We say that a subgroup $H$ is an $n$-th maximal subgroup of $G$ if there exists a chain of subgroups $G = G_0 > G_1 > \cdots > G_n = H$ such that each $G_i$ is a maximal subgroup of $G_{i-1}$, $i = 1, 2, \cdots, n$. The purpose of this note is to classify all finite simple groups with the property that every third maximal subgroup is nilpotent.

**Theorem.** If $G$ is a finite simple group such that every third maximal subgroup of $G$ is nilpotent, then $G$ is isomorphic either to a linear fractional group $\text{PSL}(2, q)$, for certain $q > 3$, or to $\text{Sz}(2^3)$, the Suzuki simple group over the field of $2^3$ elements.

**Remark.** If the group $\text{PSL}(2, q)$ satisfies the condition that all third maximal subgroups are nilpotent then it follows that

(a) $q = 2^r$, $3^s$ or $t$, where $r$, $s$, $t$ are primes, $r > 2$, and

(b) if $q = 3^s$ or $2^t$, then $(q+1)/\varepsilon$, $(q-1)/\varepsilon$, where $\varepsilon = 2$ if $q$ is odd, and $\varepsilon = 1$ if $q$ is even, are products of at most two (not necessarily distinct) primes; if $q = t$, then $(t-1)/2$ is a product of at most two primes and $(t+1)/2$ is either a product of at most two primes or a power of 2.

Conversely, the groups $\text{Sz}(2^3)$ and $\text{PSL}(2, q)$, where $q$ satisfies the conditions (a) and (b) above, have the property that every third maximal subgroup is nilpotent.

**Notations and known results.** We let $H \leq G$, $H < G$, $H \triangleleft G$ mean that $H$ is a subgroup, a proper subgroup, and a normal subgroup of $G$, respectively. We let $N(S)$, $C(S)$, for any subset $S$ of $G$, denote the normalizer and the centralizer of $S$ in $G$, respectively. We let $Z(G)$ denote the centre of $G$. If $x$ is any element of $G$ we let $\langle x \rangle$ be the group generated by $x$.

The following result of Janko [3] and Berkovič [1] is essential.

**Lemma 1.** If $G$ is a finite group all of whose second maximal subgroups are nilpotent, then $G$ is either soluble or isomorphic to $\text{PSL}(2, 5)$ or $\text{SL}(2, 5)$. 

**Proof of Theorem.** According to a definition of J. G. Thompson [6], we say that a finite group $G$ is an $N$-group if the normalizer of any non-trivial soluble subgroup of $G$ is itself soluble.

If $G$ is a non-abelian simple group all of whose third maximal subgroups are nilpotent, then $G$ is an $N$-group. For suppose that there exists a non-trivial soluble subgroup $S \leq G$ such that $N(S)$ is non-soluble. The group $H = N(S)$ is clearly a maximal subgroup of $G$. For if not, there exists a subgroup $M$, with $H \leq M$, which is second maximal in $G$. But then $M$ has the property that every proper subgroup of $M$ is nilpotent, and hence, by a result of Iwasawa [2], we see that $M$ is soluble, a contradiction. Now $H$, being maximal in $G$, has the property that every second maximal subgroup of $H$ is nilpotent. By Lemma 1 we see that $H \cong PSL(2,5)$ or $H \cong SL(2,5)$. Since $S \neq 1$, we have that $H \cong SL(2,5)$ and $S = Z(H)$. The Theorem D of Suzuki [5] p. 682, gives that $G$ is non-simple, a contradiction.

Therefore $G$ is a $N$-group. Now Thompson [6] has classified all finite $N$-groups and since $G$ is simple we see that $G$ is isomorphic to one of the following groups:

- $PSL(2, q)$, $q$ a prime power, $q > 3$,
- $Sz(2^{2n+1})$,
- $PSL(3, 3)$,
- $M_{11}$,
- $A_7$,

or

- $PSU_3(3^2)$.

**Case 1.** The groups $Sz(q)$, $q = 2^{2n+1}$, $n \geq 1$.

Let $2q = r^2$, $r = 2^{n+1}$. Then if $G = Sz(q)$, the following relations must be satisfied:

- $q-1 = \text{prime}$,
- $q+r+1 = \text{prime}$,
- $q-r+1 = \text{prime}$.

These are satisfied only if $n = 1$, since at least one is divisible by 5.

**Case 2.** The group $PSL(3, 3)$ is inadmissible since this contains the Hessian group $H$ as a subgroup. The group $H$ has subgroups $H > F > E$, in the notation of Miller-Blichfeldt-Dickson [4] p. 239 and $E$ is non-nilpotent of order 36.

**Case 3.** The groups $M_{11}$ and $A_7$.

Both these groups are inadmissible since they contain a subgroup
isomorphic to $A_4$, while non-soluble subgroups of any group with our property are either $PSL(2, 5)$ or $SL(2, 5)$.

**Case 4.** The group $PSU_3(3^2)$.

This group is inadmissible since $PSU_3(3^2) > U_2(3^2)$. The group $SU_2(3^2)$ is contained in $U_2(3^2)$ with index 4 and is non-nilpotent.

Thus we have ruled out all possibilities except the linear fractional groups and the group $Sz(2^3)$, as stated in the theorem.

Now suppose that the group $G$ is isomorphic to $PSL(2, q)$, $q > 3$. Then $G$ has the property that every third maximal subgroup is nilpotent if and only if every maximal subgroup $H$ has the property (*) every second maximal subgroup of $H$ is nilpotent.

Let $q = p^n > 3$, $p$ a prime. A $p$-Sylow normalizer $N$ is isomorphic to the groups of transformations of $GF(q)$

$$x \rightarrow \alpha^2 x + \beta, \quad \alpha, \beta \in GF(q), \quad \alpha = 0.$$  

It follows that $N$ satisfies (*) if and only if

(a) $(q-1)/e$ is a product of less than or equal to two primes (not necessarily distinct);
(b) $n = 1$ when $p > 3$;
(c) $n$ is a prime greater than 2 when $p = 3$;
(d) $n$ is a prime when $p = 2$.

The dihedral subgroup $D_{2(q+1)/e}$ satisfies (*) if and only if

(e) $(q+1)/e$ is a product of at most two primes or a power of 2.

The condition (a) implies that (*) holds for the dihedral groups $D_{2(q-1)/e}$. The only other possible maximal subgroups are $PSL(2, 5)$, $PSL(2, 3)$, $S_4$, or $PSL(2, 2)$, and these groups automatically satisfy (*)

References


Australian National University, Canberra

and

Monash University, Melbourne