The starting point of this monograph is the notion of distributional stability and infinite divisibility. Stable distributions are the celebrated class which exhibit both of the aforesaid properties and, accordingly, offer a number of remarkably explicit formulae and identities. We therefore begin our journey by addressing the robust mathematical theory that supports the characterisation of stable distributions in preparation for later chapters.

### 1.1 One-dimensional stable distributions

We begin our discussion by first restricting ourselves to the one-dimensional setting. The following definition, for which we use \(=\) to mean equality in distribution, is key to the notion of distributional stability.

**Definition 1.1** A non-degenerate random variable \(X\) has a *stable* distribution if, for any \(a > 0\) and \(b > 0\), there exists \(c > 0\), such that

\[
aX_1 + bX_2 \overset{(d)}{=} cX,
\]

(1.1)

where \(X_1\) and \(X_2\) are independent and \(X_1 \overset{(d)}{=} X_2 \overset{(d)}{=} X\). We exclude from this definition the possibility that \(X \equiv 0\).

The experienced reader will immediately spot that Definition 1.1 pertains to what is more broadly known in the literature as a *strictly stable* random variable. The notion of a *stable* random variable is reserved for a slightly broader concept. Since we will never have occasion in this book to distinguish the difference, we will depart from the traditional convention and refer only to stable random variables, using this definition.
Stable Distributions

Observe that (1.1) implies

\[ X \overset{(d)}{=} \frac{X_1 + X_2}{d_2}, \]

for some constant \( d_2 > 0 \). By induction, it is easy to see that, for any \( n \geq 0 \), there exists a constant \( d_n > 0 \) and \( n \) independent random variables \( X_i, 1 \leq i \leq n \), with the same distribution as \( X \), such that

\[ X \overset{(d)}{=} \frac{X_1 + X_2 + \cdots + X_n}{d_n}. \quad (1.2) \]

Said another way, any stable random variable \( X \) is infinitely divisible.

For convenience, let us recall the so-called Lévy–Khintchine representation, which provides a complete characterisation of infinitely divisible distributions. We first introduce some notation. Let \( \mu \) be the probability distribution of a real-valued random variable and define its characteristic function by

\[ \hat{\mu}(z) = \int_{\mathbb{R}} e^{ix} \mu(dx), \quad z \in \mathbb{R}. \]

If \( \mu \) is an infinitely divisible distribution, then it is known that its characteristic function never vanishes. As a consequence, there exists a continuous function \( \Psi: \mathbb{R} \mapsto \mathbb{C} \), called the characteristic exponent of \( \mu \), such that

\[ \exp\{-\Psi(z)\} := \hat{\mu}(z), \quad \text{for } z \in \mathbb{R}. \quad (1.3) \]

**Theorem 1.2 (Lévy–Khintchine representation)** A function \( \Psi: \mathbb{R} \mapsto \mathbb{C} \) is the characteristic exponent of an infinitely divisible random variable if and only if there exists a triple \( (a, \sigma, \Pi) \), where \( a \in \mathbb{R}, \sigma \geq 0 \) and \( \Pi \) is a measure concentrated on \( \mathbb{R} \setminus \{0\} \) satisfying \( \int_{\mathbb{R}} (1 \wedge x^2) \Pi(dx) < \infty \), such that

\[ \Psi(z) = iaz + \frac{1}{2} \sigma^2 z^2 + \int_{\mathbb{R}} \left(1 - e^{izx} + izx1_{|x|<1}\right) \Pi(dx), \quad (1.4) \]

for every \( z \in \mathbb{R} \). Moreover, the triple \( (a, \sigma^2, \Pi) \) is unique within the given arrangement on the right-hand side of (1.4).

The measure \( \Pi \) is called the Lévy measure of the distribution \( \mu \) and \( \sigma \) its Gaussian coefficient. Whilst the triple \( (a, \sigma, \Pi) \) defining \( \Psi(z) \) is unique as described, in various situations one may prefer to use a different regularising function \( h(x) \), in which case (1.4) is written as

\[ \Psi(z) = iaz + \frac{1}{2} \sigma^2 z^2 + \int_{\mathbb{R}} \left(1 - e^{izx} + izh(x)\right) \Pi(dx), \quad z \in \mathbb{R}, \quad (1.5) \]
where
\[ \tilde{a} = a - \int_{\mathbb{R}} (h(x) - x1_{|x|<1}) \, \Pi(dx), \]
which is finite.

In this chapter, we shall interchange between the two equivalent representations given by (1.4) and (1.5). For example, when the measure \( \Pi \) satisfies the stronger condition
\[ \int_{\mathbb{R}} (1 \wedge |x|) \, \Pi(dx) < \infty, \]
we may choose \( h(x) \equiv 0 \). If the distribution \( \mu \) has finite mean, we may choose \( h(x) \equiv x \). In some cases, it will be convenient to choose \( h(x) = \sin(x) \) or \( h(x) = x/(1 + x^2) \). Everywhere in this book, when we say that the distribution \( \mu \) has characteristic triple \((a, \sigma, \Pi)\) without specifying the regularising function \( h \), we assume that the characteristic exponent is given via (1.4), otherwise we will say that the distribution \( \mu \) has characteristic triple \((a, \sigma, \Pi)\) with the regularising function \( h \), in which case \( \Psi \) will be given by (1.5).

The following main result provides the explicit characteristic exponent of stable distributions. As part of its proof, which will be provided in the next section, we will also be obliged to understand the structure of the underlying triple \((a, \sigma, \Pi)\) in the associated Lévy–Khintchine formula.

**Theorem 1.3** A stable random variable \( X \) has a characteristic exponent satisfying
\[ \Psi(z) = c|z|^\alpha \left(1 - i\beta \tan \left(\frac{\pi \alpha}{2}\right) \text{sgn}(z)\right), \quad z \in \mathbb{R}, \quad (1.6) \]
where
\[ \alpha \in (0, 1) \cup (1, 2], \quad c > 0 \quad \text{and} \quad \beta \in [-1, 1] \]
or
\[ \alpha = 1, \beta = 0 \quad \text{and we understand} \quad \beta \tan \left(\frac{\pi \alpha}{2}\right) := 0. \]

The latter case is known as the symmetric Cauchy distribution.

**Remark 1.4** Note that the symmetric Cauchy distribution with drift \( \delta \in \mathbb{R} \), that is,
\[ \Psi(z) = c|z| + \delta z, \quad z \in \mathbb{R}, \]
also belongs to the class of one-dimensional stable distributions. Nonetheless, we will henceforth only deal with the case that \( \delta = 0 \) when \( \alpha = 1 \).
Remark 1.5 We also note that the case $\alpha = 2$ corresponds to the case where $X$ has a Gaussian distribution. As we shall see in Chapter 2, associated to each of the distributions discussed in this chapter is a Lévy process. As one might expect, the case $\alpha = 2$ leads to Brownian motion. For other values of $\alpha$, we will find an association with Lévy processes that do not have continuous paths, the so-called $\alpha$-stable processes (also referred to as just stable processes). It is the case of processes with path discontinuities that forms the primary concern of this book. For this reason, the overwhelming majority of this text will be restricted to the setting that $\alpha \in (0, 2)$.

1.2 Characteristic exponent of a one-dimensional stable law

We dedicate this section entirely to the proof of Theorem 1.3. As part of this process, we need to establish two key intermediary results.

Lemma 1.6 The sequence $(d_k)_{k \geq 1}$ defined by (1.2) is strictly increasing and satisfies $d_k = k^{1/\alpha}$ for some $\alpha > 0$, $k \geq 1$.

Proof Recall that $\hat{\mu}$ denotes the characteristic function of a stable distribution $X$ and, thanks to the infinite divisibility of $X$, $\hat{\mu}(z) \neq 0$ for $z \in \mathbb{R}$. From the definition of the sequence $(d_k)_{k \geq 1}$, the scaling property in (1.2) can be reworded to say

$$\Psi(d_k z) = k \Psi(z), \quad z \in \mathbb{R}, \quad k \geq 1. \quad (1.7)$$

In turn, this implies $|\hat{\mu}(d_{k+1} z)| = |\hat{\mu}(z)||\hat{\mu}(d_k z)| \leq |\hat{\mu}(d_k z)|$ and hence

$$\left| \hat{\mu}\left(\frac{d_{k+1}}{d_k} z\right) \right| \leq |\hat{\mu}(z)|, \quad k \geq 1.$$

We are now forced to conclude that $d_{k+1} \geq d_k$, for $k \geq 1$. To see why, note that

$$\left| \hat{\mu}\left(\left(\frac{d_{k+1}}{d_k}\right)^n z\right) \right| \leq |\hat{\mu}(z)|, \quad \text{for any} \quad n \geq 1,$$

with $(d_{k+1}/d_k)^n \to 0$ as $n \to \infty$, which would imply that $1 \leq |\hat{\mu}(z)|$, leading to a contradiction.

Next, we observe that for all $m, n \geq 1$ and $z \in \mathbb{R},$

$$|\Psi(d_{mn} z)| = mn|\Psi(z)| = n|m\Psi(z)| = |n\Psi(d_m z)| = |\Psi(d_n d_m z)|,$$

implying that $d_{mn} = d_n d_m$. In particular, for any positive integer $j$, $d_{mj} = d_m^j$. If $1 < n < m$, there is a positive integer $p$ such that $m^j \leq n^p < m^{j+1}$. Using these inequalities and the established monotonicity of $(d_k)_{k \geq 1}$, we have
1.2 Characteristic exponent of a one-dimensional stable law

\[ \frac{j}{j+1} \log d_m \leq \log d_n \leq \frac{j+1}{j} \log d_m. \]

Hence, taking \( j \to \infty \), we get

\[ \frac{\log d_m}{\log m} = \frac{\log d_n}{\log n} =: \frac{1}{\alpha}, \]

for some strictly positive constant \( \alpha \). Therefore \( \log d_n = \log n^{1/\alpha} \) or equivalently \( d_n = n^{1/\alpha} \), for \( n \geq 1 \) and \( \alpha > 0 \). □

Our second intermediary result characterises the form of the underlying Lévy measure of any stable distribution.

**Proposition 1.7** If \( X \) is a stable random variable, then necessarily \( \alpha \in (0, 2] \).

In the case that \( \alpha = 2 \), \( X \) is Gaussian distributed. Otherwise when \( \alpha \in (0, 2) \), then there exist \( c_1, c_2 \geq 0 \) such that \( c_1 + c_2 > 0 \) and the underlying Lévy measure \( \Pi \) satisfies

\[ \Pi(dx) = \lvert x \rvert^{-1-\alpha} \left( c_1 1_{(x>0)} + c_2 1_{(x<0)} \right) dx, \quad x \in \mathbb{R}. \quad (1.8) \]

**Proof** Recall that identity \((1.7)\) and Lemma 1.6 imply \( k\Psi(z) = \Psi(k^{1/\alpha}z) \), for \( z \in \mathbb{R} \) and \( k \geq 1 \). More precisely, we observe

\[ ikaz + \frac{1}{2} k^2 \sigma^2 z^2 + \int_{\mathbb{R}} \left( 1 - e^{izx} + izx 1_{(|x|<1)} \right) k \Pi(dx) \]

\[ = iak^{1/\alpha}z + \frac{1}{2} \sigma^2 z^{2/\alpha} + \int_{\mathbb{R}} \left( 1 - e^{izk^{1/\alpha}x} + izk^{1/\alpha}x 1_{(|x|<1)} \right) \Pi(dx), \quad (1.9) \]

for any \( k \geq 1 \) and \( z \in \mathbb{R} \). Hence if \( \sigma > 0 \), we are forced to take \( \alpha = 2 \). Moreover, still in the setting \( \alpha = 2 \), if we then let \( k \) tend to \( \infty \), the latter identity implies \( a = 0 \) and \( \Pi \equiv 0 \). In conclusion, the case that \( \alpha = 2 \) corresponds to a Gaussian random variable.

Next, we assume \( \sigma = 0 \). Again from identity \((1.9)\), by changing variables in the integral on the right-hand side, we deduce

\[ k \Pi(dx) = \Pi(k^{-1/\alpha}dx), \quad x \neq 0. \]

Therefore, for the functions \( \overline{\Pi^+}(x) := \Pi([x, \infty)), \ x > 0, \) and \( \overline{\Pi^-}(x) := \Pi((-\infty, x)), \ x < 0, \) we have

\[ \overline{\Pi^+}(x) = \frac{1}{k} \overline{\Pi^+}(k^{-1/\alpha}x) \quad \text{and} \quad \overline{\Pi^-}(x) = \frac{1}{k} \overline{\Pi^-}(k^{-1/\alpha}x). \]

From the first of these two, we have, for all \( k, n \geq 1 \),

\[ \frac{1}{n} \overline{\Pi^+} \left( \frac{k^{1/\alpha}}{n^{1/\alpha}} \right) = \overline{\Pi^+} \left( k^{1/\alpha} \right) = \frac{1}{k} \overline{\Pi^+} (1). \]
Stable Distributions

Since \( \{(k/n)^{1/\alpha}; k, n \in \mathbb{N}\} \) is dense in \([0, \infty)\) and the function \( \Pi^+ \) is non-increasing, we deduce \( \Pi^+(x) = x^{-\alpha}\Pi^+(1) \), for \( x > 0 \). Similarly, we may deduce \( \Pi^-(x) = |x|^{-\alpha}\Pi^-(1) \), for \( x < 0 \).

Now taking \( c_1 := \alpha\Pi^+(1) \) and \( c_2 := \alpha\Pi^-(1) \), we obtain
\[
\Pi(dx) = |x|^{-1-\alpha}\left(c_11_{(x>0)} + c_21_{(x<0)}\right)dx, \quad x \in \mathbb{R},
\]
as required. As \( \Pi \) is a Lévy measure, in particular, it must satisfy the integral condition
\[
\int_{\mathbb{R}} (1 \wedge |x|^2) \Pi(dx) < \infty.
\]
We thus deduce that \( \alpha \in (0, 2) \). \( \square \)

Finally, we are ready to compute the characteristic exponent \( \Psi \) as stated in Theorem 1.3.

**Proof of Theorem 1.3** Since the case \( \alpha = 2 \) has already been characterised as Gaussian in the proof of Proposition 1.7, we set \( \sigma = 0 \) and focus on the case \( \alpha \in (0, 2) \).

We first observe that, when \( \alpha \in (0, 1) \) the function \( x \mapsto |x|^{-(\alpha + 1)} \) is integrable near 0 and hence we may take the regularising function in (1.5) to satisfy \( h(x) = 0 \). From identity (1.7), we deduce that \( \tilde{a} = 0 \) in (1.5), or in other words,
\[
a = -\int_{(|x|<1)} x \Pi(dx).
\]
Using the well-known integral identity for the gamma function, see for instance (A.7) in the Appendix, we have
\[
\int_{0}^{\infty} e^{ix}x^{-s-1}dx = z^{-s}\Gamma(s)e^{\pi i s/2}, \quad z > 0, \quad 0 < s < 1,
\]
and, appealing to integration by parts, we find that
\[
\int_{0}^{\infty} \left(e^{ix} - 1\right)x^{-1-\alpha}dx = z^\alpha e^{-\pi i \alpha/2}\Gamma(-\alpha), \quad z > 0.
\]
Making the change of variable \( x \mapsto -x \) and taking the complex conjugate of both sides we find
\[
\int_{-\infty}^{0} \left(e^{ix} - 1\right)|x|^{-1-\alpha}dx = \int_{0}^{\infty} \left(e^{-ix} - 1\right)x^{-1-\alpha}dx = z^\alpha e^{\pi i \alpha/2}\Gamma(-\alpha), \quad z > 0.
\]
for \( z > 0 \). Note also that when \( z \) takes negative values, we can similarly make use of the computations leading to (1.12). Then, we apply the following simple identity
1.2 Characteristic exponent of a one-dimensional stable law

\[ c_1 e^{-\pi i/2} + c_2 e^{\pi i/2} = (c_1 + c_2) \cos(\pi \alpha/2) \left( 1 - i \frac{c_1 - c_2}{c_1 + c_2} \tan(\pi \alpha/2) \right), \]

and observe that

\[ c = -(c_1 + c_2) \Gamma(-\alpha) \cos(\pi \alpha/2) > 0, \]

since \(-\Gamma(-\alpha)\) is positive for \(\alpha \in (0, 1)\). This completes the proof of the case \(\alpha \in (0, 1)\).

When \(\alpha \in (1, 2)\), the function \(x \mapsto |x|^{-(\alpha+1)}\) integrates \(x^2\) in a neighbourhood of 0 and hence we may take the regularising function in (1.5) as \(h(x) = x\). Again identity (1.7) implies \(\tilde{a} = 0\) in (1.5), and therefore

\[ a = \int_{|x| \geq 1} x \Pi(dx). \]

Similarly, we use (1.10) and apply integration by parts twice to find

\[ \int_0^\infty \left( e^{ix} - 1 - ix \right) x^{-1-\alpha} \, dx = z^\alpha e^{-\pi i/2} \Gamma(-\alpha), \tag{1.13} \]

for \(z > 0\), and the rest of the proof proceeds in the same way as in the case \(\alpha \in (0, 1)\).

Finally, the case \(\alpha = 1\) must be treated differently. In this case, we observe

\[ \int_0^\infty \left( 1 - e^{ix} + ix \chi_{|x| < 1} \right) \frac{dx}{x^2} = \int_0^\infty \left( 1 - \cos(x) \right) \frac{dx}{x^2} \]

\[ - i \int_0^\infty \left( \sin(x) - x \chi_{|x| < 1} \right) \frac{dx}{x^2}. \tag{1.14} \]

A change of variables followed by integration by parts gives us

\[ \int_0^\infty (1 - \cos(zx)) \frac{dx}{x^2} = |z| \int_0^\infty \frac{\sin(x)}{x} \, dx = |z| \int_0^\infty \int_0^\infty \sin(x) e^{-ux} \, du \, dx. \]

Since

\[ \int_0^\infty e^{-ux} \sin(x) \, dx = \frac{1}{u^2 + 1}, \tag{1.15} \]

we get

\[ \int_0^\infty (1 - \cos(zx)) \frac{dx}{x^2} = \frac{|z|\pi}{2}. \tag{1.16} \]
Next, for simplicity, we assume that \( z > 0 \). Observe that

\[
\int_0^\infty (\sin(zx) - zxI_{|x| < 1}) \frac{dx}{x^2} = \int_0^{1/z} (\sin(zx) - zx) \frac{dx}{x^2} + \int_{1/z}^\infty \sin(zx) \frac{dx}{x^2} - z \log z
\]

\[
= z \left( \int_0^1 (\sin(x) - x) \frac{dx}{x^2} + \int_1^\infty \sin(x) \frac{dx}{x^2} \right) - z \log z.
\]

(1.17)

Hence by defining

\[
K := \int_0^1 (\sin(x) - x) \frac{dx}{x^2} + \int_1^\infty \sin(x) \frac{dx}{x^2},
\]

and putting all the pieces in (1.16) and (1.17) back into (1.14), we deduce

\[
\int_0^\infty \left( 1 - e^{ix} + izxI_{|x| < 1} \right) \frac{dx}{x^2} = \frac{|z|\pi}{2} - iKz + iz \log |z|, \quad z \in \mathbb{R} \setminus \{0\}.
\]

Therefore, from Proposition 1.7 and the above reasoning, the characteristic exponent \( \Psi \) satisfies

\[
\Psi(z) = iaz + (c_1 - c_2)iKz + (c_1 + c_2)|z|\frac{\pi}{2} + (c_1 - c_2)iz \log |z|, \quad z \in \mathbb{R} \setminus \{0\}.
\]

As we must have \( \Psi(k^{1/\alpha}z) = k\Psi(z) \), \( z \in \mathbb{R}, k \in \mathbb{N} \), albeit now \( \alpha = 1 \), from Lemma 1.6, we deduce that \( c_1 = c_2 \) and then

\[
\Psi(z) = iaz + (c_1 + c_2)|z|\frac{\pi}{2}, \quad z \in \mathbb{R}.
\]

Taking note of Remark 1.4, by taking \( a = 0 \), we get the desired result. \( \square \)

Reviewing the proof here, we also get some information about the constants \( c_1 \) and \( c_2 \), appearing in Proposition 1.7, in relation to the parameters \( c \) and \( \beta \) in (1.6).

**Corollary 1.8** When \( \alpha \in (0, 2) \), the constants \( c_1, c_2 \) appearing in the Lévy measure (1.8) satisfy

\[
c = -(c_1 + c_2)\Gamma(-\alpha) \cos(\pi\alpha/2) \quad \text{and} \quad \beta = \frac{c_1 - c_2}{c_1 + c_2},
\]

(1.18)

when \( \alpha \in (0, 1) \cup (1, 2) \). Moreover, \( c_1 = c_2 \) with \( c = c_1\pi \), when \( \alpha = 1 \).

We also get from the proof of Theorem 1.3 the values of \( a \) in the Lévy–Khintchine triple 1.4. As such, the following corollary completes the statement of Proposition 1.7.
Corollary 1.9 When $\alpha \in (0, 1)$, the constant $a$ in the Lévy–Khintchine triple is equal to $-\int_{|x|<1} x\Pi(dx)$, when $\alpha \in (1, 2)$, we have $a = \int_{|x|\geq 1} x\Pi(dx)$ and when $\alpha = 1$, we have $a = 0$.

1.3 Moments

An important feature of stable distributions when $\alpha \in (0, 2)$, which is one of their signature properties that differs from the setting that $\alpha = 2$, is that they do not possess second moments (and hence no other greater moments). The precise cut-off where positive moments exist is the concern of the next main result.

Theorem 1.10 Suppose that $X$ is a stable distribution with index $\alpha \in (0, 2)$. Then $\mathbb{E}[|X|^\beta] < \infty$, for $0 \leq \beta < \alpha$, and for $\beta \geq \alpha$, we have $\mathbb{E}[|X|^\beta] = \infty$.

Proof We start by noting that, irrespective of the symmetry in the distribution of $X$, thanks to the shape of $\Pi$ given in Theorem 1.7, we have

$$\int_{|x|\geq 1} |x|^\beta \Pi(dx) < \infty,$$

for $\beta \in [0, \alpha)$ and infinite for $\beta \in [\alpha, \infty)$.

Next, note that the Lévy–Khintchine exponent (1.4), written here as $\Psi$, has $\sigma = 0$ and can be decomposed in the form $\Psi = \Psi^{(1)} + \Psi^{(2)}$, where

$$\Psi^{(1)}(z) = iaz + \int_{|x|\geq 1} \left(1 - e^{ixz}\right) \Pi(dx), \quad z \in \mathbb{R},$$

and

$$\Psi^{(2)}(z) = \int_{|x|<1} \left(1 - e^{ixz} + ixz\right) \Pi(dx), \quad z \in \mathbb{R},$$

with

$$a = \begin{cases} 
-\int_{|x|<1} x\Pi(dx) & \text{if } \alpha \in (0, 1), \\
0 & \text{if } \alpha = 1, \\
\int_{|x|\geq 1} x\Pi(dx) & \text{if } \alpha \in (1, 2).
\end{cases}$$

For the first of these two, we note that it corresponds to the characteristic exponent of a compound Poisson random variable, say

$$X^{(1)} = -a + \sum_{i=1}^{\mathbb{N}} \Xi_i,$$

where $\mathbb{N}$ is an independent Poisson distributed random variable with rate $\Pi(|x| \geq 1)$ and $(\Xi_i, i \geq 1)$ are i.i.d. with distribution $\Pi(|x| \geq 1)^{-1}\Pi(dx)I_{|x| \geq 1}$. 

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(We use the usual convention that $\sum_{i=1}^{0} := 0$.) We want to consider the moments of $X^{(1)}$. It is already clear from the tail of $\Pi$ that $\Xi_1$ has a finite $\beta$-moment if $\beta \in [0, \alpha)$ and infinite $\beta$-moment if $\beta \geq \alpha$. In particular, $\Xi_1$ has a first moment (and hence all smaller positive moments) if and only if $\alpha \in (1, 2)$.

When $\Xi_1$ has a first moment, that is, $\alpha \in (1, 2)$, we observe that $X^{(1)}$ can be rewritten as

$$X^{(1)} = \sum_{i=1}^{N} \Xi_i,$$

where each of the $\Xi_i$ has zero mean. In that case, we may appeal to an inequality for martingale differences, which states that, for $\beta \in [1, \alpha)$ and $n \geq 1$,

$$\mathbb{E}\left[\left|\sum_{i=1}^{n} \Xi_i\right|^\beta\right] \leq 2^\beta \sum_{i=1}^{n} \mathbb{E}[|\Xi_i|^\beta]. \quad (1.19)$$

As the right-hand side is equal to $2^\beta n \mathbb{E}[|\Xi_1|^\beta]$, it follows by an independent randomisation of $n$ by the Poisson distribution of $N$ that $\mathbb{E}[|X^{(1)}|^\beta] < \infty$.

When $X^{(1)}$ has no first moment, that is, $\alpha \in (0, 1]$, we can use the inequality

$$\left(\sum_{i=1}^{n} u_i\right)^q \leq \sum_{i=1}^{n} u_i^q, \quad u_1, \ldots, u_n \geq 0, \quad (1.20)$$

for $q \in (0, 1]$, to deduce that

$$\mathbb{E}\left[\left|\sum_{i=1}^{n} \Xi_i\right|^\beta\right] \leq \mathbb{E}\left[\left(\sum_{i=1}^{n} |\Xi_i|\right)^\beta\right] \leq \sum_{i=1}^{n} \mathbb{E}\left[|\Xi_i|^\beta\right] = n \mathbb{E}\left[|\Xi_1|^\beta\right] < \infty,$$

for $\beta \in [0, \alpha)$. Hence, again following an independent randomisation of $n$ by the distribution of $N$, $\mathbb{E}[|X^{(1)}|^\beta] < \infty$, for $\beta \in [0, \alpha)$.

Next, we want to show that $\mathbb{E}[|X^{(2)}|^\beta] < \infty$, for $\beta \in [0, \alpha)$ and $\alpha \in (0, 2)$, where $X^{(2)}$ is the random variable whose characteristic exponent is given by $\Psi^2$. To this end, we write

$$\Psi^2(z) = -\int_{|x|<1} \sum_{k \geq 0} \frac{(izx)^{k+2}}{(k+2)!} \Pi(dx). \quad (1.21)$$

The sum and the integral may be exchanged using Fubini’s Theorem and the estimate

$$\sum_{k \geq 0} \int_{|x|<1} \frac{|x|^{k+2}}{(k+2)!} \Pi(dx) \leq \sum_{k \geq 0} \frac{|z|^{k+2}}{(k+2)!} \int_{|x|<1} x^2 \Pi(dx) < \infty.$$

Hence, the right-hand side of (1.21) can be written as a power series for all $z \in \mathbb{C}$ and is thus entire. In turn this guarantees that $\hat{\mu}^2(z) := \exp\{-\Psi^2(z)\}$
1.4 Normalised one-dimensional stable distributions

is also an entire function. Note that \( \hat{\mu}^{(2)}(z) \) is nothing more than the Fourier transform of the measure \( \mu^{(2)}(dx) = 1(X^{(2)} \in dx) \), for \( x \in \mathbb{R} \). Since \( \hat{\mu}^{(2)}(z) \) is an entire function, it follows that all the moments of \( \mu^{(2)} \), and hence of \( X^{(2)} \), exist.

To complete the proof for the case \( \beta \in [0, \alpha) \), we can appeal again to (1.19), when \( \alpha \in (1, 2) \) and (1.20) when \( \alpha \in (0, 1] \) to ensure that \( X = X^{(1)} + X^{(2)} \) has the required moment structure.

For the case \( \beta \geq \alpha \), suppose that \( X \) has \( \beta \)-moments. Without loss of generality we may assume that \( \beta \in (0, 2) \), as we will shortly rule out any moments for \( \beta \geq \alpha \). Recalling that \( X^{(2)} \) always has finite moments, using the inequalities (1.19), when \( \beta \geq 1 \), and (1.20), when \( \beta \in (0, 1] \), together with the simple relation \( X^{(1)} = X - X^{(2)} \), we have that the \( \beta \)-moment of \( X^{(1)} \) exists. As \( X^{(1)} \geq \Xi_1 \) on the event \( \{ N \geq 1 \} \), it follows that \( \Xi_1 \) has \( \beta \)-moments. We have already concluded that this can happen when \( \beta \in [0, \alpha) \) and hence the required condition follows.

\[ \square \]

1.4 Normalised one-dimensional stable distributions

In the sequel, we denote by \( S(\alpha, \beta, c) \) a stable distribution, meaning that its characteristic exponent satisfies (1.6). It appears that there are three parameters naturally associated with stable distributions. However, we want to work with a normalised version of such distributions, reducing the number of parameters from three down to two.

**Definition 1.11** Let \( \tilde{X} \) be distributed according to \( S(\alpha, \beta, c) \). Define

\[
b := c \sqrt{1 + \beta^2 \tan(\pi\alpha/2)^2}, \quad \rho := \frac{1}{2} + \frac{1}{\pi\alpha} \tan^{-1}(\beta \tan(\pi\alpha/2)),
\]

where \( \tan^{-1}(\cdot) \) denotes the inverse function of \( \tan(\cdot) \) restricted to its principal branch \((-\pi/2, \pi/2)\). Then we say that the random variable \( X := b^{-\frac{1}{2}} \tilde{X} \) is distributed as a normalised stable distribution with parameters \((\alpha, \rho)\) or simply \( X \sim S_{\text{norm}}(\alpha, \rho) \).

Observe from (1.22) that \( \beta \), and hence \( b \), can be written in terms of \( \rho \) as follows

\[
\beta = \cot\left(\frac{\pi\alpha}{2}\right) \tan\left(\pi\alpha \left(\rho - \frac{1}{2}\right)\right), \quad b = \frac{c}{\cos\left(\pi\alpha \left(\rho - \frac{1}{2}\right)\right)}.
\]

When \( \alpha \in (0, 1) \), by varying \( \beta \in [-1, 1] \), the parameter \( \rho \) ranges over \([0, 1]\), where the boundary points \( \rho = 0 \) and \( \rho = 1 \) correspond to the cases \( \beta = -1 \) and \( \beta = 1 \), respectively.
The case $\alpha \in (1, 2)$ is slightly different. In order to deduce the range of $\rho$, we first recall the following trigonometric identity
\[
\cot\left(\frac{\pi \alpha}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{\pi \alpha}{2}\right),
\]
which implies that, by varying $\beta \in [-1, 1]$, the range of $\rho$ is $[1 - 1/\alpha, 1/\alpha]$. Note that the boundary points $\rho = 1 - 1/\alpha$ and $\rho = 1/\alpha$ correspond to the cases $\beta = 1$ and $\beta = -1$, respectively.

When $\alpha = 1$, we know that $X$ is symmetric and hence $\rho = 1/2$. Therefore, we introduce the set of admissible parameters
\[
\mathcal{A} := \{\alpha \in (0, 1), \, \rho \in [0, 1]\} \cup \{\alpha = 1, \, \rho = 1/2\} \cup \{\alpha \in (1, 2), \, \rho \in [1 - \alpha^{-1}, \alpha^{-1}]\}. \tag{1.24}
\]

**Proposition 1.12** Let $(\alpha, \rho) \in \mathcal{A}$ and assume that $X$ is distributed as $S_{\text{norm}}(\alpha, \rho)$. Then its characteristic exponent is given by
\[
\Psi(z) = |z|^\alpha \left( e^{\pi \alpha \left(\frac{1}{2} - \rho\right)}1_{(z>0)} + e^{-\pi \alpha \left(\frac{1}{2} - \rho\right)}1_{(z<0)} \right). \tag{1.25}
\]

The Lévy measure of $X$ satisfies (1.8) with
\[
c_1 = \Gamma(1 + \alpha) \frac{\sin(\pi \alpha \rho)}{\pi}, \quad c_2 = \Gamma(1 + \alpha) \frac{\sin(\pi \alpha \hat{\rho})}{\pi}, \tag{1.26}
\]
where $\hat{\rho} = 1 - \rho$.

**Proof** We first prove identity (1.25). In order to do so, we take $\tilde{X}$ with the same distribution as $S(\alpha, \beta, c)$ and define $X = b^{-\frac{1}{2}} \tilde{X}$, where $b$ was defined in (1.22). We also let $\tilde{\Psi}$ and $\Psi$ denote their respective characteristic exponents. It is then clear that
\[
\Psi(z) = \tilde{\Psi}(b^{-\frac{1}{2}} z) = b^{-1/2} \tilde{\Psi}(z), \quad z \in \mathbb{R}.
\]

Using (1.23), we note that
\[
c \left[ 1 - i \beta \tan(\pi \alpha/2) \text{sgn}(z) \right] \\
= c \left[ 1 - i \tan\left(\pi \alpha \left(\rho - \frac{1}{2}\right)\right) \text{sgn}(z) \right] \\
= \frac{c}{\cos\left(\pi \alpha \left(\rho - \frac{1}{2}\right)\right)} \left[ \cos\left(\pi \alpha \left(\rho - \frac{1}{2}\right)\right) - i \sin\left(\pi \alpha \left(\rho - \frac{1}{2}\right)\right) \text{sgn}(z) \right] \\
= \frac{c}{\cos\left(\pi \alpha \left(\rho - \frac{1}{2}\right)\right)} \left[ e^{\pi \alpha \left(\frac{1}{2} - \rho\right)}1_{(z>0)} + e^{-\pi \alpha \left(\frac{1}{2} - \rho\right)}1_{(z<0)} \right].
\]
Using (1.6) and (1.23) again, we deduce that $\Psi(z)$ is given in the form of (1.25), up to a multiplicative constant.

For the given expressions of $c_1$ and $c_2$ in (1.26), using standard trigonometric identities and the reflection formula for the gamma function (see identity (A.12) in the Appendix), we obtain that $\beta$ and $c$, defined in (1.18), satisfy

$$\beta = \frac{c_1 - c_2}{c_1 + c_2} = \cot \left( \frac{\pi \alpha}{2} \right) \tan \left( \pi \alpha \left( \rho - \frac{1}{2} \right) \right),$$

and

$$c = -(c_1 + c_2) \Gamma(-\alpha) \cos(\pi \alpha / 2) = \cos \left( \pi \alpha \left( \rho - \frac{1}{2} \right) \right), \quad (1.27)$$

as required. With the choices of $c_1$ and $c_2$ in (1.26), it is obvious that the first equation in (1.22) gives us $b = 1$, while the second equation in (1.22) becomes an identity, that is, both the left- and right-hand sides are equal to $\rho$. In conclusion, the choices in (1.26) necessarily hold if $S_{\text{norm}}(\alpha, \rho)$. □

### 1.5 Distributional identities

We now consider the probability distribution of stable random variables. This includes understanding where the distribution is supported for the different parameter regimes of $\alpha$ and $\rho$.

Let $p(x, \alpha, \rho)$ denote the density of $S_{\text{norm}}(\alpha, \rho)$, where $x$ belongs to its support. Note that, because stable random variables are infinitely divisible but do not belong to the class of compound Poisson distributions, their support is either in the positive half-line, the negative half-line or in the whole real line. Moreover, it is easy to verify that the density $p(x, \alpha, \rho)$ exists and that it is infinitely differentiable. Indeed, observe, for example, from (1.27) that for all values of admissible parameters $(\alpha, \rho)$, we have

$$\left| \alpha \left( \frac{1}{2} - \rho \right) \right| < \frac{1}{2}.$$  

Since for $z \in \mathbb{R}$, we necessarily have that

$$\text{Re}(\Psi(z)) = \cos \left( \pi \alpha \left( \frac{1}{2} - \rho \right) \right) |z|^{\alpha},$$

one can deduce that the function $\exp\{-\Psi(z)\}$ is integrable and decays to zero faster than $|z|^{-n}$ for any $n \geq 2$. Therefore, the inverse Fourier transform, which gives $p(\cdot, \alpha, \rho)$, is well defined as follows
Stable Distributions

\[ p(x, \alpha, \rho) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\Psi(z)-izx} \, dz = \frac{1}{\pi} \text{Re} \left[ \int_{0}^{\infty} e^{-\Psi(z)-izx} \, dz \right], \quad (1.28) \]

for \( x \) in the support of the distribution of \( X \). Moreover, with the given decay of \( \exp(-\Psi(z)) \), one can similarly write the derivatives of \( p \) as inverse Fourier transforms. The next theorem provides the Mellin transform of the positive part of a stable random variable. This identity will be very useful in the sequel.

**Theorem 1.13** Assume that \( X \sim S_{\text{norm}}(\alpha, \rho) \). Then for all \( s \in \mathbb{C} \) in the strip \( -1 < \text{Re}(s) < \alpha \), we have

\[ \mathbb{E}[X^s 1_{(X>0)}] = \frac{\sin(\pi \rho s)}{\sin(\pi s)} \frac{\Gamma(1-s/\alpha)}{\Gamma(1-s)}. \quad (1.29) \]

**Proof** Assume that \( -1 < \text{Re}(s) < 0 \) and \( \alpha \neq 1 \). Using (1.28), we obtain

\[
\mathbb{E}[X^s 1_{(X>0)}] = \int_{0}^{\infty} x^s p(x) \, dx \\
= \frac{1}{\pi} \text{Re} \left( \int_{0}^{\infty} x^s \int_{0}^{\infty} e^{-\Psi(z)-izx} \, dz \, dx \right) \\
= \frac{\Gamma(s+1)}{\pi} \text{Re} \left( e^{-\pi i(s+1)/2} \int_{0}^{\infty} e^{-\Psi(z)z^{-s-1}} \, dz \right) \\
= \frac{\Gamma(s+1)}{\pi \alpha} \Gamma\left(-\frac{s}{\alpha}\right) \text{Re} \left( e^{-\frac{\pi i}{2}(s+1)+\pi i(\frac{1}{2}-\rho)s} \right) \\
= -\frac{\Gamma(s+1)}{\pi \alpha} \Gamma\left(-\frac{s}{\alpha}\right) \sin(\pi \rho s). 
\]

The last expression is equivalent to the right-hand side of (1.29), after applying the recursion formulae (Eq. A.8 in the Appendix) and the reflection formula for the gamma function (see identity (A.12) in the Appendix). We have proved (1.29) for \( \text{Re}(s) \in (-1, 0) \) and now we need to use an analytic continuation argument to ensure that it holds for \( \text{Re}(s) \in [0, \alpha) \). To this end, we first observe from Theorem 1.10 that

\[ \mathbb{E}[|X|^s] < \infty \quad \text{when} \quad 0 \leq s < \alpha. \]

This implies that \( \mathbb{E}[X^s 1_{(X>0)}] \) is analytic in the strip \( -1 < \text{Re}(s) < \alpha \). It is not difficult to see that the right-hand side of (1.29) is also an analytic function in the aforesaid domain. The identity thus holds by a standard analytic continuation argument. The case \( \alpha = 1 \) also follows from continuity properties of both sides of (1.29) in the parameters \( \alpha \) and \( \rho \). \( \square \)

**Remark 1.14** As there is explosion on the right-hand side of (1.29) at the critical values \( s = \alpha \) and \( s = -1 \), the above result, in fact, gives us necessary
and sufficient conditions for the existence of finite absolute moments. Indeed, $E[|X|^s] < \infty$ if and only if $-1 < s < \alpha$. This extends the result of Theorem 1.10.

The following corollary to Theorem 1.13 determines whether the support is the negative or positive half-line or the whole real line.

**Corollary 1.15** Assume that $X$ has the same distribution as $S_{\text{norm}}(\alpha, \rho)$. Then

$$
\mathbb{P}(X > 0) = \rho.
$$

In particular, if $\alpha \in (0, 1)$ and $\rho = 1$ (resp. $\rho = 0$), the support of $X$ is the positive half-line (resp. negative half-line) and, in any other case, the support of any stable law is $\mathbb{R}$.

**Proof** The first conclusion follows by taking limits, as $s$ goes to 0, in (1.29). Combining it with the comments before Proposition 1.12, the remaining statement in the corollary follows. □

The following result, known as Zolotarev's duality, relates the density of a stable distribution with parameters $(\alpha, \rho)$ to the density of a stable distribution with parameters $(1/\alpha, \alpha \rho)$ whenever they are admissible; cf. (1.24).

**Theorem 1.16** (Zolotarev’s duality) Assume that both pairs $(\alpha, \rho)$ and $(1/\alpha, \alpha \rho)$ are admissible. Then for $X$ and $\tilde{X}$ which are distributed as $S_{\text{norm}}(\alpha, \rho)$ and $S_{\text{norm}}(1/\alpha, \alpha \rho)$, respectively, we have

$$
\mathbb{P}(X^{-\alpha} \in B, X > 0) = \frac{1}{\alpha} \mathbb{P}(\tilde{X} \in B, \tilde{X} > 0),
$$

(1.30)

for all Borel sets $B$.

**Proof** Let us denote the function on the right-hand side of (1.29) by $m(s; \alpha, \rho)$. By applying the reflection formula (A.12) and the recursion formula (A.8) for the gamma function, it is easy to see that, for all $s$ in the strip $-1 < \text{Re}(s) < \alpha$, we have

$$
m(-\alpha s; \alpha, \rho) \equiv \alpha^{-1} m(s; \alpha^{-1}, \alpha \rho),
$$

which implies the statement of the Theorem. □

The result of Theorem 1.16 can also be expressed in terms of the density functions as follows

$$
z^{-\frac{1}{\alpha}} p(z^{-\frac{1}{\alpha}}, \alpha, \rho) = z \rho \left( z, \frac{1}{\alpha}, \alpha \rho \right),
$$

(1.31)

for $z$ in the support of $X$. 

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Next, observe that, if the support of the distribution $S_{\text{norm}}(1/\alpha, \alpha \rho)$ is the real line, then
\[ p(-x, \alpha, \rho) = p(x, \alpha, 1 - \rho), \]
thus it is enough to study this function for $x > 0$ in this case. Below we give expressions for the density $p(x, \alpha, \rho)$. We start by treating the case of $\alpha = 1$, the Cauchy distribution, separately.

**Theorem 1.17** When $\alpha = 1$ and $\rho = 1/2$, we have
\[ p(x, 1, 1/2) = \frac{1}{\pi(1 + x^2)}, \quad x \in \mathbb{R}. \]

**Proof** Recalling that the distribution is symmetric, we can appeal to Theorem 1.13 and check that the Mellin transform of $p(x, 1, 1/2)$ on the positive half-line is equal to $\sin(\pi s/2)/\sin(\pi s)$, for $-1 < s < 1$. To this end, we note that, for $-1 < s < 1$,
\[ \frac{\sin(\pi s/2)}{\sin(\pi s)} = 2 \frac{\Gamma(s)\Gamma(-s)}{\Gamma(s/2)\Gamma(-s/2)} = \frac{1}{2\pi} \Gamma\left(\frac{1}{2} + \frac{s}{2}\right) \Gamma\left(\frac{1}{2} - \frac{s}{2}\right) = \frac{1}{2} \int_0^\infty \frac{y^{s-1/2}}{\pi(1 + y)} \, dy = \int_0^\infty \frac{x^s}{\pi(1 + x^2)} \, dx, \]
where the first equality uses the recursion formula (A.8) and the reflection formula for gamma functions (A.12), the second follows from the duplication formula for gamma functions (A.14), the third uses the definition of the beta function in (A.18) (all of the last four identities found in the Appendix) and the final equality is the result of a change of variables. \qed

In all other cases we have a convergent power series representation, as described in the next theorem.

**Theorem 1.18** If $\alpha \in (0, 1)$, then
\[ p(x, \alpha, \rho) = \frac{1}{\pi} \sum_{n \geq 1} (-1)^{n-1} \frac{\Gamma(1 + \alpha n)}{n!} \sin(n \pi \alpha \rho) x^{-n \alpha - 1}, \quad x > 0, \quad (1.32) \]
and if $\alpha \in (1, 2)$ then
\[ p(x, \alpha, \rho) = \frac{1}{\pi} \sum_{n \geq 1} (-1)^{n-1} \frac{\Gamma(1 + n/\alpha)}{n!} \sin(n \pi \rho) x^{n-1}, \quad x > 0. \quad (1.33) \]
Moreover, when $\alpha \in (0, 1)$ (resp. $\alpha \in (1, 2)$) and $|\beta| \neq 1$ (i.e. $0 < \alpha \rho, \alpha \rho < 1$) formula (1.33) (resp. (1.32)) provides complete asymptotic expansion as $x$ goes to $0^+$ (resp. as $x$ goes to $\infty$).

**Remark 1.19** Before passing to the proof, it is worth emphasising to the unfamiliar reader that the statement above for the asymptotic expansions in the two regimes $\alpha \in (0, 1)$ and $\alpha \in (1, 2)$ do indeed rely in the series expansion for the opposite regime.

**Proof of Theorem 1.18** According to Theorem 1.13, the Mellin transform of $p(x, \alpha, \rho)$ on $(0, \infty)$, satisfies

$$M(z) := \int_0^\infty p(x, \alpha, \rho)x^{z-1} \, dx = \frac{\sin(\pi \rho (z-1))}{\sin(\pi (z-1))} \frac{\Gamma(1-(z-1)/\alpha)}{\Gamma(2-z)}. \quad (1.34)$$

Observe that this function has simple poles at points $z = 1 + n\alpha, n \geq 1$, and $z = -m, m \geq 0$. Then by applying Proposition A.1 and identity (A.11) (both in the Appendix), we find that

$$\text{Res}(M, 1 + n\alpha) = \left[ \frac{\sin(\pi \rho (z-1))}{\sin(\pi (z-1))\Gamma(2-z)} \right]_{z=1+n\alpha} \times \text{Res}(\Gamma(1-(z-1)/\alpha), z = 1 + n\alpha)$$

$$= \frac{\sin(n\pi \alpha \rho)}{\sin(n\pi \alpha)\Gamma(1-n\alpha)} \times (-1)^n \frac{\alpha}{(n-1)!}.$$

Finally using the reflection formula for the gamma function (A.12) and simplifying the result, we arrive at

$$\text{Res}(M, 1 + n\alpha) = \frac{1}{\pi} (-1)^{n-1} \frac{(1 + n\alpha)}{n!} \sin(n\pi \alpha \rho).$$

On the other hand, from (A.16), we deduce

$$|\sin(x + iy)| \sim \frac{\exp(|y|)}{2}, \quad \text{as} \quad y \to \infty,$$

and

$$\left| \frac{\Gamma((x + iy)/\alpha)}{\Gamma(x + iy)} \right| \sim \exp \left\{ -\frac{\pi}{2} \left( \frac{1}{\alpha} - 1 \right) \right\} |y|^{x\left(\frac{1}{\alpha}-1\right)} \alpha^{-\frac{1}{\alpha}+\frac{1}{2}}, \quad \text{as} \quad y \to \infty,$$

uniformly in any finite interval $-\infty < a \leq x \leq b < \infty$. This gives us the estimate

$$|M(x + iy)| \leq C|y|^{x\left(\frac{1}{\alpha}-1\right)} \exp \left\{ -\frac{\pi}{2} \left( \frac{1}{\alpha} + 1 - 2\rho \right) \right\} |y|, \quad (1.35)$$

as for all $y$ sufficiently large, where $C > 0$ is an unimportant constant. Note that, when $\alpha \in (0, 1)$, the exponential term in (1.35) is decreasing on account
of the fact that $1/\alpha > 1 > \rho$. Moreover, when $\alpha \in (1, 2)$, the exponential term is again decreasing on account of the fact that $\alpha \rho \leq 1$, in which case

$$\frac{1}{\alpha} + 1 - 2\rho \geq 1 - \frac{1}{\alpha} > 0.$$ 

As such, $M(z)$ is absolutely integrable on the vertical line $c + iR$, where $c$ is a constant in $(0, 1 + \alpha)$, therefore we may use the Mellin transform inversion formula

$$p(x, \alpha, \rho) = \frac{1}{2\pi i} \int_{c+iR} M(z)x^{-z} \, dz.$$ 

Let us define $b_k = 1 + \alpha(2k + 1)/2$ and set $\ell$ to be an integer. We also consider the contour $L = L_1 \cup L_2 \cup L_3 \cup L_4$, defined as

- $L_1 := \{\text{Re}(z) = c, -\ell \leq \text{Im}(z) \leq \ell\}$,
- $L_2 := \{\text{Im}(z) = \ell, c \leq \text{Re}(z) \leq b_k\}$,
- $L_3 := \{\text{Re}(z) = b_k, -\ell \leq \text{Im}(z) \leq \ell\}$,
- $L_4 := \{\text{Im}(z) = -\ell, c \leq \text{Re}(z) \leq b_k\}$.

It is clear that $L$ is the rectangle bounded by vertical lines $\text{Re}(z) = c, \text{Re}(z) = b_k$ and by horizontal lines $\text{Im}(z) = \pm \ell$. We assume that $L$ is oriented counterclockwise; see Figure 1.1.

![Figure 1.1 The contour $L = L_1 \cup L_2 \cup L_3 \cup L_4$](https://doi.org/10.1017/9781108648318.002)Published online by Cambridge University Press

The function $M(z)$ is analytic in the interior of $L$, except for simple poles at $s_j = 1 + \alpha j$, for $1 \leq j \leq k$ and is continuous on $L$. Using the residue theorem we find
1.5 Distributional identities

\[ \frac{1}{2\pi i} \int_{L} M(z) x^{-z} \, dz = \sum_{j=1}^{k} \text{Res}(M, s_j) \times x^{-s_j}. \]

Next, we estimate the integrals over the horizontal side \( L_2 \) as follows

\[ \left| \int_{L_2} M(z) x^{-z} \, dz \right| < (b_k - c) \times x^{-b_k} \max_{z \in L_2} |M(z)|. \]

When \( \ell \) increases, we have \( \max_{z \in L_2} |M(z)| \) goes to 0. Therefore,

\[ \int_{L_2} M(z) x^{-z} \, dz \to 0 \quad \text{as} \quad \ell \to \infty. \]

Similarly, we deduce that the integral on the contour \( L_4 \) goes to 0 as \( \ell \) goes to \( \infty \). Thus putting all the pieces together, we have

\[ -\frac{1}{2\pi i} \int_{c+i\mathbb{R}} M(z) x^{-z} \, dz + \frac{1}{2\pi i} \int_{b_k+i\mathbb{R}} M(z) x^{-z} \, dz = \sum_{j=1}^{k} \text{Res}(M, s_j) \times x^{-s_j}. \]

In other words, we have deduced

\[ p(x, \alpha, \rho) = -\sum_{n=1}^{k} \text{Res}(M, 1 + n\alpha) x^{-(1+an)} + \frac{1}{2\pi i} \int_{b_k+i\mathbb{R}} M(z) x^{-z} \, dz. \quad (1.36) \]

Now suppose that \( \alpha \in (0, 1) \). Our aim is to prove that as \( k \) goes to \( \infty \), the integral of the right-hand side of (1.36) converges to 0 for \( x > 0 \). Intuitively this is clear, since the Mellin transform (1.34) can be rewritten with the help of the reflection formula (A.12) as

\[ \frac{-\sin(\pi \rho(z - 1)) \Gamma(z - 1)}{\sin(\pi(z - 1)/\alpha) \Gamma((z - 1)/\alpha)}. \]

Noting that, for \( z = b_k + iu, u \in \mathbb{R} \), the ratio of sine functions above, say \( H(z) \), is periodic and uniformly bounded in \( u \) and \( b_k \) as \( k \to \infty \). The integral in the right-hand side of (1.36) can thus be estimated as follows

\[ \left| \int_{b_k+i\mathbb{R}} M(z) x^{-z} \, dz \right| < x^{-b_k} \int_{\mathbb{R}} \left| \frac{\Gamma(b_k - 1 + iu)}{\Gamma((b_k - 1 + iu)/\alpha)} \right| H(b_k - 1 + iu) \, du. \quad (1.37) \]

To see why the integral on the right-hand side of (1.37) is finite, we can appeal to (A.15) and deduce

\[ \frac{\Gamma(z)}{\Gamma(z/\alpha)} = \frac{1}{\sqrt{\alpha}} \exp \left\{ -z \left( \frac{1}{\alpha} \ln(z) - A \right) + O(z^{-1}) \right\}, \quad |z| \to \infty, \text{Re}(z) \geq 0, \]

where \( A = (1 + \ln(\alpha) + \alpha) / \alpha \), which is negative as we have assumed \( \alpha \in (0, 1) \). This implies that the right-hand side of (1.37) is finite. Observe, moreover, that \( \Gamma(z)/\Gamma(z/\alpha) \) is continuous in the half-plane \( \text{Re}(z) \geq 0 \), thus
\[ x^{-b_k} \left| \frac{\Gamma(b_k - 1 + iu)}{\Gamma((b_k - 1 + iu)/\alpha)} \right| \]

converges to 0 as \( k \) goes to \( \infty \) (uniformly in \( u \in \mathbb{R} \)), implying that the integral in the right-hand side of (1.37) vanishes as \( k \) goes to \( \infty \) and the series representation in (1.32) for the case \( \alpha \in (0, 1) \) follows.

We can also pick out elements of the previous arguments to help us prove the asymptotic expansion for \( \alpha \in (1, 2) \), as \( x \) goes to \( \infty \). More precisely, we take \( x \in (1, \infty) \) and observe that the integral in the right-hand side in (1.36) can be estimated as follows

\[ \left| \int_{b_k+i\mathbb{R}} M(z)x^{-z} \, dz \right| < x^{-b_k} \int_{\mathbb{R}} \left| M(b_k + ir) \right| \, dr = O(x^{-b_k}) \quad \text{as} \quad x \to \infty, \quad (1.38) \]

where the integral on the right-hand side of (1.38) is finite thanks to the estimate (1.35).

The series representation in (1.33) follows from (1.31) (Zolotarev’s duality) and (1.32). The asymptotic expansion (1.33) for \( \alpha \in (0, 1) \), as \( x \) goes to 0 follows from similar arguments for the case \( \alpha \in (1, 2) \) as \( x \to \infty \). Indeed, one has an identity in the spirit of (1.36), which is constructed from a rectangular contour integral which contains, for example, the first \( k \) negative poles. Then an argument similar to (1.38) provides the desired asymptotic. \( \square \)

### 1.6 Stable distributions in higher dimensions

Similarly to the one-dimensional case, one can define infinitely divisible distributions in \( \mathbb{R}^d \). Just as in one dimension, the characteristic exponent of an \( \mathbb{R}^d \) infinitely divisible distribution also has a Lévy–Khintchine representation. Replacing scalar products by Euclidean inner products and understanding \( |\cdot| \) as the associated norm, we have, for \( z \in \mathbb{R}^d \),

\[
\Psi(z) = i a \cdot z + \frac{1}{2} z \cdot Q z + \int_{\mathbb{R}^d} \left( 1 - e^{i z \cdot x} + i z \cdot x 1_{|x|<1} \right) \Pi(dx), \quad (1.39)
\]

where \( a \in \mathbb{R}^d \), \( Q \) is a symmetric nonnegative-definite \( d \times d \) matrix and \( \Pi \) is a measure on \( \mathbb{R}^d \) satisfying

\[
\Pi(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (1 \wedge |x|^2) \, \Pi(dx) < \infty.
\]

We say that \( X \) is a \( d \)-dimensional stable distribution if it takes values on \( \mathbb{R}^d \) and satisfies (1.1), where addition is understood in the vectorial sense. For the same reason as in the one-dimensional setting, stable distributions on \( \mathbb{R}^d \) are
1.6 Stable distributions in higher dimensions

also infinitely divisible distributions. Theorem 1.20 identifies them in terms of a polar decomposition of their Lévy measure.

**Theorem 1.20** An infinitely divisible \( \mathbb{R}^d \)-valued distribution is a stable distribution if and only if the Lévy–Khintchine representation of its characteristic exponent satisfies either

(i) \( Q \neq 0, \Pi \equiv 0 \) and \( a = 0 \), where \( \alpha = 2 \), or

(ii) \( Q = 0 \) and there is a finite measure \( \Lambda \) on \( \mathbb{S}^{d-1} = \{ x \in \mathbb{R}^d : |x| = 1 \} \) such that

\[
\Pi(B) = \int_{\mathbb{S}^{d-1}} \Lambda(d\phi) \int_{(0,\infty)} 1_B(r\phi) \frac{dr}{r^{\alpha+1}}, \quad \text{for } B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}),
\]

where \( \mathcal{B}(\mathbb{R}^d \setminus \{0\}) \) denotes the Borel \( \sigma \)-algebra of \( \mathbb{R}^d \setminus \{0\} \) and \( \alpha \in (0,2) \). If \( \alpha \in (0,1) \cup (1,2) \), then \( a = 0 \).

**Proof** We first observe that Lemma 1.6 also holds in this case, implying that for \( \alpha > 0 \) and \( k \geq 1 \),

\[
k\Psi(z) = \Psi(k^{1/\alpha}z), \quad \text{for } z \in \mathbb{R}^d.
\]

The above identity and similar arguments to those used in the proof of Proposition 1.7 imply that if \( Q \neq 0 \), then necessarily \( \alpha = 2 \), \( \Pi \equiv 0 \) and \( a = 0 \).

If we assume \( Q = 0 \), then identity (1.41) implies that

\[
k\Pi(B) = \Pi(k^{-1/\alpha}B) \quad \text{for } B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).
\]

In particular for \( \Gamma_\ell = \{ x \in \mathbb{R}^d : |x| \geq \ell, x/|x| \in D \} \), where \( \ell > 0 \) and \( D \in \mathcal{B}(\mathbb{S}^{d-1}) \), we deduce

\[
k\Pi(\Gamma_\ell) = \Pi(k^{-1/\alpha}\Gamma_\ell) = \Pi(\Gamma_{\ell k^{-1/\alpha}}),
\]

and by taking \( \ell = (k/n)^{1/\alpha} \), we obtain

\[
\Pi\left(\Gamma_{(k/n)^{1/\alpha}}\right) = \frac{1}{k} \Pi(\Gamma_{n^{-1/\alpha}}) = \frac{n}{k} \Pi(\Gamma_1).
\]

Since the mapping \( y \mapsto \Pi(\Gamma_y) \) is monotone decreasing, an argument appealing to denseness of the rational numbers again (see the proof of Proposition 1.7) allows us to obtain

\[
\Pi(\Gamma_x) = x^{-\alpha}\Pi(\Gamma_1), \quad \text{for } x > 0.
\]

Next, we introduce a finite measure on \( \mathbb{S}^{d-1} \) as follows

\[
\Lambda(D) = \alpha \Pi(\Gamma_1), \quad \text{for } D \in \mathcal{B}(\mathbb{S}^{d-1}),
\]
and define $\Gamma_{\ell_1, \ell_2} = \{ x \in \mathbb{R}^d : \ell_1 \leq |x| < \ell_2, x/|x| \in D \}$. Therefore, by identity (1.42), we have

$$
\Pi(\Gamma_{\ell_1, \ell_2}) = \Pi(\Gamma_{\ell_1}) - \Pi(\Gamma_{\ell_2}) = \frac{\ell_1^{-\alpha} - \ell_2^{-\alpha}}{\alpha} \Lambda(D)
$$

$$
= \int_D \int_{\ell_1}^{\ell_2} \frac{dr}{r^{1+\alpha}} \Lambda(d\phi)
$$

$$
= \int_{S^{d-1}} \Lambda(d\phi) \int_{(0,\infty)} 1_{\Gamma_{\ell_1, \ell_2}}(r\phi) \frac{dr}{r^{\alpha+1}}.
$$

Since the sets $\Gamma_{\ell_1, \ell_2}$ fulfill the conditions of Dynkin’s Lemma (or $\pi - \lambda$ Theorem) then (1.40) holds for any Borel set of $\mathbb{R}^d \setminus \{0\}$.

Finally, we observe

$$
\int_{\{|x| < 1\}} |x|^2 \Pi(dx) = \Lambda(S^{d-1}) \int_0^1 r^{-\alpha} dr.
$$

Since $\Pi$ is a Lévy measure, we necessarily have in addition to $\alpha > 0$ that $\alpha < 2$. $\square$

**Remark 1.21** As in the one-dimensional case, we will proceed ignoring the Gaussian setting in part (i) of Theorem 1.20, allowing us to focus on the regime $\alpha \in (0, 2)$.

Appealing to equation (1.40) and undertaking computations similar in spirit to those done in the proof of Theorem 1.3, one readily finds that the characteristic exponent of a stable distribution on $\mathbb{R}^d$ can be written as follows, for $z \in \mathbb{R}^d$,

$$
\Psi(z) = -\Gamma(-\alpha \cos \left(\frac{\pi \alpha}{2}\right)) \int_{S^{d-1}} |z \cdot \phi|^\alpha \left(1 - i \tan \left(\frac{\pi \alpha}{2}\right) \text{sgn}(z \cdot \phi)\right) \Lambda(d\phi),
$$

for $\alpha \neq 1$, and

$$
\Psi(z) = iz \cdot a + \frac{\pi}{2} \int_{S^{d-1}} \left(|z \cdot \phi| + i z \cdot \phi \log(z \cdot \phi)\right) \Lambda(d\phi),
$$

for $\alpha = 1$, where $a \in \mathbb{R}^d$ and $\Lambda$ satisfies

$$
\int_{S^{d-1}} \phi \Lambda(d\phi) = 0.
$$

The *isotropic* case will be of particular interest in what follows. Recall that a measure $\mu$ on $\mathbb{R}^d$ is symmetric if $\mu(B) = \mu(-B)$ for $B \in \mathcal{B}(\mathbb{R}^d)$ and isotropic if $\mu(B) = \mu(U^{-1}B)$ for every orthogonal matrix $U$. Observe that when $d = 1$, the notion of isotropy is equivalent to symmetry.
1.6 Stable distributions in higher dimensions

**Definition 1.22** We say that \( X \) is a *symmetric stable* random variable if it is stable distributed and its law is symmetric.

In particular, if \( X \) is symmetric we have that its characteristic function is equal to that of \(-X\), that is, \( \Psi(z) = \Psi(-z) \) for \( z \in \mathbb{R}^d \). By considering the real and imaginary parts of \( \Psi \), the latter clearly implies that

\[
\Psi(z) = \int_{S^{d-1}} |z \cdot \phi|^\alpha \Lambda_0(d\phi) \quad \text{for} \quad \alpha \in (0, 2), \tag{1.43}
\]

where

\[
\Lambda_0(d\phi) = \begin{cases} 
-\Gamma(-\alpha) \cos \left( \frac{\pi \alpha}{2} \right) \Lambda(d\phi) & \text{if } \alpha \neq 1, \\
\frac{\pi}{2} \Lambda(d\phi) & \text{if } \alpha = 1.
\end{cases}
\]

Using the double angle trigonometric identity and Euler’s reflection formula (A.12), we have for \( \alpha \neq 1 \),

\[
-\Gamma(-\alpha) \cos \left( \frac{\pi \alpha}{2} \right) = -\Gamma(-\alpha) \frac{\sin(\pi\alpha)}{2 \sin \left( \frac{\pi \alpha}{2} \right)}
= -\frac{\Gamma(-\alpha) \Gamma(\alpha/2) \Gamma(1 - \alpha/2)}{2 \Gamma(\alpha) \Gamma(1 - \alpha)}
= \frac{\Gamma(\alpha/2) \Gamma(1 - \alpha/2)}{2 \Gamma(\alpha + 1)}.
\]

It is important to note that the right-hand side in the third equality is equal to \( \pi/2 \) when \( \alpha = 1 \) thanks to the special values \( \Gamma(1/2) = \sqrt{\pi} \) and \( \Gamma(2) = 1 \). Therefore, in the sequel we define

\[
\Lambda_0(d\phi) = \frac{\Gamma(\alpha/2) \Gamma(1 - \alpha/2)}{2 \Gamma(\alpha + 1)} \Lambda(d\phi),
\]

for \( \alpha \in (0, 2) \).

**Definition 1.23** We say that \( X \) is an *isotropic stable* random variable if it is stable distributed and, for all orthogonal transforms \( B: \mathbb{R}^d \mapsto \mathbb{R}^d \), \( BX \) is equal in law to \( X \).

Observe that if \( X \) belongs to this class then \( X \) is also symmetric and, consequently, its characteristic function satisfies (1.43) and it can be written as follows

\[
\Psi(z) = |z|^\alpha \int_{S^{d-1}} ||z|^{-1} z \cdot \phi|^\alpha \Lambda_0(d\phi).
\]

Since the law of \( X \) is isotropic then \( \Psi(z) = \Psi(Uz) \), for any orthogonal matrix, \( U \), and therefore, without loss of generality, we may take \( \Lambda \) equal to Lebesgue (surface) measure on \( S^{d-1} \) and, thanks to symmetry,
Stable Distributions

\[ c = \int_{\mathbb{S}^{d-1}} \|z\|^{-\alpha} z \cdot \phi \Lambda_0(d\phi), \]

should be a positive constant, that is, for \( \alpha \in (0, 2) \), we have

\[ \Psi(z) = c|z|^\alpha \quad \text{for} \quad z \in \mathbb{R}^d. \]

Similarly as in the one dimensional case, the explicit computation of the constant \( c \) is important for our purposes. In that case, we may take

\[ c = \int_{\mathbb{S}^{d-1}} |1 \cdot \phi|^\alpha \Lambda_0(d\phi), \]

where \( 1 = (1, 0, \cdots, 0) \in \mathbb{R}^d \) is the ‘North Pole’ on \( \mathbb{S}^{d-1} \). Hence, using skew product coordinates (also called generalised polar coordinates) in \( \mathbb{R}^d \), we deduce

\[
\int_{\mathbb{S}^{d-1}} |1 \cdot \phi|^\alpha \Lambda_0(d\phi) = \frac{\Gamma(\alpha/2)\Gamma(1 - \alpha/2)}{2\Gamma(\alpha + 1)} \frac{2\pi^{(d-1)/2}}{\Gamma((d - 1)/2)} \int_0^\pi \sin^{d-2}(\theta)|\cos(\theta)|^\alpha d\theta
\]

\[
= 2 \frac{\Gamma(\alpha/2)\Gamma(1 - \alpha/2)}{\Gamma(\alpha + 1)} \frac{\pi^{(d-1)/2}}{\Gamma((d - 1)/2)} \int_0^{\pi/2} \sin^{d-2}(\theta) \cos^\alpha(\theta) d\theta
\]

\[
= \pi^{(d-1)/2} \frac{\Gamma(\alpha/2)\Gamma(1 - \alpha/2)}{\Gamma(\alpha + 1)} \frac{\Gamma((\alpha + 1)/2)}{\Gamma((d + \alpha)/2)}
\]

\[
= 2^{1-\alpha} \pi^{d/2} \frac{\Gamma(1 - \alpha/2)}{\Gamma(\alpha + 1)} \frac{\Gamma((d + \alpha)/2)}{\alpha \Gamma((d + \alpha)/2)}
\]

\[
= 2^{-\alpha} \pi^{d/2} \frac{\left|\Gamma(-\alpha/2)\right|}{\Gamma((d + \alpha)/2)},
\]

where, from the Appendix, we have used (A.19) in the third equality and the duplication formula (A.14) in the fourth equality. We thus have

\[ c = 2^{-\alpha} \pi^{d/2} \frac{\left|\Gamma(-\alpha/2)\right|}{\Gamma((d + \alpha)/2)}. \]

We may now say the isotropic \( d \)-dimensional stable process is \textit{normalised} if

\[ \Psi(z) = |z|^\alpha, \quad z \in \mathbb{R}^d. \]

This corresponds to setting

\[
\Pi(B) = \frac{1}{c} \int_{\mathbb{S}^{d-1}} \Lambda(d\phi) \int_0^\infty 1_B(r\phi) \frac{dr}{r^{\alpha+1}}, \quad \text{for} \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}),
\]

where \( \Lambda(d\phi) \) is the surface measure on \( \mathbb{S}^{d-1} \).
1.7 Comments

It is also worth noting for later that the Lévy measure can be written as absolutely continuous with respect to $d$-dimensional Lebesgue measure as well as with respect to the skew product measure $\sigma_1(d\phi)r^{d-1}dr$, where $\sigma_1(d\phi)$ is the surface measure on $S^{d-1}$ normalised to have unit mass.

**Theorem 1.24** For a normalised isotropic stable distribution, that is, having characteristic exponent $\Psi(z) = |z|^\alpha$, $z \in \mathbb{R}$, we have

\[
\Pi(B) = 2^{\alpha-1} \pi^{-d/2} \Gamma((d + \alpha)/2) \int_{|z|<1} \frac{dz}{|z|^\alpha + d} \left( \int_{|\phi| < 1} \sigma_1(d\phi) \right) \left( \int_0^\infty 1_B(r\phi) \frac{1}{r^{\alpha+1}} dr \right),
\]

and in (generalised) polar coordinates,

\[
\Pi(B) = 2^{\alpha-1} \pi^{-d} \frac{\Gamma((d + \alpha)/2)\Gamma(d/2)}{\Gamma(-\alpha/2)} \int_{S^{d-1}} \sigma_1(d\phi) \left( \int_0^\infty 1_B(r\phi) \frac{1}{r^{\alpha+1}} dr \right),
\]

where $B$ is a Borel set in $\mathbb{R}^d$.

**Remark 1.25** Also taking inspiration from the one-dimensional case, in particular the computations in the proof of Theorem 1.3, and the statement in Corollary 1.9, we note that, the value of $a$ in (1.39) can be identified explicitly within the choice of normalisation in Theorem 1.24. Specifically, when $\alpha \in (0, 1)$ we have $a = -\frac{1}{2} \int_{\mathbb{R}^d} x1_{\{|x|<1\}} \Pi(dx)$; when $\alpha \in (1, 2)$ we have $a = \frac{1}{2} \int_{\mathbb{R}^d} x1_{\{|x|<1\}} \Pi(dx)$; and when $\alpha = 1$, we have $a = 0$.

1.7 Comments

The class of stable distributions appeared for the first time in the celebrated monograph of Paul Lévy [143] *Calcul de probabilités*. They were introduced by Lévy as limits for normalised sums of independent identically distributed random variables that do not satisfy a finite second moment condition. The concept of stable distributions was fully developed in 1937 with the appearance of the monographs of Lévy [144] and Khintchine [106]. Both authors characterised them as a class of infinitely divisible distributions. Since their debut, stable distributions have appeared in a vast array of probabilistic models motivated by physics, biology and economics, to name but some of the many areas of influence.

There are several monographs where stable distributions are treated in detail, see for instance Gnedenko and Kolmogorov [80], Linde [146], Sato [190] and Uchaikin and Zolotarev [208]. There are also other monographs that deal specifically with stable distributions or stochastic processes which are associated to them, such as Samorodnitsky and Taqqu [189] and Zolotarev [220].
Zolotarev [220] (see also Uchaikin and Zolotarev [208]) gave a complete treatment to real-valued stable distributions and in particular the series representation and asymptotic expansions of their densities were described in a concise way for the first time. The multidimensional case is treated in Chapter 2 of Samorodnitsky and Taqqu [189] and in Uchaikin and Zolotarev [208]. In Chapter 1 of Samorodnitsky and Taqqu [189], the one-dimensional case is also treated.

The explicit form of the characteristic exponent of stable distributions (Theorem 1.3) was first treated by Lévy [142] and by Khintchine and Lévy [107]. The formulation of Theorem 1.3 follows Gnedenko and Kolmogorov [80] with the corrections of Hall [86]. We stress that the article of Hall corrected a number of different derivations of Theorem 1.3 that have appeared in the literature up to the beginning of the 1980’s. Lemma 1.6 is based on similar arguments used in Pitman and Pitman [169]. The proof of Proposition 1.7 and Theorem 1.20 follows from Kuelbs [114], where stable distributions are defined on Hilbert spaces. Theorem 1.10 is based on a more general result for moments of infinitely divisible random variables found in Section 25 of Sato [190]. The moment inequality (1.19) is taken from Lemma 1 of Biggins [30].

The existence of a distributional density thanks to infinite divisibility (that predicates Section 1.5) can be found in Section 24 of Sato [190]. All the results that appear in Section 1.5 can be found in the monographs of Uchaikin and Zolotarev [208] and Zolotarev [220] and we adopt a similar approach, for example in Theorem 1.13. On a final note, we mention that it was shown in Hoffmann-Jørgensen [108] that the densities derived in Theorem 1.18 can be expressed in terms of incomplete hypergeometric functions.