{Im}\lambda+1)t}|h(t)|\widetilde{\Delta}(t)\,dt\Big)\widetilde{\Delta}(s)\,ds\\ &\leq \frac{C}{|\lambda|}(1+|\lambda|)\int_1^\infty |f(a_s)|e^{(\mathrm{Im}\lambda-1)s}\frac{e^{(2/p-1-\mathrm{Im}\lambda)s}}{(\frac{2}{p}-1)-\mathrm{Im}\lambda}\|h\|_{p',\infty}\widetilde{\Delta}(s)\,ds \quad \text{(using Equation (4-7))}\end{aligned}$$

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$$\leq \frac{C(1+|\lambda|)}{|\lambda|(\frac{2}{p}-1-\mathrm{Im}\lambda)}\int_1^\infty |f(a_s)|e^{-2/p's}\|h\|_{p',\infty}\widetilde{\Delta}(s)\,ds$$
$$\leq \frac{C(1+|\lambda|)}{|\lambda|}d(\lambda,\partial\mathbf{S}_{\gamma_p})^{-1}\|f\|_{p,1}\|h\|_{p',\infty}.$$

Similarly, we can prove

$$I_4 \leq \frac{C(1+|\lambda|)}{|\lambda|}d(\lambda,\partial\mathbf{S}_{\gamma_p})^{-1}\|f\|_{p,1}\|h\|_{p',\infty}.$$

Since $\lambda \notin B_{\gamma_p}(0)$, $1/|\lambda| \leq C$, and so adding the estimates of I_1, I_2, I_3 and I_4, the lemma follows. □

Now that we have $T_\lambda f \in L^{p,1}(G//K)$ for $0 < \mathrm{Im}\lambda < \gamma_p$, next we need the spherical transform of $T_\lambda f$ to prove that $T_\lambda f$ is a representative of B_λ in $L^{p,1}(G//K)$. We find the spherical transform of $T_\lambda f$, using the same calculation in Lemma 3.9 to prove the following lemma.

LEMMA 4.6. *Suppose* $0 < \mathrm{Im}\lambda < \gamma_p$ *and* $f \in L^{p,1}(G//K)$. *Then,*

$$\widehat{T_\lambda f}(z) = \frac{\widehat{f}(\lambda) - \widehat{f}(z)}{z^2 - \lambda^2} \quad \textit{for all } z \in \mathbf{S}_{\gamma_p} \setminus \{\pm\lambda\}.$$

PROOF OF THEOREM 1.3. We have gathered all the details to find the explicit formula of the resolvent transform for the outside and inside of the strip $\mathbf{S}_{\gamma_p}$, as in Lemma 3.10. Using the spherical Fourier transform of b_λ and its $L^{p,1}$ norm estimates, we can show that the associated resolvent transform is the zero polynomial for each $g \in L^{p',\infty}(G//K)$ that annihilates the ideal I. This gives a proof of Theorem 1.3. □

REMARK 4.7. We could not prove the Wiener Tauberian theorem for $L^{p,1}(G//K)$, $1 \leq p < 2$, for a real-rank-one semisimple Lie group G (other than $\mathrm{SL}(2,\mathbb{R})$), because of the following reason. It is known that for $\lambda \in \mathbb{C}_+$, $b_\lambda(a_t)$ is asymptotic to $t^{-(m_1+m_2-1)}$ when $m_1 + m_2 > 1$. By a direct calculation, it follows that $\{b_\lambda : \mathrm{Im}\lambda > \gamma_p\}$ does not belong to $L^{p,1}(G//K)$ unless $p < (m_1+m_2+1)/(m_1+m_2-1)$. So one cannot define the resolvent transform as in Section 4.3 for all $p \in [1,2)$. Even for $p < (m_1+m_2+1)/(m_1+m_2-1)$, we are unable to prove $\|b_\lambda\|_{p,1}$ goes to zero as $\lambda \to \infty$, which was crucially used to show that $\mathcal{R}[g]$ is the zero polynomial.

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