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GLOBAL EXISTENCE AND COMPARISON THEOREM FOR A NONLINEAR PARABOLIC EQUATION

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In this paper we consider a nonlinear parabolic equation with gradient dependent nonlinearities of the form

$$u_t - \Delta u = a|u|^p + b|\nabla u|^q,$$

$0 < p, q$ and $a, b \in \mathbb{R}$, with homogeneous boundary condition in a bounded domain $\Omega \subseteq \mathbb{R}^N$. In the case $0 < p, q \leqslant 1$ we prove the existence of solution for suitable initial data. A comparison theorem for the solutions with respect to supersolutions and subsolutions is proved. Using these result, uniqueness and boundedness of solutions is studied.

1. INTRODUCTION

We consider the nonlinear parabolic problem with gradient dependent nonlinearities:

$$\text{(1.1)} \qquad \begin{cases} u_t - \Delta u = a|u|^p + b|\nabla u|^q & \text{in} \quad Q_T = \Omega \times (0,T), \\ u = 0 & \text{on} \quad S = \partial\Omega \times (0,T), \\ u(x,0) = \phi(x) & \text{in} \quad \Omega, \end{cases}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $p > 0$, $q > 0$ and $a, b \in \mathbb{R}$.

In the case of $q = 1, b > 0$ and $a = 0$ the problem was considered by Ben Artzi in [**3, 4**]. He showed the existence and decay of the global solution when $\Omega = \mathbb{R}^n$. He also introduced the open problem of the existence of the solution when $\Omega \neq \mathbb{R}^N$. In this paper we consider this problem. We shall show the existence, uniqueness and boundedness of the solution for suitable initial data, where Ω is a bounded domain in $\mathbb{R}^N$.

The problem (1.1) without the gradient term, that is, the equation

$$u_t - \Delta u = |u|^{p-1}u,$$

with $p > 1$, has been extensively studied by many authors providing various sufficient conditions for blow up and global existence. Moreover some qualitative properties, such

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as, the nature of the blow up set, the rate and profile of blow up, maximum existence time and continuation after blow up, boundedness of global solutions, and convergence to a stationary state were investigated. For these topics we refer the reader to the books and survey articles [**7, 19, 12, 22, 16, 2, 13, 3, 8**].

In the case $p, q > 1$ and $b > 0$ several authors have studied the existence of nonglobal positive solutions by giving some conditions for blow up, under certain assumptions on p, q, N and Ω (see for instance [**6, 9, 8, 14, 15, 17, 18, 20, 21**]). The problem (1.1) was considered by Souplet in [**18**] for $p > 1$ and $q > 1$. He proposed a model in population dynamics, where this type of equations describes the evolution of the population density of some biological species under the effect of a certain natural mechanism.

The aim of this paper is to prove the existence and uniqueness of global weak solutions for initial data in $L^2(\Omega)$ for $0 < p, q \leqslant 1$.

The following problem has been considered in [**1**],

$$\begin{cases} u_t = \triangle u^m - |\nabla u^\alpha|^q + u^p & \text{in} \quad Q = \Omega \times (0, \infty), \\ u = 0 & \text{on} \quad S = \partial\Omega \times (0, \infty), \\ u(x,0) = \phi(x) \geqslant 0 & \text{in} \quad \Omega, \end{cases}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$. It has been shown that a global weak solution exists for nonnegative initial data in $L^{m+1}(\Omega)$, under the assumptions $m \geqslant 1$, $\alpha \geqslant m/2$, $1 \leqslant q < 2$ and $1 < p \leqslant \alpha q$. The authors in [**1**] also introduced the open problem of the uniqueness of the solution. Our result, in part, gives a solution to this problem in the case $\alpha = m = q = p = 1$.

This paper is organised as follows. In section two we establish the existence of global weak solution for initial data in $L^2(\Omega)$. In the third section we consider subsolutions and supersolutions and prove a comparison theorem for the case $p = q = 1$. In section four, by using our comparison theorem, we prove the boundedness of solutions.

2. Existence of global solutions

In this section we prove the existence of global weak solution of problem (1.1), when the initial data is in $L^2(\Omega)$ and $0 < p, q \leqslant 1$. The techniques in [**20, 6**] for the existence in the case $p > 1, q > 1$ rely on the differentiabilty of $J_1(u) = u^p$ and $J_2(u) = |\triangle u|^q$, and are not applicable here for the case $0 < p, q \leqslant 1$. Our technique is based on Galerkin's method.

In the following we give some notations and definitions which will be used later. Let $\Omega \subset \mathbb{R}^N$ be a domain with smooth boundary, $T > 0$ and $Q_T = \Omega \times (0, T)$.

Definition. Let the initial data $\phi(x) \in L^2(\Omega)$. By a weak solution of the problem

(1.1) on Q_T, we mean a function $u(x,t) \in L^2(0,T;H_0^1(\Omega))$ such that

$$(2.2)\qquad \int_{Q_T}\Big[-u\zeta_t + \nabla u.\nabla\zeta - \big(a|u|^p + b|\nabla u|^q\big)\zeta\Big]\,dx\,dt \\ + \int_\Omega u(x,T)\zeta(x,T)\,dx - \int_\Omega \phi(x)\zeta(x,0)\,dx = 0,$$

for every test function $\zeta(x,t) \in W^{1,2}(0,T;H_0^1(\Omega))$. We shall say that u is a global weak solution of the problem (1.1), if u is a weak solution on Q_T for all positive T.

The following lemmas in [11] are crucial in our work. In these lemmas $\Omega' \subseteq \Omega$ is an arbitrary measurable subset of Ω and $\operatorname{mes}\Omega'$ means the Lebesgue measure of the set Ω'.

Lemma 2.1. *If $\|u_k\|_r \leqslant M, r > 1, M > 0$, for a given sequence of functions $u_k(x)$, $k = 1,2,\dots$ then it is possible to extract a subsequence from $\{u_k\}$, that is weakly convergent in $L^r(\Omega)$. If in addition, $\|u_k\|_{r,\Omega'} \leqslant \mu(\operatorname{mes}\Omega')$ for every mensurable subset Ω' of Ω, where $\mu(\tau)$ is a continuous function for $\tau \geqslant 0$ and $\mu(0) = 0$, then it is possible to extract a subsequence from $\{u_k\}$ which is strongly convergent in $L^r(\Omega)$. If $\{u_k(x)\}$ converges to u almost everywhere on Ω and $\|u_k\|_{r,\Omega} \leqslant M$, $r > 1$, $M > 0$ then $\{u_k\}$ converges to u strongly in $L^{r^*}(\Omega)$, for every $r^* < r$, and weakly in $L^r(\Omega)$.*

Lemma 2.2. *Let $f(x,u)$ be a measurable function on the set $\{x \in \Omega, u \in (-\infty,\infty)\}$, which is continuous in u for almost all x from Ω. If a sequence of functions $\{u_k(x)\}$ from $L^1(\Omega)$ converges almost everywhere to $u(x) \in L^1(\Omega)$ and $\big\|f(x,u_k(x))\big\|_{r,\Omega} \leqslant M$, $r > 1$, then the functions $f(x,u_k(x))$ converge to $f(x,u(x))$ in the norm of $L^{r^*}(\Omega)$, for every $r^* < r$ and weakly in $L^r(\Omega)$. If in addition, it is known that $\big\|f(x,u_k(x))\big\|_{r,\Omega'} \leqslant \mu(\operatorname{mes}\Omega')$, where $\mu(\tau)$ is a continuous function of $\tau \geqslant 0$ and $\mu(0) = 0$, then $\big\{f(x,u_k(x))\big\}$ converges to $f(x,u(x))$ strongly in $L^r(\Omega)$.*

The following theorem is the main result of this section.

Theorem 2.3. *Let the initial data $\phi(x)$ be in $L^2(\Omega)$, then the problem (1.1) has a weak solution in $W^{1,2}(0,T;H_0^1(\Omega))$.*

Proof: We take a fundamental system $\{\psi_k(x)\}$, $k = 1,2,\dots$ in the space $H_0^1(\Omega)$ such that $\int_\Omega \psi_i\psi_j\,dx = \delta_{ij}$ and $\max_\Omega\{|\psi_k|, |\nabla\psi_k|\} = C_k < \infty$. An approximate solution $u^n(x,t)$ for the problem (1.1) will be sought in the usuall form $u^n(x,t) = \sum_{k=1}^n C_k^n(t)\psi_k(x)$, where $C_k^n(t)$, $k = 1,2,\dots,n$ are determined by the system of ordinary differential equations

$$(2.3)\qquad \begin{cases} \displaystyle\int_\Omega u_t^n\psi_k\,dx + \int_\Omega \nabla u^n.\nabla\psi_k\,dx - \int_\Omega \big[a|u^n|^p + b|\nabla u^n|^q\big]\psi_k\,dx = 0, \\ C_k^n(0) = \displaystyle\int_\Omega \psi_k\phi(x)\,dx \qquad k = 1,2,\dots,n. \end{cases}$$

On the other hand for each n, there is a T_n with $0 < T_n \leqslant T$ such that $C_k^n(t)$, k

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$= 1, 2, \ldots, n$ is a solution of (2.3), and

$$\max_{0 \leqslant t \leqslant T_n} \sum_{k=1}^{n} \left[C_k^n(t)\right]^2 = \max_{0 \leqslant t \leqslant T_n} \|u^n\|^2_{L^2(\Omega)}.$$

For simplicity in writing in the next paragraph we write $u(x,t)$ instead of $u^n(x,t)$. First of all notice that from the equations (2.3) for $0 \leqslant t \leqslant T_n$ we have

$$\int_\Omega u_t u\, dx + \int_\Omega |\nabla u|^2\, dx - \int_\Omega \left[a|u|^p + b|\nabla u|^q\right] u\, dx = 0. \tag{2.4}$$

Now by using Young's inequality, we get

$$|bu|\ |\nabla u|^q \leqslant \frac{1}{2}|\nabla u|^2 + C_1 |u|^{2/(2-q)},$$

and

$$|au|\ |u|^p \leqslant C_1(1+u^2),$$

where C_1 is a positive constant. Thus

$$\frac{1}{2}|\nabla u|^2 + b|u|\ |\nabla u|^q + au|u|^p \leqslant |\nabla u|^2 + C_1|u|^{2/2-q} + C_1(1+u^2).$$

Again by using Young's inequality for the second term on the left hand side, we obtain

$$\begin{aligned} \frac{1}{2}|\nabla u|^2 + b|u|\ |\nabla u|^q + au|u|^p &\leqslant |\nabla u|^2 + C_2(1+u^2) + C_1(1+u^2) \\ &= |\nabla u|^2 + (C_1+C_2)(1+u^2), \end{aligned} \tag{2.5}$$

where C_2 is a positive constant. Integrating (2.5) over Ω yields

$$\begin{aligned} \frac{1}{2}\textstyle\int_\Omega |\nabla u|^2\, dx + b\int_\Omega |u|\ |\nabla u|^q\, dx + a\int_\Omega u|u|^p\, dx &\leqslant (C_1+C_2)\textstyle\int_\Omega (1+u^2)\, dx \\ &\quad + \textstyle\int_\Omega |\nabla u|^2\, dx. \end{aligned}$$

Thus

$$\begin{aligned} &\int_\Omega |\nabla u|^2\, dx - a\int_\Omega u|u|^p\, dx - b\int_\Omega u|\nabla u|^q\, dx \\ &\qquad\qquad \geqslant \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - (C_1+C_2)\int_\Omega (1+u^2)\, dx. \end{aligned}$$

By using (2.4) we obtain

$$-\int_\Omega u_t u\, dx \geqslant \frac{1}{2}\int_\Omega |\nabla u|^2 dx - (C_1+C_2)\int (1+u^2)\, dx. \tag{2.6}$$

Integrating (2.6) in time over $[0,t]$ gives,

$$-\int_0^t \int_\Omega u_t u\, dx\, dt \geqslant \frac{1}{2}\int_0^t \int_\Omega |\nabla u|^2 dx\, dt - (C_1+C_2)\int_0^t \int_\Omega (1+u^2)\, dx\, dt.$$

Therefore

$$-\frac{1}{2}\|u(x,t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u(x,0)\|^2_{L^2(\Omega)} \geqslant \frac{1}{2}\int_0^t\int_\Omega |\nabla u|^2\,dx\,dt - (C_1+C_2)\int_0^t\int_\Omega (1+u^2)\,dx\,dt.$$

But $\|u(x,0)\|^2_{L^2(\Omega)} = \sum_{k=1}^{n} C_k^n(0)^2 \leqslant \|\phi(x)\|^2_{L^2(\Omega)}$. Hence,

$$\|u(x,t)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega |\nabla u|^2\,dx\,dt \leqslant \|\phi(x)\|^2_{L^2(\Omega)} + 2(C_1+C_2)\int_0^t\int_\Omega (1+u^2)\,dx\,dt. \tag{2.7}$$

Thus

$$\|u(x,t)\|^2_{L^2(\Omega)} \leqslant \|\phi(x)\|^2_{L^2(\Omega)} + 2(C_1+C_2)t\ \operatorname{mes}(\Omega) + 2(C_1+C_2)\int_0^t \|u(x,t)\|^2_{L^2(\Omega)}\,dt.$$

Now by applying Gronwall's inequality, we get,

$$\|u^n(x,t)\|_{L^2(\Omega)} = \|u(x,t)\|_{L^2(\Omega)} \leqslant C_3 = C_3(t,\phi,\Omega). \tag{2.8}$$

Finally, from (2.7) and (2.8) we obtain,

$$\|u^n(x,t)\|_{L^2(Q_t)} + \|\nabla u^n(x,t)\|_{L^2(Q_t)} \leqslant C_4 = C_4(t,\phi,\Omega), \tag{2.9}$$

where $C_4(t,\phi,\Omega)$ is a continuous function for $t \geqslant 0$. In particular (2.8) implise that $T_n = T$.

Now we shall show that the sequence $\{u^n(x,t)\}$ converges to a function $u(x,t)$, which is a weak solution of the problem (1.1).

By considering the uniform estimate (2.9) and Lemma 2.1 it is possible to choose a subsequence from $\{u^n\}$ which is weakly convergent in $L^2(Q_T)$ to a function u, moreover the derivatives sequence $\left\{\frac{\partial u^n}{\partial x_i}\right\}$ is convergent weakly in $L^2(Q_T)$ to $\frac{\partial u}{\partial x_i}$.

Let Q' be an arbitrary meansurable subset of Q_T and let $0 < \alpha \leqslant 2$, then by Holder's inequality we have:

$$\begin{aligned}\int_{Q'} |u^n(x,t)|^\alpha\,dx\,dt &\leqslant \left(\int_{Q'} |u^n(x,t)|^2\,dx\,dt\right)^{\alpha/2} \operatorname{mes}(Q')^{2-\alpha/2}\\ &\leqslant C_4(T)^\alpha \operatorname{mes}(Q')^{2-\alpha/2},\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}\int_{Q'} |\nabla u^n(x,t)|^\alpha\,dx\,dt &\leqslant \left(\int_{Q'} |\nabla u^n(x,t)|^2\,dx\,dt\right)^{\alpha/2} \operatorname{mes}(\Omega')^{2-\alpha/2}\\ &\leqslant C_4(T)^\alpha \operatorname{mes}(Q')^{2-\alpha/2}.\end{aligned} \tag{2.11}$$

Hence for the special case $\alpha = 3/2$

$$\|u^n(x,t)\|_{L^{3/2}(Q')} \leqslant \big(C_4(T)\big)\,\mathrm{mes}(Q')^{1/6}. \quad .$$

By setting $\mu\big(\mathrm{mes}(Q')\big) = \big(C_4(T)\big)\,\mathrm{mes}(Q')^{1/6}$ in Lemma 2.1, it follows that, there is a subsequence of $\{u^n\}$ which is convergent to a function $u(x,t)$ in $L^{3/2}(Q_T)$. Then there is a subsequence of $\{u^n\}$ which is convergent to u almost everywhere. Now if we let $f(x,u^n) = |u^n|^p$ in Lemma 2.2, it follows that $f(x,u^n)$ is convergent weakly to $f(x,u)$ in $L^2(Q_T)$.

By using a simillar argument one can conclude that there is a subsequence of $\{u^n\}$ such that $|\nabla u^n|^q$ is convergent weakly to $|\nabla u|^q$ in $L^2(Q_T)$.

Therefore there is a subsequence of $\{u^n\}$, say again $\{u^n\}$, such that

$$(2.12)\qquad \begin{array}{ll} u^n \to u & \text{almost everywhere} \\ u^n \to u & \text{weakly in } L^2(Q_T), \\ \dfrac{\partial u^n}{\partial x_i} \to \dfrac{\partial u}{\partial x_i} & \text{weakly in } L^2(Q_T), \\ |u^n|^p \to |u|^p & \text{weakly in } L^2(Q_T), \\ |\nabla u^n|^q \to |\nabla u|^q & \text{weakly in } L^2(Q_T). \end{array}$$

Now we can prove u is a weak solution of the problem (1.1). Let $\phi^m = \sum_{k=1}^{m} d_k(t)\psi_k(x)$ where $d_k(t), k = 1,2,\dots,m$ are continuous functions with weak derivatives $d_k'(t)$ in $L^2(0,T)$. We denote the set of such ϕ^m by A_m. From the equality (2.3) for $\phi^m \in A_m$, where $m \leqslant n$, we have:

$$\int_{Q_T} [-u^n\phi_t^m + \nabla u^n.\nabla \phi^m]\,dx\,dt - \int_{Q_T} \big[a|u^n|^p + b|\nabla u^n|^q\big]\phi^m\,dx\,dt + \int_\Omega u^n\phi^m\,dx\big|_0^T = 0.$$

Now for a fixed $m \geqslant 1$, let n tend to ∞, then from (2.12) and the above equality we get

$$(2.13)\quad \int_{Q_T} [-u\phi_t^m + \nabla u.\nabla \phi^m]\,dx\,dt - \int_{Q_T} \big[a|u|^p + b|\nabla u|^q\big]\phi^m\,dx\,dt + \int_\Omega u\phi^m\,dx\big|_0^T = 0.$$

Since $\bigcup_{m=1}^{\infty} A_m$ is dense in $W^{1,2}\big(0,T;H_0^1(\Omega)\big)$, (2.13) is valid for every function $\phi(x,t)$ in $W^{1,2}\big(0,T;H_0^1(\Omega)\big)$.

In the next step we are going to see that $u(x,t) \in W^{1,2}\big(0,T;H_0^1(\Omega)\big)$. If we multiply the equation in (2.3) by $\dfrac{\partial C_k^n(t)}{\partial t}$ and summing over k, we get,

$$(2.14)\qquad \int_\Omega (u_t^n)^2\,dx + \int_\Omega \nabla u^n.(\nabla u^n)_t - \int_\Omega \big[a|u^n|^p + b|\nabla u^n|^q\big]u_t^n\,dx = 0.$$

For simplicity in the next paragraph we let $u^n(x,t) = u(x,t)$. From (2.14), by using Young's inequality, we obtain,

$$\int_\Omega u_t^2\,dx \leqslant -\int_\Omega \nabla u.(\nabla u)_t\,dx + \frac{1}{2}\int_\Omega u_t^2\,dx + C\int_\Omega \big(|u|^{2p} + |\nabla u|^{2q}\big)\,dx,$$

where C is a positive constant. It follows that

$$0 \leqslant \frac{1}{2}\int_\Omega u_t^2\,dx \leqslant -\int_\Omega \nabla u.(\nabla u)_t\,dx + C\int_\Omega \left(|u|^{2p} + |\nabla u|^{2q}\right)dx,$$

Consequently

$$\int_\Omega \nabla u.(\nabla u)_t\,dx \leqslant C\int_\Omega \left(|u|^{2p} + |\nabla u|^{2q}\right)dx.$$

Hence

$$\int_\Omega u_t^2\,dx \leqslant 4C\int_\Omega \left(|u|^{2p} + |\nabla u|^{2q}\right)dx. \tag{2.15}$$

Integrating (2.15) in time over $[0,T]$ and using (2.9) we get

$$\int_{Q_T} (u_t^n)^2\,dx\,dt = \int_{Q_T} u_t^2\,dx\,dt \leqslant C_0, \tag{2.16}$$

where C_0 is a positive constant.

By considering (2.16) we can suppose, possibly by passing to a subsequnce, that

$$(u^n)_t \to w \qquad \text{weakly in } L^2(Q_T).$$

Since $u \in L^2(Q_T)$, we have $u \in D'\big(]0,T[;H_0^1(\Omega)\big)$, where $D'\big(]0,T[;H_0^1(\Omega)\big)$ is the space of the $H_0^1(\Omega)$-valued distribution on $]0,T[$. Thus, $\partial_t u \in D'\big(]0,T[;X\big)$. Moreover for $\zeta \in C_0^\infty(Q_T)$, we have

$$\begin{aligned}\int_{Q_T} \partial_t u\zeta\,dx\,dt &= -\int_{Q_T} u\zeta_t\,dx\,dt = -\lim_{n\to\infty}\int_{Q_T} u_n\zeta_t\,dx\,dt \\ &= \lim_{n\to\infty}\int_{Q_T} (u_n)_t\zeta\,dx\,dt = \int_{Q_T} w\zeta\,dx\,dt,\end{aligned}$$

and consequently, $w = \partial_t u$. Therefore, $u \in L^2\big(0,T;H_0^1(\Omega)\big)$ and $\partial_t u \in L^2\big(0,T;H_0^1(\Omega)\big)$. Hence as a consequence of [5, Proposition A.6], there exists $\tilde{u} \in W^{1,2}\big(0,T;H_0^1(\Omega)\big)$ such that

$$u = \tilde{u} \text{ and } \partial_t u = d\tilde{u}/dt \qquad \text{almost everywhere in }]0,T[.$$

This completes the proof of theorem. ∎

3. Comparison Theorem and Uniqueness

In this section, we consider subsolutions and supersolutions for the problem (1.1), and we prove a comparison theorem for these kind of solutions.

Definition. We say that $u(x,t) \in W^{1,2}\big(0,T;H_0^1(\Omega)\big)$ is a subsolution (supersolution) of the problem (1.1), if $u(x,0) \leqslant (\geqslant)\phi(x)$ on Ω and the inequality

$$\int_{Q_T} u_t\zeta\,dx\,dt + \int_{Q_T} \nabla u.\nabla\zeta\,dx\,dt - \int_{Q_T} \left(a|u|^p + b|\nabla u|^q\right)\zeta\,dx\,dt \leqslant 0(\geqslant 0),$$

holds for every nonnegative test function $\zeta(t,x) \in L^2(0,T;H_0^1(\Omega))$.

THEOREM 3.1. *If u_- is a subsolution and u_+ a supersolution of problem* (1.1) *and $p=q=1$ then $u_-(x,t) \leqslant u_+(x,t)$ on Q_T.*

PROOF: We prove the theorem for the case $a \geqslant 0$ and $b \geqslant 0$, the proof of the other cases is simillar. Let $v(x,t) = u_+(x,t) - u_-(x,t)$. By the definition of supersolution and subsolution, $v(x,t)$ must satisfy in the following inequality:

(3.17)
$$\int_{Q_T} v_t\zeta\,dx\,dt + \int_{Q_T} \nabla v.\nabla\zeta\,dx\,dt[12pt] \geqslant \int_{Q_T} \big(a|u_+| + b|\nabla u_+|\big)\zeta\,dx\,dt - \int_{Q_T} \big(a|u_-| + b|\nabla u_-|\big)\zeta\,dx\,dt,$$

for every test function $\zeta \geqslant 0$. Hence

$$\begin{aligned}\int_{Q_T} v_t\zeta\,dx\,dt &+ \int_{Q_T} \nabla v.\nabla\zeta\,dx\,dt\\ &\geqslant \int_{Q_T} \big(a|u_+| + b|\nabla u_+|\big)\zeta\,dx\,dt\\ &- \int_{Q_T} \big(a|u_-| + a|v| + b|\nabla u_+| + b|\nabla v|\big)\nabla\psi\,dx\,dt\\ &= -a\int_{Q_T} |v|\zeta\,dx\,dt - b\int_{Q_T} |\nabla v|\zeta\,dx\,dt.\end{aligned} \tag{3.18}$$

Let

$$\zeta(t,x) = \begin{cases} \dfrac{|v| - v}{2} & 0 \leqslant t \leqslant T_0,\\ 0 & T_0 < t \leqslant T,\end{cases}$$

where $0 \leqslant T_0 \leqslant T$ is arbitrary.

If we set

$$\Omega_1(t) = \{x \in \Omega \mid v(x,t) < 0\}, \Omega_2(t) = \{x \in \Omega \mid v(x,t) \geqslant 0\},$$

where $0 \leqslant t \leqslant T_0$, it follows that

$$\begin{aligned}\int_\Omega v_t(x,t)\zeta(x,t)\,dx &= \int_{\Omega_1(t)} v_t\zeta\,dx + \int_{\Omega_2(t)} v_t\zeta\,dx\\ &= \int_{\Omega_1(t)} v_t\zeta\,dx = -\int_{\Omega_1(t)} \zeta_t\zeta\,dx = -\int_\Omega \zeta_t\zeta\,dx.\end{aligned} \tag{3.19}$$

On the other hand

$$\begin{aligned}\int_\Omega \nabla v(x,t).\nabla\zeta(x,t)\,dx &= \int_{\Omega_1(t)} \nabla v.\nabla\zeta\,dx + \int_{\Omega_2(t)} \nabla v.\nabla\zeta\,dx\\ &= -\int_{\Omega_1(t)} |\nabla v|^2\,dx\\ &= -\int_{\Omega_1(t)} |\nabla\zeta|^2\,dx\\ &= -\int_\Omega \big|\nabla\zeta(x,t)\big|^2\,dx,\end{aligned} \tag{3.20}$$

$$
\text{(3.21)} \qquad \begin{aligned} |a|\int_\Omega |v|\zeta\,dx = |a|\int_{\Omega_1(t)} |v|\zeta\,dx + |a|\int_{\Omega_2(t)} |v|\zeta\,dx \ &= |a|\int_{\Omega_1(t)} \zeta^2\,dx \\ &\leqslant |a|\int_\Omega \zeta^2\,dx, \end{aligned}
$$

and

$$
\text{(3.22)} \qquad \begin{aligned} |b|\int_\Omega |\nabla v|\zeta\,dx \ &= |b|\int_{\Omega_1(t)} |\nabla v|\zeta\,dx + |b|\int_{\Omega_2(t)} |\nabla v|\zeta\,dx \\ &= |b|\int_{\Omega_1(t)} |\nabla \zeta|\zeta\,dx \\ &\leqslant \int_{\Omega_1(t)} |\nabla \zeta|^2\,dx + b^2\int_{\Omega_1(t)} \zeta^2\,dx \\ &\leqslant \int_\Omega |\nabla \zeta|^2\,dx + b^2\int_\Omega \zeta^2\,dx. \end{aligned}
$$

By considering (3.17), (3.18), (3.19), (3.20), (3.21), (3.22), we get

$$
\begin{aligned} \int_0^{T_0}\int_\Omega \zeta_t(x,t)\zeta(x,t)\,dx\,dt \ &+ \int_0^{T_0}\int_\Omega \left|\nabla\zeta(x,t)\right|^2 dx\,dt \\ &\leqslant \int_0^{T_0}\int_\Omega |\nabla\zeta|^2\,dx\,dt + C\int_0^{T_0}\int_\Omega \zeta^2\,dx\,dt, \end{aligned}
$$

where C is a positive constant. Hence

$$
\int_0^{T_0}\int_\Omega \zeta_t\zeta\,dx\,dt \leqslant C\int_0^{T_0}\int_\Omega \zeta^2\,dx\,dt.
$$

On the other hand

$$
\int_0^{T_0}\int_\Omega \zeta_t\zeta\,dx\,dt = \frac{1}{2}\int_\Omega \zeta^2(x,T_0)\,dx - \frac{1}{2}\int_\Omega \zeta^2(x,0)\,dx = \frac{1}{2}\int_\Omega \zeta^2(x,T_0)\,dx.
$$

Therefore

$$
\int_\Omega \zeta^2(x,T_0)\,dx \leqslant 2C\int_0^{T_0}\int_\Omega \zeta^2\,dx\,dt.
$$

Hence

$$
\int_\Omega \left[\frac{|v|-v}{2}\right]^2 dx \leqslant 2C\int_0^{T_0}\int_\Omega \left[\frac{|v|-v}{2}\right]^2 dx\,dt.
$$

Then Gronwall's inequality implies that

$$
\int_\Omega \left[\frac{|v|-v}{2}\right]^2 dx = 0.
$$

Hence $\big(|v|-v\big)/2 = 0$, and $v \geqslant 0$. This completes the proof of the theorem. ∎

Corollary 3.2 *If in problem* (1.1), $p = q = 1$ *and* $\phi(x) \geqslant 0$, *then there is at least one positive solution.*

Corollary 3.3. *The solution of problem* (1.1) *is unique for* $p = q = 1$.

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4. Boundedness of solutions

In this section by using the comparison Theorem 3.1, we prove the boundedness of solutions in $L^\infty(Q_T)$ for $p = q = 1$.

Consider the problem

$$\Delta\psi + \lambda\psi = 0, \quad x \in \Omega, \quad \psi = 0, x \in \partial\Omega, \tag{4.23}$$

where Ω is a bounded domain in $\mathbb{R}^N$. The following lemma is well-known, the reader is referred to [**10**] for the proof.

Lemma 4.1. *There exists a positive eigenvalue of the problem* (4.23) *which has a positive eigenfunction on* Ω. *Moreover this eigenvalue is simple.*

Proposition 4.2. *Let in the problem* (1.1), Ω *be a regular domain of class* C^2 *in* $\mathbb{R}^N$, $b < 0$, $\phi(x) \in L^2(\Omega)$ *and* $0 \leqslant \phi(x) \leqslant \psi(x)$, *where* $\psi(x)$ *is the above eigenfunction.*

(i) *If* $a \leqslant 0$ *and* $p = q = 1$, *then* $u(x,t)$ *is bounded in* $L^\infty(Q_T)$.

(ii) *If* $a > 0$ *and small,* $q = p = 1$, *then* $u(x,t)$ *is bounded in* $L^\infty(Q_T)$.

Proof: (i) Consider the function $W(x,t) = e^{-\lambda t/2}\psi(x)$ on $\overline{\Omega} \times [0,T]$. For this function we have

$$W_t - \Delta W - a|W| - b|\nabla W| = \frac{\lambda}{2}\psi(x)e^{-\lambda t/2} - a|W| - b|\nabla W| \geqslant 0, \tag{4.24}$$

in $\Omega \times [0,T]$ and $W(t,x) = 0$ on $\partial\Omega \times [0,T]$.

Let $\zeta(t,x)$ be a nonnegative test function. By multiplying (4.24) by $\zeta(t,x)$ and integrating over Q_T, we get

$$\int_{Q_T} W_t\zeta\,dx\,dt + \int_{Q_T} \nabla W.\nabla\zeta\,dx\,dt - a\int_{Q_T} |W|^p\zeta(t,x)\,dx\,dt \\ - b\int_{Q_T} |\nabla W|^q\zeta(t,x)\,dx\,dt \geqslant 0.$$

But $0 \leqslant \phi(x) \leqslant W(x,0) = \psi(x)$. Thus by Comparison Theorem 3.1

$$0 \leqslant u(x,t) \leqslant Ce^{-\lambda t/2}.$$

(ii) Again if we let $W(x,t) = e^{-\lambda t/2}\psi(x)$ and $a \leqslant \lambda/2$, we have:

$$W_t - \Delta W - a|W| - b|\nabla W| \geqslant 0.$$

Hence by a simillar argument, we get

$$0 \leqslant u(x,t) \leqslant Ce^{-\lambda t/2}.$$

∎

Remark. By Considering (2.9), it follows that $\|u\|_{L^2(Q_T)}$ is bounded for $0 < p,\ q \leqslant 1$.

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