



Smoothing of Limit Linear Series of Rank One on Saturated Metrized Complexes of Algebraic Curves

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Abstract. We investigate the smoothing problem of limit linear series of rank one on an enrichment of the notions of nodal curves and metrized complexes called saturated metrized complexes. We give a finitely verifiable full criterion for smoothability of a limit linear series of rank one on saturated metrized complexes, characterize the space of all such smoothings, and extend the criterion to metrized complexes. As applications, we prove that all limit linear series of rank one are smoothable on saturated metrized complexes corresponding to curves of compact-type, and we prove an analogue for saturated metrized complexes of a theorem of Harris and Mumford on the characterization of nodal curves contained in a given gonality stratum. In addition, we give a full combinatorial criterion for smoothable limit linear series of rank one on saturated metrized complexes corresponding to nodal curves whose dual graphs are made of separate loops.

1 Introduction

A saturated metrized complex is an object that encodes information about a degenerating family of smooth curves. Roughly speaking, a saturated metrized complex \mathcal{C} over a field κ consists of a metric graph and an algebraic curve over κ associated with each point of the metric graph (Definition 2.1). A limit linear series of rank r on \mathcal{C} consists of a linear series of rank r on each associated algebraic curve of \mathcal{C} satisfying certain compatibility conditions. The main purpose of this paper is to provide a full criterion for lifting a limit linear series of rank one to a linear series of the same rank on a smooth curve C .

1.1 Context and Motivation

Degeneration to singular curves has been one of the most important tools in the theory of smooth algebraic curves. Fundamental results on algebraic curves such as the Brill–Noether Theorem and Gieseker–Petri Theorem were established via degeneration to singular curves [16, 22, 23][25, Chapter 5]. While degeneration to irreducible curves such as cuspidal curves and nodal curves was considered by Castelnuovo [12] and several researchers, Eisenbud and Harris [16] subsequently developed a theory of

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degeneration of linear series to certain reducible curves called limit linear series. A limit linear series is usually denoted by limit g_d^r , where the integers d and r are called the degree and rank of the limit linear series, respectively. The theory of limit linear series has numerous applications, for instance, a proof of non-unirationality of $M_{2,3}$ [17], a detailed study of the monodromy of Weierstrass points [18], and a proof of irreducibility of certain families of special linear series of curves [19]. However, until recently the notion of limit linear series was largely restricted to curves of compact-type, *i.e.*, nodal curves whose dual graph is a tree. We refer to Osserman's survey for a more recent treatment of limit linear series on curves of compact type [30].

Recently, Amini and Baker [3] defined a notion of limit linear series on nodal curves in general. In fact, instead of working with nodal curves per se, they considered an enrichment of nodal curves called metrized complexes (nodal curves with a metric assigned to the corresponding dual graph) and formulated a notion of limit linear series on a metrized complex. They show that on a metrized complex associated with a curve of compact-type, their notion of limit linear series coincides with that of Eisenbud and Harris. Independently, Ossermann [31] has generalized the notion of limit linear series of Eisenbud–Harris to curves of non-compact type.

The notions of limit linear series of Eisenbud and Harris, Amini and Baker and Osserman satisfy two key properties.

- (1) For any family of smooth curves degenerating to a curve of compact type (a metrized complex or a nodal curve), any family of linear series on each smooth curve in the family degenerates to a limit linear series on the curve of compact type (a metrized complex or a nodal curve respectively). This property is called the specialization property.
- (2) The limit linear series is formulated in terms of linear series on each irreducible component and with relations between the linear series on each irreducible component that depend on the dual graph (see Definition 2.5 for a precise definition).

However, even in the case of curves of compact type, the converse of Property (1) does not hold in general. In other words, not every limit g_d^r arises as a limit of linear series. A limit linear series is said to be smoothable if it arises as a limit of linear series (see Definition 3.8 for a precise definition of smoothability). Eisenbud and Harris also considered a refinement of the notion of limit linear series called refined limit linear series. They showed that every refined limit g_d^1 is smoothable. For $r \geq 2$, they constructed a moduli space of limit g_d^r and showed that any limit g_d^r in irreducible components of the expected dimension in the moduli space of limit g_d^r is smoothable, which they call the regeneration theorem [16, Theorem 3.4].

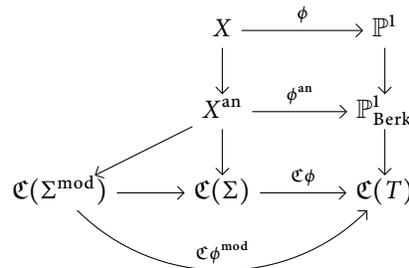
1.2 Smoothing Criterion for Limit Linear Series of Rank One on Saturated Metrized Complexes

In this paper, we consider a refinement of the notion of a metrized complex called *saturated metrized complexes* (Definition 2.1) and undertake a detailed study of smoothability of limit g_d^1 on saturated metrized complexes of algebraic curves. Roughly speaking, a saturated metrized complex can be considered as a metric graph Γ with an algebraic curve associated with each point of Γ . On the other hand, for a metrized

complex, only the points in a finite subset A of Γ are associated with algebraic curves. Therefore, a saturated metrized complex can be derived from a metrized complex by inserting curves at the points in $\Gamma \setminus A$ (the process is called a saturation of the metrized complex). Conversely, a metrized complex can be derived from a saturated metrized complex by ignoring the curves associated with the points in $\Gamma \setminus A$.

The main goal in this paper is to provide a full smoothing criterion for limit g_d^1 on saturated metrized complexes. We briefly explain our approach first and state the criterion in Theorem 1.1.

In a recent work Amini, Baker, Brugallé, and Rabinoff [4, 5] showed that harmonic morphisms between metrized complexes can essentially be lifted to finite morphisms between curves. Let \mathbb{K} be an algebraically closed field of characteristic 0 that is complete with respect to a non-archimedean valuation of value group \mathbb{R} . Let the corresponding residue field be κ . Suppose that X is a smooth proper curve over \mathbb{K} . Let Σ be a skeleton of the Berkovich analytification X^{an} of X . Let $\mathcal{C}(\Sigma)$ be the saturated metrized complex over the residue field κ associated with Σ (see Appendix A.1 for a precise construction). A base point free g_d^1 on X induces a morphism $\phi: X \rightarrow \mathbb{P}^1$ of degree d . By the functoriality of analytification, we have an induced map $\phi^{\text{an}}: X^{\text{an}} \rightarrow \mathbb{P}_{\text{Berk}}^1$, where $\mathbb{P}_{\text{Berk}}^1$ is the Berkovich projective line. The retraction from X^{an} to the skeleton Σ induces a pseudo-harmonic morphism $\mathcal{C}\phi$ (see §3.1 for a precise definition) from the saturated metrized complex $\mathcal{C}(\Sigma)$ to a saturated metrized complex $\mathcal{C}(T)$ of genus zero where the underlying metric tree T of $\mathcal{C}(T)$ is a skeleton of $\mathbb{P}_{\text{Berk}}^1$. On the other hand, by [5, Theorem A] there is a harmonic morphism $\mathcal{C}\phi^{\text{mod}}$ from a modification $\mathcal{C}(\Sigma^{\text{mod}})$ of $\mathcal{C}(\Sigma)$ to $\mathcal{C}(T)$ which is compatible with ϕ^{an} . More precisely, we have the following commutative diagram.



Recall that we can represent a g_d^1 on X by (D, H) where D is an effective divisor of degree d and H is a two-dimensional linear subspace of $H^0(X, \mathcal{L}(D))$. The specialization $(\mathcal{D}, \mathcal{H})$ of (D, H) to $\mathcal{C}(\Sigma)$ represents a limit g_d^1 on $\mathcal{C}(\Sigma)$ that is smoothable. On the other hand, the lifting theorem [5, Theorem B] guarantees that any harmonic morphism onto a genus-0 (saturated) metrized complex can be lifted to a finite morphism onto a projective line. Therefore, to investigate whether a limit g_d^1 represented by $(\mathcal{D}, \mathcal{H})$ on a saturated metrized complex is smoothable, we must characterize when there exist a genus-0 saturated metrized complex $\mathcal{C}(T)$ and a modification $\mathcal{C}(\Sigma^{\text{mod}})$ of $\mathcal{C}(\Sigma)$ such that there is a harmonic morphism $\mathcal{C}\phi^{\text{mod}}: \mathcal{C}(\Sigma^{\text{mod}}) \rightarrow \mathcal{C}(T)$ which is “compatible” with the data $(\mathcal{D}, \mathcal{H})$.

An important aspect is that, starting from a smoothable limit g_d^1 represented by $(\mathcal{D}, \mathcal{H})$ on $\mathfrak{C}(\Sigma)$, there generally exist different choices of $\mathfrak{C}(\Sigma)^{\text{mod}}$, $\mathfrak{C}(T)$, and $\mathfrak{C}\phi^{\text{mod}}: \mathfrak{C}(\Sigma^{\text{mod}}) \rightarrow \mathfrak{C}(T)$ in the above commutative diagram. Therefore, to give a full smoothing criterion, the main challenge is to determine all possible $\mathfrak{C}(\Sigma^{\text{mod}})$, $\mathfrak{C}(T)$, and $\mathfrak{C}\phi^{\text{mod}}$ that are compatible with the data $(\mathcal{D}, \mathcal{H})$.

In this paper, the key to addressing this challenge is to first reorganize the information in $(\mathcal{D}, \mathcal{H})$ and associate with it a combinatorial object called a *global diagram* on Γ . A global diagram consists of a “piecewise-constant differential form” defined on Γ and a partition of the set of tangent directions at every point in Γ . It turns out that the failure of exactness of this differential form is an obstruction to smoothing $(\mathcal{D}, \mathcal{H})$. We say that $(\mathcal{D}, \mathcal{H})$ is solvable if this differential form is exact. Hence, by integration of the exact differential form for a solvable $(\mathcal{D}, \mathcal{H})$, we derive a rational function ρ on Γ with everywhere nonzero slopes. The bifurcation tree \mathcal{B} associated with ρ is a metric tree whose points correspond to superlevel components of Γ at all values of ρ . Moreover, there is a canonical projection $\pi_{\mathcal{B}}$ from Γ to \mathcal{B} . By properly gluing the bifurcation tree \mathcal{B} along its branches, we can derive a metric tree that is called a partition tree with respect to \mathcal{B} . (See §4 for precise definitions of bifurcation trees and partition trees.) We denote by $\Lambda_{\mathcal{D}, \mathcal{H}}^{(1)}$ the space of all such partition trees with respect to \mathcal{B} . One observation is that in the above commutative diagram, the underlying metric tree T of $\mathfrak{C}(T)$ can only possibly be a partition tree.

More smoothing obstructions arise from the compatibility between $\pi_{\mathcal{B}}$ and the data in $(\mathcal{D}, \mathcal{H})$. We organize these obstructions into three additional levels and associate a subspace (denoted by $\Lambda_{\mathcal{D}, \mathcal{H}}^{(2)}$, $\Lambda_{\mathcal{D}, \mathcal{H}}^{(3)}$, and $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$, respectively) of $\Lambda_{\mathcal{D}, \mathcal{H}}^{(1)}$ with each of these three additional levels (Section 5). In particular,

$$\Lambda_{\mathcal{D}, \mathcal{H}}^{(1)} \supseteq \Lambda_{\mathcal{D}, \mathcal{H}}^{(2)} \supseteq \Lambda_{\mathcal{D}, \mathcal{H}}^{(3)} \supseteq \Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}.$$

We will show that any genus zero saturated metrized complex $\mathfrak{C}(T)$ that can arise in the above commutative diagram must have an underlying metric tree T in $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$, and conversely we can build such a $\mathfrak{C}(T)$ from any metric tree T in $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$. See Figure 11 for an example of $\Lambda_{\mathcal{D}, \mathcal{H}}^{(1)}$, $\Lambda_{\mathcal{D}, \mathcal{H}}^{(2)}$, $\Lambda_{\mathcal{D}, \mathcal{H}}^{(3)}$, and $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$.

In summary, we have the following theorem as the smoothing criterion.

Theorem 1.1 (Smoothing Criterion) *A pre-limit g_d^1 (or a limit g_d^1) represented by $(\mathcal{D}, \mathcal{H})$ on a saturated metrized complex is smoothable if and only if the space of metric trees $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$ associated with $(\mathcal{D}, \mathcal{H})$ is nonempty.*

One feature that distinguishes the study of smoothability of limit g_d^1 in comparison with the lifting of harmonic morphisms of [4, 5] is the appearance of the space of trees $\Lambda_{\mathcal{D}, \mathcal{H}}^{(1)}$, $\Lambda_{\mathcal{D}, \mathcal{H}}^{(2)}$, $\Lambda_{\mathcal{D}, \mathcal{H}}^{(3)}$, and $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$. These spaces are naturally endowed with a partial order (see Appendix B) and lead to several directions for future work. For example, they are essentially spaces of phylogenetic trees and seem to be related to similar spaces occurring in other contexts such as the space of trees [9] and in the context of

dynamics [15]. Another direction of investigation is that the work of Abramovich, Caporaso, and Payne [1] suggests that $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$ can be interpreted in terms of the Berkovich analytification of the moduli space of all smoothings of the underlying limit g_d^1 .

Furthermore, the above criterion is an effective characterization of smoothable limit g_d^1 , since it is not only a full criterion but also finitely verifiable given the data $(\mathcal{D}, \mathcal{H})$ (for finite verifications, see §2.7 and an equivalent form of the smoothing criterion in Theorem 2.6).

We also expect the global diagram technique employed in setting up the smoothing criterion to be a useful tool for characterizing the gonality stratum and for studying moduli spaces of metrized complexes (and tropical curves) with a given gonality. These topics will be pursued in the future. We would like to mention that Omid Amini has independently obtained a smoothing theorem on limit g_d^1 for limit linear series in the framework developed in an upcoming paper [2], which is a refinement of [3]. Cartwright has also recently studied the problem of smoothing divisors of rank two and higher [10]. Cartwright considered lifting of rank two divisors on tropical curves (and metrized complexes). Given a matroid and an infinite field κ , he constructed a graph (and a metrized complex) and a divisor of rank two, called a matroid divisor, on it and showed that the problem of lifting the matroid divisor to a smooth curve over $\kappa[[t]]$ is equivalent to realizability of the underlying matroid over κ . These results show that the lifting problem for higher rank divisors (rank two and higher) is sensitive to the underlying field and is evidence for the difficulty in generalizing our smooth criterion for higher rank divisors. In addition, Cartwright, Jensen, and Payne [11] and Jensen and Ranganathan [27] have proved higher rank lifting theorems for divisors on tropical curves which are general and special chain of loops respectively.

1.3 Applications

We present four concrete applications of the smoothing criterion.

1.3.1 Application 1: Saturated Metrized Complexes of Compact Type

In Section 7.1, we prove a version of the smoothing theorem of Eisenbud and Harris [16] for curves of compact type in the setting of saturated metrized complexes with underlying metric graphs being trees (we call them saturated metrized complexes of compact type) as follows: on a saturated metrized complex of compact type, every diagrammatic pre-limit linear series of rank one must be smoothable.

1.3.2 Application 2: Saturated Metrized Complexes with Genus- g Underlying Metric Graphs Containing g Separate Loops

In Section 7.2, we follow the same philosophy in the proof of the above Eisenbud–Harris Theorem and generalize it to a full combinatorial characterization of smoothable limit g_d^1 on a saturated metrized complex whose underlying metric graph Γ contains g separate loops. Note that the case for metric graphs made of chains of loops that are used for tropical proofs of the Brill–Noether Theorem [14] and Gieseker–Petri Theorem [26] falls into this category.

1.3.3 Application 3: Saturated Metrized Complexes of Harris–Mumford Type

Harris and Mumford [24, Theorem 5] considered specific families of nodal curves and proved a smoothing theorem for limit g_d^1 for curves in these families. Using this, they obtained a partial characterization of the gonality stratum. We refer to saturated metrized complexes corresponding to these families of nodal curves as Harris–Mumford saturated metrized complexes and discuss them in detail in Section 7.3. The setting of Theorem 5 of [24] is slightly different from ours since they worked with nodal curves rather than saturated metrized complexes. In particular, the Harris–Mumford types of saturated metrized complexes have flower-like underlying metric trees (see Figure 9). In Theorem 7.6 we prove an analogue of [24, Theorem 5], while Theorem 1.1 can actually be considered as a generalization of [24, Theorem 5] in the above sense.

1.3.4 Application 4: Extending the Smoothing Criterion to Metrized Complexes

Theorem 1.1 also suggests the following approach to an analogous smoothing criterion for limit g_d^1 on metrized complexes: a limit g_d^1 on a metrized complex is smoothable if and only if this limit g_d^1 can be extended to a smoothable limit g_d^1 on a saturated metrized complex that is a saturation of the original metrized complex. We have more detailed discussions and give a concrete example in Section 7.4 to show that even in the setting of metrized complexes, the smoothing criterion is still finitely verifiable.

1.4 Outline of the Rest of the Paper

The rest of the paper is organized as follows. In Section 2, we explain some notions and terminologies essential to the paper. In Section 3, we study the relation between smoothable limit g_d^1 and harmonic morphisms between saturated metrized complexes. In Section 4, we discuss the notions of bifurcation trees and partition trees. In Section 5, we study properties of the spaces $\Lambda_{\mathcal{D}, \mathcal{H}}^{(1)}$, $\Lambda_{\mathcal{D}, \mathcal{H}}^{(2)}$, $\Lambda_{\mathcal{D}, \mathcal{H}}^{(3)}$, and $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$ that arise from different levels of obstructions of the limit g_d^1 from being smoothable. In Section 6, we prove the smoothing criterion. Section 7 is devoted to examples and applications of our smoothing criterion.

2 Some Notions and Terminologies Related to the Smoothing Criterion

In this section, we first present the basic notions and terminologies, and then state an alternative version of the smoothing criterion that is directly verifiable.

Let \mathbb{K} be an algebraically closed field of characteristic 0 complete with respect to a non-trivial non-archimedean absolute value. We assume that the value group of \mathbb{K} is the group of real numbers $(\mathbb{R}, +)$. Let R be the valuation ring of \mathbb{K} , and κ be the residue field of \mathbb{K} that we also assume to be algebraically closed and of characteristic 0. (On the other hand, starting from κ , we can also construct such a field \mathbb{K} with value group \mathbb{R} and residue field κ using generalized Puiseux series [29].) For an algebraically

closed field K , we let $K_\infty = K \cup \{\infty\}$ be the projective line \mathbb{P}_K^1 with coordinates. The above notations and assumptions will be applied throughout this paper.

2.1 Saturated Metrized Complexes

Definition 2.1 A saturated metrized complex \mathcal{C} over an algebraically closed field κ consists of the following data.

- (i) A metric graph Γ .
- (ii) A smooth complete irreducible algebraic curve C_p over κ associated with each point $p \in \Gamma$ such that C_p is a projective line except for points in a finite subset of Γ . (For simplicity, we also use C_p to represent the set of closed points of C_p .)
- (iii) For every point $p \in \Gamma$, an injection $\text{red}_p: \text{Tan}_\Gamma(p) \rightarrow C_p$ called the reduction map at p where $\text{Tan}_\Gamma(p)$ is the set of tangent directions on Γ incident to p , $\text{red}_p(t)$ is called the *marked point* in C_p associated with the tangent direction $t \in \text{Tan}_\Gamma(p)$, and $A_p = \text{Im}(\text{red}_p)$ is the set of marked points of C_p . For convenience, we let $\text{Red}: \coprod_{p \in \Gamma} \text{Tan}_\Gamma(p) \rightarrow \coprod_{p \in \Gamma} C_p$ be defined as $\text{Red}(t) = \text{red}_p(t)$ when $t \in \text{Tan}_\Gamma(p)$.

The *genus* $g(\mathcal{C})$ of \mathcal{C} is defined as $g(\Gamma) + \sum_{p \in \Gamma} g(C_p)$, where $g(\Gamma)$ is the genus (the first Betti number) of the metric graph Γ and $g(C_p)$ is the genus of the algebraic curve C_p . The genus of a saturated metrized complex is finite since C_p has genus zero for all but a finite number of points $p \in \Gamma$.

Remark 2.2 The notion of saturated metrized complex follows the same philosophy as that of metrized complex in [3], except there, only the points in a finite vertex set A of Γ are associated with curves. Hence, by ignoring the curves associated with the points in $\Gamma \setminus A$ for a vertex set A of Γ , we can derive a metrized complex \mathcal{C}' from a saturated metrized complex \mathcal{C} . We say that \mathcal{C}' is the *restriction* of \mathcal{C} to A . Conversely, given a metrized complex \mathcal{C}' and a saturated metrized complex \mathcal{C} , we say that \mathcal{C} is a *saturation* of \mathcal{C}' if \mathcal{C}' is a restriction of \mathcal{C} and they have the same genus (the newly inserted curves in \mathcal{C} are all projective lines).

Remark 2.3 Saturated metrized complexes appear as the inverse image of a skeleton in the map from the Huber adic space to the Berkovich analytification of a curve [20], and are closely related to “exploded tropicalizations” [32].

2.2 Divisor Theory on a Saturated Metrized Complex

Here we give a natural extension of the divisor theory on metrized complexes in [3] to a divisor theory on saturated metrized complexes. This divisor theory on metrized complexes introduced in [3] combines the conventional divisor theory on algebraic curves and an analogous divisor theory on metric graphs [21, 28].

Let \mathcal{C} be a saturated metrized complex with underlying metric graph Γ and algebraic curve C_p at point p . A *pseudo-divisor* \mathcal{D} on a saturated metrized complex \mathcal{C} is the data $(D_\Gamma, \{D_p\}_{p \in \Gamma})$, where D_Γ is a divisor on Γ and D_p is a divisor on the curve C_p satisfying the relation $D_\Gamma(p) = \deg(D_p)$ for every point $p \in \Gamma$. We also say that D_Γ is the tropical part of \mathcal{D} and D_p is the C_p -part of \mathcal{D} . We say that a point $u \in C_p$

is a *supporting point* of \mathcal{D} if $D_q(u) \neq 0$. The *degree* of \mathcal{D} is defined to be the degree of D_Γ . If in addition $D_p = 0$ for all but finitely many points $p \in \Gamma$, then we call \mathcal{D} a *divisor* on \mathcal{C} . We say \mathcal{D} is *effective* if D_p is effective for all points $p \in \Gamma$.

Note that for any pseudo-divisor \mathcal{D} on a saturated metrized complex, D_p will have degree 0 for all but finitely many points $p \in \Gamma$. However, D_p can be non-zero for infinitely many points in Γ , which is unconventional from the viewpoint of divisor theory. The notion of divisors on a saturated metrized complex rectifies this aspect.

Remark 2.4 All divisors on \mathcal{C} form a group isomorphic to $\bigoplus_{p \in \Gamma} \text{Div}(C_p)$ (the free abelian group on $\coprod_{p \in \Gamma} C_p$). Therefore, we also write a divisor \mathcal{D} on \mathcal{C} formally as $\mathcal{D} = \sum_{p \in \Gamma} D_p$, where D_p is the C_p -part of \mathcal{D} .

A *pseudo-rational function* $\mathfrak{f} = (f_\Gamma, \{f_p\}_{p \in \Gamma})$, where f_Γ is a tropical rational function (piecewise-linear function with integer slopes) on Γ and f_p is a rational function on the algebraic curve C_p . We also say that f_Γ is the *tropical part* of \mathfrak{f} and f_p is the C_p -part of \mathfrak{f} . We say that \mathfrak{f} is *nonzero* if f_p is nonzero for all $p \in \Gamma$. The *principal pseudo-divisor* $\text{div}(\mathfrak{f})$ associated with a nonzero pseudo-rational function \mathfrak{f} is defined as $(\text{div}(f_\Gamma), \{\text{div}(f_p) + \text{div}_p(f_\Gamma)\}_{p \in \Gamma})$ where $\text{div}_p(f_\Gamma) = \sum_{t \in \text{Tan}_\Gamma(p)} \text{sl}_t(f_\Gamma)(\text{red}_p(t))$ and $\text{sl}_t(f_\Gamma)$ is the outgoing slope of the function f_Γ along the tangent direction t . A *rational function* is a pseudo-rational function whose associated principal pseudo-divisor is a divisor. Divisors \mathcal{D}_1 and \mathcal{D}_2 are *linearly equivalent* if they differ by a principal divisor.

As in the case of principal divisors on an algebraic curve or on a metric graph, the set of the principal divisors on a saturated metrized complex forms an abelian group under addition.

2.3 Limit Linear Series on a Saturated Metrized Complex

Definition 2.5 Let \mathcal{C} be a saturated metrized complex. A *pre-limit linear series* of rank r and degree d (also known as a pre-limit g_d^r) on \mathcal{C} is represented by the data $(\mathcal{D}, \mathcal{H})$ where $\mathcal{D} = (D_\Gamma, \{D_p\}_{p \in \Gamma})$ is an effective divisor of degree d on \mathcal{C} and $\mathcal{H} = \{H_p\}_{p \in \Gamma}$, where H_p is an $(r+1)$ -dimensional subspace of the function field of C_p .

A *limit linear series* of rank r and degree d (also known as a limit g_d^r) on \mathcal{C} is a pre-limit g_d^r represented by $(\mathcal{D}, \mathcal{H})$ with $\mathcal{H} = \{H_p\}_{p \in \Gamma}$ that satisfies the following additional condition: for every effective divisor $\mathcal{E} = (E_\Gamma, \{E_p\}_{p \in \Gamma})$ on \mathcal{C} of degree r such that the support of E_p does not intersect the set A_p of marked points of C_p for every $p \in \Gamma$, there exists a rational function $\mathfrak{f} = (f_\Gamma, \{f_p\}_{p \in \Gamma})$ such that $f_p \in H_p$ for all points $p \in \Gamma$ and $\mathcal{D} - \mathcal{E} + \text{div}(\mathfrak{f}) \geq 0$.

Our definition of limit linear series is a slight modification of the notion of crude limit linear series in [3, §5.3]. For instance, we impose the additional restriction that the support of E_p does not intersect the set A_p . This restriction is justified by Theorem 3.7, which states that a specialization of linear series is a limit linear series.

Throughout this paper, unless otherwise specified, we let $\mathcal{D} = (D_\Gamma, \{D_p\}_{p \in \Gamma})$ and $\mathcal{H} = \{H_p\}_{p \in \Gamma}$ for convenience. When we say $F = \{f_p\}_{p \in \Gamma}$ is an element of \mathcal{H} , it means

$f_p \in H_p$ for all $p \in \Gamma$. In addition, we may also consider F as a function from $\coprod_{p \in \Gamma} C_p$ to κ which sends $u \in C_p$ to $f_p(u)$.

We say a pre-limit g_d^r (respectively, a limit g_d^r) represented by $(\mathcal{D}, \mathcal{H})$ is *refined* if $(\mathcal{D}, \mathcal{H})$ has the following additional properties.

- The constant function is contained in H_p for every point $p \in \Gamma$.
- For every point p , the support of D_p is disjoint from the set A_p of marked points of C_p .

The two conditions are justified by Lemma 3.11, stating that a smoothable pre-limit g_d^r must be refined.

2.4 Alternative Version of the Smoothing Criterion

We have the following equivalent version of the smoothing criterion in Theorem 1.1. This version is finitely verifiable (see §2.7).

Theorem 2.6 (Smoothing Criterion, Version II) *A pre-limit g_d^1 (or a limit g_d^1) represented by $(\mathcal{D}, \mathcal{H})$ on a saturated metrized complex is smoothable if and only if $(\mathcal{D}, \mathcal{H})$ is diagrammatic, is solvable, and satisfies the intrinsic global compatibility conditions.*

We primarily employ this version in the rest of the paper, in particular in Section 7. The equivalence of these two versions of the smoothing criterion follows from Proposition 5.6. In the rest of this section, we explain the terms *diagrammatic*, *solvable*, and *intrinsic global compatibility* conditions that appear in the smoothing criterion. In Section 7.4, we extend this smoothing criterion on saturated metrized complexes to the case for metrized complexes in a natural way.

2.5 Diagrams Induced by $(\mathcal{D}, \mathcal{H})$ and Solvability

For a pre-limit g_d^1 represented by $(\mathcal{D}, \mathcal{H})$ to be refined, we extract the information in the space H_p and associate a combinatorial object called a local diagram with a point $p \in \Gamma$. A global diagram is formed by assembling the local diagrams at all points in Γ in a “continuous” way.

First let us give a more precise description of local and global diagrams on a metric graph Γ .

Definition 2.7 (i) A *local diagram* at a point p of Γ is made of following data.

- A nonzero integer $m(p, t)$ called the *multiplicity* associated with each tangent direction $t \in \text{Tan}_\Gamma(p)$, where $\text{Tan}_\Gamma(p)$ is the set of tangent directions emanating from p . We refer to those tangent directions with negative multiplicities as incoming tangent directions and denote the set they form by $\text{In}_\Gamma(p)$. Similarly, we refer to those tangent directions with positive multiplicities as outgoing tangent directions and denote the set they form by $\text{Out}_\Gamma(p)$.
- The elements in $\text{Tan}_\Gamma(p)$ are partitioned into equivalence classes with one of the equivalence classes being exactly $\text{In}(p)$. We refer to this partition of $\text{Tan}_\Gamma(p)$ as the *local partition* at p and refer to the tangent directions that belong to the same equivalence class as *locally equivalent*.

(ii) An open neighborhood of a point $p \in \Gamma$ is called a simple neighborhood if it is simply connected and every point in the neighborhood, except possibly p , has valence two. For a simple neighborhood U of a point $p \in \Gamma$, a local diagram at a point p induces a local diagram at any point q in $U \setminus \{p\}$ as follows. Suppose that q lies along the tangent direction $t \in \text{Tan}_\Gamma(p)$. Assign the integer $-m(p, t)$ to the tangent direction at q corresponding to the edge joining p and q and assign the integer $m(p, t)$ to the other tangent direction at q . Assign the two tangent directions at q to different equivalence classes.

(iii) A *global diagram* on a metric graph Γ is a collection of local diagrams at all the points in Γ such that the local diagrams satisfy the following continuity property: for every point p , there is a simple neighborhood U of p such that for every point $q \in U \setminus \{p\}$ the local diagram induced by p at q coincides with the local diagram at q .

(iv) A global diagram on Γ is called *solvable* if there exists a tropical rational function ρ on Γ such that the outgoing slope $\text{sl}_t(\rho)$ of ρ along the tangent direction $t \in \text{Tan}_\Gamma(p)$ for any point $p \in \Gamma$ coincides with the multiplicity. Formally, this means that the differential equation $\text{sl}_t(\rho) = m(p, t)$ is satisfied. This equation is referred to as the *characteristic equation* of the global diagram.

Remark 2.8 Recall that a vertex set of Γ (the set of all points of valence at least three) induces an edge-weighted graph called a *model* of Γ (see [6]). Since Γ is compact, the multiplicity aspect of a global diagram can be represented in terms of the following data: a model for Γ , an orientation of each edge of the model, and a non-zero integer associated with each edge called the *multiplicity* of that edge. In other words, a global diagram induces a piecewise-constant integer-valued differential form on Γ and this differential form is exact if the global diagram is solvable.

When a global diagram is solvable, the solution of the characteristic equation is unique up to a translation by a constant. Since $m(p, t)$ is nowhere zero for all p and $t \in \text{Tan}_\Gamma(p)$, the solution must have everywhere nonzero slopes.

For a two-dimensional linear space H_p of rational functions on C_p that contains the constant functions, we can naturally associate a local diagram defined with each point of Γ . The details of such a procedure are presented in the following construction. This construction translates the algebraic data encoded in \mathcal{H} to combinatorial data on Γ . Furthermore, the two aspects of the definition of local diagrams, multiplicity and local partition, are both motivated from the notion of harmonic morphisms (§3.1).

Construction of Local Diagrams From \mathcal{H} In the following, given a refined pre-limit g_d^1 represented by $(\mathcal{D}, \mathcal{H})$, we associate a local diagram with every point on the metric graph using the data \mathcal{H} . From the definition of a refined pre-limit g_d^1 , we know that the two-dimensional linear space H_p of rational functions on C_p has a basis $\{1, f_p\}$ where f_p is a nonconstant rational function on C_p . Let $\bar{f}_p: C_p \rightarrow \kappa_\infty$ be the function on C_p extending f_p to its poles. We construct the local diagram at p as follows: for a tangent direction t of C_p , we let the multiplicity $m(p, t)$ be the ramification index of \bar{f}_p at the marked point corresponding to the tangent direction t with sign “−” if $\text{red}_p(t)$ is a pole of f_p and sign “+” otherwise. The local partition at p

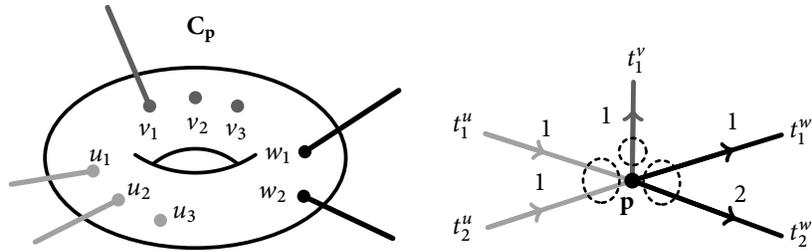


Figure 1: A local diagram (right) at p derived from H_p on the curve C_p (left) of genus 1.

is defined by declaring that two tangent directions are locally equivalent if and only if their marked points are in the same level set of f_p . Hence $\text{In}_\Gamma(p)$ is the set of tangent directions whose corresponding marked points are the poles of f_p and the elements of $\text{In}_\Gamma(p)$ are all locally equivalent. In this way, we construct a local diagram at p from f_p . Moreover, since different choices of nonconstant functions f_p in H_p afford the same local diagram at p , we also say that this local diagram at p is induced by H_p .

Compatibility Between \mathcal{D} and \mathcal{H} For a nonconstant function $f_p \in H_p$, we say that $D_{f_p}^+$ and $D_{f_p}^-$ are the effective and non-effective parts of $\text{div}(f_p)$, respectively, if $\text{div}(f_p) = D_{f_p}^+ - D_{f_p}^-$ and both $D_{f_p}^+$ and $D_{f_p}^-$ are effective with no overlapping supporting points. Note that by the construction of the local diagrams from f_p , the marked point associated with an incoming tangent direction in $\text{In}_\Gamma(p)$ is a pole of f_p and we must have $D_{f_p}^- + \sum_{t \in \text{In}_\Gamma(p)} m(p, t)(\text{red}_p(t)) \geq 0$ (note that $m(p, t) < 0$ for $t \in \text{In}_\Gamma(p)$). By comparing D_p with $D_{f_p}^- + \sum_{t \in \text{In}_\Gamma(p)} m(p, t)(\text{red}_p(t))$, we introduce the compatibility between D_p and H_p . More precisely, we say that H_p is compatible with D_p or \mathcal{D} if $D_p \geq D_{f_p}^- + \sum_{t \in \text{In}_\Gamma(p)} m(p, t)(\text{red}_p(t))$. We say that a supporting point of $D_p - \sum_{t \in \text{In}_\Gamma(p)} m(p, t)(\text{red}_p(t)) - D_{f_p}^-$ is a base point of $(\mathcal{D}, \mathcal{H})$ when D_p is compatible with H_p . In this sense, a supporting point of D_p is either a base point of $(\mathcal{D}, \mathcal{H})$ or a pole of a nonconstant rational function in H_p . We say \mathcal{D} and \mathcal{H} are compatible if D_p and H_p are compatible for all $p \in \Gamma$. Note that the total number of base points must be finite when \mathcal{D} and \mathcal{H} are compatible.

Example 2.9 Figure 1 illustrates the construction of a local diagram at $p \in \Gamma$ using a two-dimensional linear space H_p of rational functions on a genus-1 curve C_p . We suppose that each H_p contains a constant function. Consider a nonconstant rational function $f_p \in H_p$. Suppose f_p has degree 3 and the poles of f_p are u_1, u_2 , and u_3 with all ramification indices being 1, and points u_1 and u_2 are marked points with associated tangent directions t_1^u and t_2^u . Then t_1^u and t_2^u are incoming tangent directions with $m(p, t_1^u) = m(p, t_2^u) = -1$. Let c and c' be distinct values in κ and suppose $f_p^{-1}(c) = \{v_1, v_2, v_3\}$ with all ramification indices being 1 and $f_p^{-1}(c') = \{w_1, w_2\}$ with ramification indices of w_1 and w_2 being 1 and 2, respectively. Suppose v_1, w_1 , and w_2 are marked points with associated tangent directions t_1^v, t_1^w , and t_2^w , respectively. Then t_1^v, t_1^w , and t_2^w are outgoing tangent directions with $m(p, t_1^v) = 1, m(p, t_1^w) = 1$, and $m(p, t_2^w) = 2$.

Moreover, the corresponding local partition of $\text{Tan}_\Gamma(p)$ is $\{\{t_1^u, t_2^u\}, \{t_1^v\}, \{t_1^w, t_2^w\}\}$. Moreover, a divisor D_p on C_p is compatible with H_p if and only if u_3 is a supporting point of D_p .

Definition 2.10 We say that a refined pre-limit g_d^1 (respectively, limit g_d^1) represented by $(\mathcal{D}, \mathcal{H})$ on a saturated metrized complex $\mathfrak{C} = (\Gamma, \{C_p\}_{p \in \Gamma})$ is *diagrammatic* if \mathcal{D} and \mathcal{H} are compatible and the local diagrams induced by \mathcal{H} form a global diagram. In addition, if the global diagram induced by $(\mathcal{D}, \mathcal{H})$ is solvable, then we say that $(\mathcal{D}, \mathcal{H})$ is *solvable*.

2.6 Intrinsic Global Compatibility Conditions

In general, the extra condition in the definition of a (diagrammatic) limit g_d^1 over that of a (diagrammatic) pre-limit g_d^1 represented by $(\mathcal{D}, \mathcal{H})$ does not guarantee solvability (see the example in §5.1). However, if $(\mathcal{D}, \mathcal{H})$ is smoothable, then it must be solvable (see Theorem 2.6).

On the other hand, solvability is only a necessary condition for $(\mathcal{D}, \mathcal{H})$ being smoothable. In particular, it does not fully utilize information in H_p . In order to determine the smoothability of $(\mathcal{D}, \mathcal{H})$, we will construct the bifurcation tree \mathcal{B} and the natural surjection $\pi_{\mathcal{B}}: \Gamma \rightarrow \mathcal{B}$ from a solution to the characteristic equation and introduce the intrinsic global compatibility conditions which are compatibility conditions between H_p 's and the map $\pi_{\mathcal{B}}$.

Let ρ be a rational function on Γ with everywhere nonzero slopes. For a point p in Γ , recall that $\text{Tan}_\Gamma(p)$ is the set of tangent directions emanating from p . By $\text{Tan}_\Gamma^{\rho+}(p)$, we denote the set of tangent directions in $\text{Tan}_\Gamma(p)$, where ρ locally increases. We canonically associate ρ with a pair $(\mathcal{B}, \pi_{\mathcal{B}})$, where \mathcal{B} is a metric tree called the *bifurcation tree* with respect to ρ , and $\pi_{\mathcal{B}}: \Gamma \rightarrow \mathcal{B}$ is a canonical projection from Γ onto \mathcal{B} (see details of the construction in §4.1). In addition, \mathcal{B} has a distinguished point called the *root*. Moreover, $\pi_{\mathcal{B}}$ induces a push forward map $\pi_{\mathcal{B}*}$ from the tangent directions at all points in Γ to the tangent directions at all points in \mathcal{B} . More precisely, if we let $\text{Tan}_{\mathcal{B}}^+(x)$ be the set of forward tangent directions at $x \in \mathcal{B}$ (meaning that the distance function from the root increases along these directions), then by the construction in Section 4.1, we have $\pi_{\mathcal{B}*}(t) \in \text{Tan}_{\mathcal{B}}^+(\pi_{\mathcal{B}}(p))$ for all $p \in \Gamma$ and $t \in \text{Tan}_\Gamma^{\rho+}(p)$.

Using the above notions, we formulate the intrinsic global compatibility conditions. For a solvable $(\mathcal{D}, \mathcal{H})$ with a solution ρ , let \mathcal{B} be the bifurcation tree and let $\pi_{\mathcal{B}}$ be the corresponding projection from Γ onto \mathcal{B} .

Definition 2.11 A collection of non-constant rational functions $G = \{g_p\}_{p \in \Gamma} \in \mathcal{H}$ is called *admissible* if it has one of the following equivalent properties.

- There is a function $\eta: \coprod_{x \in \mathcal{B}} \text{Tan}_{\mathcal{B}}^+(x) \rightarrow \kappa$ such that $G \circ \text{Red}(t) = \eta \circ \pi_{\mathcal{B}*}(t)$ for all $t \in \coprod_{p \in \Gamma} \text{Tan}_\Gamma^{\rho+}(p)$.
- For each pair of tangent directions $t_1 \in \text{Tan}_\Gamma^{\rho+}(p_1)$ and $t_2 \in \text{Tan}_\Gamma^{\rho+}(p_2)$ such that $\pi_{\mathcal{B}}(p_1) = \pi_{\mathcal{B}}(p_2)$ and $\pi_{\mathcal{B}*}(t_1) = \pi_{\mathcal{B}*}(t_2)$, we have

$$g_{p_1}(\text{red}_{p_1}(t_1)) = g_{p_2}(\text{red}_{p_2}(t_2)).$$

We say that $(\mathcal{D}, \mathcal{H})$ satisfies the *intrinsic global compatibility conditions* if \mathcal{H} contains an *admissible* collection of non-constant rational functions.

2.7 Finite Verification of Intrinsic Global Compatibility Conditions

In this section, we will show that the intrinsic global compatibility conditions can be verified in finitely many steps, which makes Theorem 2.6 an effective smoothing criterion.

A point $p \in \Gamma$ is called an *ordinary point* of ρ if its valence is two, the slopes of ρ at p in the two tangent directions have the same magnitude and opposite signs, and the curve C_p has genus 0. Denote the set of ordinary points by \mathcal{O}_ρ . The points in $\mathcal{E}_\rho := \Gamma \setminus \mathcal{O}_\rho$ and the values in the image of ρ restricted to \mathcal{E}_ρ are called *exceptional points* and *exceptional values* of ρ , respectively. Note that \mathcal{E}_ρ is a finite set.

The intrinsic global compatibility conditions are finitely verifiable since the verification can be restricted to the set \mathcal{E}_ρ of exceptional points, which is finite. This is because for any ordinary point $p \in \mathcal{O}_\rho$, there is only one forward tangent direction t in $\text{Tan}_\Gamma^{\rho+}(p)$, and by Lemma 2.12 we can always find some non-constant $g_p \in H_p$ such that $g_p(\text{red}_p(t)) = c$ for whatever desirable $c \in \kappa$.

Lemma 2.12 *Let C be a curve over κ and H the linear subspace of the function field of C spanned by $\{1, f\}$, where f is a non-constant rational function on C .*

- (i) *Suppose u is a point on C that is not a pole of f . Then for any value $c \in \kappa$, the space of all rational functions in H taking value c at the point u is a line in H .*
- (ii) *Suppose u_1 and u_2 are points on C that are not poles of f and $f(u_1) \neq f(u_2)$. Then for any distinct values $c_1, c_2 \in \kappa$, we can always find a unique non-constant $g \in H$ such that $g(u_1) = c_1$ and $g(u_2) = c_2$.*

Proof All rational functions in $g \in H$ can be expressed as a linear combination of 1 and f , i.e., $g = \alpha + \beta f \in H$ for some $\alpha, \beta \in \kappa$. For (i), the space $\{g \in H \mid g(u) = c\} = \{\alpha + \beta f \mid \alpha + \beta f(u) = c\}$ is a line in H . For (ii), the linear equations $\alpha + \beta f(u_1) = c_1$ and $\alpha + \beta f(u_2) = c_2$ have a unique solution for α and β . ■

More accurately, we have the following algorithm to determine whether $(\mathcal{D}, \mathcal{H})$ is smoothable (see also Example 5.5).

Algorithm 2.13 **Input:** A diagrammatic pre-limit g_d^1 represented by $(\mathcal{D}, \mathcal{H})$.

Output: Whether $(\mathcal{D}, \mathcal{H})$ is smoothable.

- 1 Determine whether $(\mathcal{D}, \mathcal{H})$ is solvable. If not, then $(\mathcal{D}, \mathcal{H})$ is not smoothable. If yes, let \mathcal{E}_ρ be the set of exceptional points.
- 2 Fix a basis $\{1, f_p\}$ of H_p for all exceptional points $p \in \mathcal{E}_\rho$ and consider a collection of finitely many variables $\{\alpha_p, \beta_p\}_{p \in \mathcal{E}_\rho}$.
- 3 For each pair (t_1, t_2) such that $t_1 \in \text{Tan}_\Gamma^{\rho+}(p_1)$, $t_2 \in \text{Tan}_\Gamma^{\rho+}(p_2)$, $p_1, p_2 \in \mathcal{E}_\rho$, $\pi_{\mathcal{B}}(p_1) = \pi_{\mathcal{B}}(p_2)$, and $\pi_{\mathcal{B}*}(t_1) = \pi_{\mathcal{B}*}(t_2)$, add a linear restriction

$$\alpha_{p_1} + f_{p_1}(\text{red}_{p_1}(t_1))\beta_{p_1} = \alpha_{p_2} + f_{p_2}(\text{red}_{p_2}(t_2))\beta_{p_2}.$$

- 4 $(\mathcal{D}, \mathcal{H})$ satisfies the intrinsic global compatibility conditions and thus is smoothable if and only if there exists a solution for $\{\alpha_p, \beta_p\}_{p \in \mathcal{E}_\rho}$ satisfying the linear restrictions in (3) and $\beta_p \neq 0$ for all $p \in \mathcal{E}_\rho$. In particular, in the case $\{\alpha_p, \beta_p\}_{p \in \mathcal{E}_\rho}$ is a solution. If we let $g_p = \alpha_p + \beta_p f_p$ for $p \in \mathcal{E}_\rho$ and $g_p = f_p$ for $p \in \mathcal{O}_\rho$, then $\{g_p\}_{p \in \Gamma}$ is admissible.

3 Morphisms of Saturated Metrized Complexes and Their Relations to Smoothability

In this section, we give precise definitions of pseudo-harmonic morphisms and harmonic morphisms of saturated metrized complexes and smoothability of a pre-limit g_d^1 (Definition 3.8) and study their close relations to smoothable limit g_d^1 s (§3.3). We use notions from the theory of Berkovich analytic spaces with explanations in Appendix A. The reader is urged to use references [6, 8] with an elaborate treatment of this analytical construction.

3.1 Pseudo-harmonic Morphisms and Harmonic Morphisms

We give a natural extension of the notion of harmonic morphism of metrized complexes from [4, 5] to saturated metrized complexes. We start with the notion of pseudo-harmonic morphism of saturated metrized complexes.

Let \mathcal{C} and \mathcal{C}' be saturated metrized complexes. The underlying metric graphs of \mathcal{C} and \mathcal{C}' are Γ and Γ' respectively, and the associated curves of \mathcal{C} and \mathcal{C}' are $\{C_p\}_{p \in \Gamma}$ and $\{C'_q\}_{q \in \Gamma'}$, respectively.

Definition 3.1 A pseudo-harmonic morphism between \mathcal{C} and \mathcal{C}' consists of the data $\{\phi_\Gamma, \{\phi_p\}_{p \in \Gamma}\}$, where $\phi_\Gamma: \Gamma \rightarrow \Gamma'$ is a continuous finite surjective piecewise-linear map with integral slopes (that is called a pseudo-harmonic morphism between Γ and Γ') and $\phi_p: C_p \rightarrow C'_{\phi_\Gamma(p)}$ is a finite morphism of curves that satisfies the following compatibility conditions.

- For all $p \in \Gamma$, two tangent directions $t_1, t_2 \in \text{Tan}_\Gamma(p)$ are mapped to the same tangent direction $t' \in \text{Tan}_{\Gamma'}(\phi_\Gamma(p))$ by the induced map of ϕ_Γ if and only if the marked points corresponding to t_1 and t_2 are mapped to the marked point corresponding to the tangent direction t' by ϕ_p .
- For all $p \in \Gamma$ and all tangent directions $t \in \text{Tan}_\Gamma(p)$, the expansion factor $d_t(\phi_\Gamma)$ of ϕ_Γ along t coincides with the ramification index of ϕ_p at the marked point corresponding to the tangent direction t . Here the expansion factor $d_t(\phi_\Gamma)$ is the absolute value of the slope of ϕ_Γ along t , i.e., the ratio of the length between $\phi_\Gamma(p)$ and $\phi_\Gamma(q)$ over the length between p and q , where q is near p in the direction t .

Definition 3.2 A pseudo-harmonic morphism $\{\phi_\Gamma, \{\phi_p\}_{p \in \Gamma}\}$ between saturated metrized complexes \mathcal{C} and \mathcal{C}' is called a harmonic morphism at a point $p \in \Gamma$ if ϕ_Γ is harmonic at p and the degree of ϕ_Γ at $p \in \Gamma$ is the same as the degree of ϕ_p . More explicitly, we say ϕ_Γ is harmonic at p if it is a pseudo-harmonic morphism of metric graphs satisfying the following balancing condition: for any tangent direction $t' \in \text{Tan}_{\Gamma'}(\phi(p))$, the sum of the expansion factors $d_t(\phi_\Gamma)$ over all tangent directions t in

$\text{Tan}_\Gamma(p)$ that map to t' , i.e., the integer

$$\sum_{\substack{t \in \text{Tan}_\Gamma(p) \\ t \rightarrow t'}} d_t(\phi_\Gamma),$$

is independent of t' ; it is called the *degree* of ϕ_Γ at q .

We say ϕ_Γ is a harmonic morphism of metric graphs if ϕ_Γ is harmonic at each $p \in \Gamma$, and $\{\phi_\Gamma, \{\phi_p\}_{p \in \Gamma}\}$ is a harmonic morphism of saturated metrized complexes if $\{\phi_\Gamma, \{\phi_p\}_{p \in \Gamma}\}$ is harmonic at each $p \in \Gamma$.

Remark 3.3 More precisely, our notion of harmonic morphisms of saturated metrized complexes corresponds to the notion of *finite* harmonic morphisms of metrized complexes [5].

The notion of harmonic morphism allows us to define the notion of isomorphism of saturated metrized complexes. Two saturated metrized complexes \mathcal{C}_1 and \mathcal{C}_2 are *isomorphic* if there is a harmonic morphism $\mathcal{C}\phi_1: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ and a harmonic morphism $\mathcal{C}\phi_2: \mathcal{C}_2 \rightarrow \mathcal{C}_1$ such that $\mathcal{C}\phi_2 \circ \mathcal{C}\phi_1$ and $\mathcal{C}\phi_1 \circ \mathcal{C}\phi_2$ are identity maps on \mathcal{C}_1 and \mathcal{C}_2 , respectively.

The following theorem summarizes the lifting results of saturated metrized complexes that is a direct corollary of the lifting theorems for metrized complexes in [5]. See Appendix A.1 for the association of a saturated metrized complex with a skeleton of a Berkovich analytic curve.

Theorem 3.4 *We have the following lifting properties for saturated metrized complexes and harmonic morphisms of saturated metrized complexes.*

- (i) *Let \mathcal{C} be a saturated metrized complex of curves over κ . There exists a smooth curve X/\mathbb{K} and a skeleton Σ of the Berkovich analytification X^{an} of X such that \mathcal{C} is isomorphic to the associated saturated metrized complex of Σ .*
- (ii) *If $\mathcal{C}\phi: \mathcal{C} \rightarrow \mathcal{C}'$ is a harmonic morphism of saturated metrized complexes where \mathcal{C}' is isomorphic to the saturated metrized complex associated with a skeleton of the Berkovich analytification X'^{an} of a smooth curve X'/\mathbb{K} , then there exists a finite morphism $\phi: X \rightarrow X'$ of curves over \mathbb{K} such that \mathcal{C} is isomorphic to the associated saturated metrized complex of a skeleton of X^{an} and ϕ induces $\mathcal{C}\phi$.*

Proof For part (i), we can choose a vertex set V of the underlying metric graph Γ of \mathcal{C} such that the curves associated with points in $\Sigma \setminus V$ are all projective lines over κ with two marked points. Let \mathcal{C}_0 be the metrized complex derived from restricting \mathcal{C} to V (Remark 2.2). We can lift \mathcal{C}_0 to a smooth curve X over \mathbb{K} with skeleton Σ by the lifting theorem of metrized complexes. It is then straightforward to verify that \mathcal{C} is isomorphic to the associated saturated metrized complex of Σ .

For part (ii), we choose vertex sets of \mathcal{C} and \mathcal{C}' fine enough to derive metrized complexes \mathcal{C}_0 and \mathcal{C}'_0 , respectively, such that the harmonic morphism $\mathcal{C}\phi$ of saturated metrized complex can be restricted to harmonic morphism $\mathcal{C}\phi_0: \mathcal{C}_0 \rightarrow \mathcal{C}'_0$ of metrized complexes. By the lifting theorem of harmonic morphism of metrized complexes, we can lift $\mathcal{C}\phi_0$ to a finite morphism $\phi: X \rightarrow X'$. Then $\mathcal{C}\phi$ is induced by ϕ . ■

The rest of this subsection contains the notion of pullback of a harmonic morphism and the notion of modification of saturated metrized complex. This will be used in the proof of the smoothing criterion (§6).

Remark 3.5 (Pullback divisor and pullback function of a harmonic morphism) Let $\mathcal{C}\phi = (\phi_\Gamma, \{\phi_p\}_{p \in \Gamma})$ be a harmonic morphism between saturated metrized complexes \mathcal{C} and \mathcal{C}' whose underlying metric graphs are Γ and Γ' , respectively. Let p' be a point in Γ' and u' be a point in the associated curve $C'_{p'}$ of p' in \mathcal{C}' . Let $E_{u'}$ be the degree one effective divisor on \mathcal{C}' whose only supporting point is u' . Then we can naturally associate a *pullback divisor* $\mathcal{C}\phi^*(E_{u'}) \in \text{Div}(\mathcal{C})$ of $E_{u'}$ defined as follows: (1) the tropical part of $\mathcal{C}\phi^*(E_{u'})$ is the pullback divisor $\phi_\Gamma^*((p')) \in \text{Div}(\Gamma)$ of the divisor $(p') \in \text{Div}(\Gamma')$, (2) the C_p -part of $\mathcal{C}\phi^*(E_{u'})$ is the pullback divisor $\phi_p^*((u')) \in \text{Div}(C_p)$ of the divisor $(u') \in \text{Div}(C'_{p'})$ if $p \in \phi_\Gamma^{-1}(p')$, and (3) the C_p -part of $\mathcal{C}\phi^*(E_{u'})$ is 0 if $p \notin \phi_\Gamma^{-1}(p')$. Note that the properties of harmonic morphisms guarantee that $\mathcal{C}\phi^*(E_{u'})$ is a well-defined divisor on \mathcal{C} . We may sometimes also simply refer to the pullback divisor of $E_{u'}$ as the pullback divisor of the point u' . Since we can formally write $E_{u'} = (u')$ (Remark 2.4), we can also formally write

$$\mathcal{C}\phi^*((u')) = \sum_{p \in \phi_\Gamma^{-1}(p')} \phi_p^*((u')).$$

Moreover, by letting $\mathcal{C}\phi^*$ preserve linear combinations, we can naturally associate a pullback divisor $\mathcal{C}\phi^*(\mathcal{D}')$ on \mathcal{C} with each divisors \mathcal{D}' on \mathcal{C}' .

On the other hand, if $f' = (f'_{\Gamma'}, \{f'_{p'}\}_{p' \in \Gamma'})$ is a rational function on \mathcal{C}' , then we can also pullback f' to a rational function $\mathcal{C}\phi^*(f')$ on \mathcal{C} in a natural way: the tropical part of $\mathcal{C}\phi^*(f')$ is the pullback function $\phi_\Gamma^*(f'_{\Gamma'})$ of $f'_{\Gamma'}$, and the C_p -part of $\mathcal{C}\phi^*(f')$ is the pullback function $\phi_p^*(f'_{\phi_\Gamma(p)})$ of the rational function $f'_{\phi_\Gamma(p)}$. It is straightforward to verify that the principal divisor associated with the pullback function of f' is the same as the pullback divisor of the principal divisor associated to f' .

A metric graph Γ^{mod} is called a *modification* of a metric graph Γ if Γ is isometric to a subgraph of Γ^{mod} and the genus of Γ^{mod} is the same as the genus of Γ .

A saturated metrized complex \mathcal{C}^{mod} is called a *modification* of a saturated metrized complex \mathcal{C} if (1) the metric graph Γ^{mod} underlying \mathcal{C}^{mod} is a modification of the metric graph Γ underlying \mathcal{C} , (2) $g(\mathcal{C}^{\text{mod}}) = g(\mathcal{C})$, and (3) for each point $p \in \Gamma$, when p is also considered as a point in Γ^{mod} , the curve associated with p in \mathcal{C} is identical to the curve associated with p in \mathcal{C}^{mod} and the reduction map of \mathcal{C} at p is identical to the reduction map of \mathcal{C}^{mod} at p restricted to $\text{Tan}_\Gamma(p)$ (note that in this setting we have $\text{Tan}_{\Gamma^{\text{mod}}}(p) \supseteq \text{Tan}_\Gamma(p)$).

Remark 3.6 If \mathcal{C}^{mod} is a modification of \mathcal{C} as saturated metrized complexes with underlying metric graphs Γ^{mod} and Γ , respectively then by the retraction map from Γ^{mod} to Γ , a divisor on Γ^{mod} also naturally retracts to a unique divisor on Γ . In addition, a divisor on \mathcal{C}^{mod} naturally retracts to a divisor on \mathcal{C} and any specialization map of divisors factors through the retraction map of divisors.

3.2 Smoothability

The following theorem is the analogue of the specialization theorem for saturated metrized complexes [3, Theorem 5.9].

In Appendix A, we show that for a smooth curve X/\mathbb{K} , we can associate a saturated metrized complex $\mathfrak{C}(\Sigma)$ with a Berkovich skeleton Σ of X^{an} , and there exists a specialization map τ_* that takes a divisor on X to a divisor on $\mathfrak{C}(\Sigma)$ and a reduction map that takes a rational function on X to a rational function on $\mathfrak{C}(\Sigma)$.

Theorem 3.7 For any g_d^r on X represented by the pair (D, H) , where H is an $(r+1)$ -dimensional linear space of rational functions on X , the data $(\tau_*(D), \{H_p\}_{p \in \Gamma})$, where H_p is the image of H under the reduction map at p , is a limit g_d^r on $\mathfrak{C}(\Sigma)$.

Proof By Lemma A.4, the dimension of the space H is preserved by the specialization map. From Lemma A.3, we know that for any effective divisor $\mathcal{E} = (E_\Gamma, \{E_p\}_{p \in \Gamma})$ such that E_p has support in S_p for every $p \in \Gamma$, there exists an effective divisor E on X such that $\tau_*(E) = \mathcal{E}$. Since (D, H) represents a g_d^r on X , there must be a rational function $f \in H$ such that $D - E + \text{div}(f) \geq 0$. We apply the specialization map to this inequality. Using the property that the specialization map is a homomorphism between divisor groups that preserves effective divisors, combined with Theorem A.5, we conclude that $\tau_*(D) - \mathcal{E} + \text{div}(f) \geq 0$. ■

The following definition of smoothability of a pre-limit (or limit) g_d^r represented by $(\mathcal{D}, \mathcal{H})$ accounts for whether $(\mathcal{D}, \mathcal{H})$ can be “lifted” to some g_d^r represented by (D, H) .

Definition 3.8 A pre-limit g_d^r (respectively, a limit g_d^r) represented by $(\mathcal{D}, \mathcal{H})$ on a saturated metrized complex \mathfrak{C} is said to be *smoothable* if there exists a smooth proper curve X over \mathbb{K} and a skeleton Σ of the Berkovich analytification X^{an} of X such that \mathfrak{C} is isomorphic to the saturated metrized complex associated with Σ and there exists a g_d^r on X that is represented by (D, H) such that the associated pre-limit g_d^r (respectively, a limit g_d^r) on \mathfrak{C} is represented by $(\mathcal{D}, \mathcal{H})$.

Remark 3.9 Since \mathbb{K} is a large field with value group \mathbb{R} , we have no restrictions on the edge lengths of the underlying metric graph of \mathfrak{C} , and the above definition of smoothability is in the most general form.

Remark 3.10 Theorem 3.7 actually tells us that we do not need to distinguish the notion of a smoothable pre-limit linear series and the notion of a smoothable limit linear series. The extra restriction on limit linear series over pre-limit linear series is guaranteed by smoothability.

3.3 Smoothable Pre-Limit g_d^1 and Harmonic Morphisms

Let K be any algebraically closed field. Let H be a two-dimensional linear space of rational functions on a smooth proper curve X over K . Assume constant functions are contained in H . Then all nonconstant rational functions in H have the same poles and same order on the poles. We say the degree of H is the degree of any nonconstant

function in H . Therefore, H defines a morphism $\phi: X \rightarrow \mathbb{P}^1_K$ of the same degree, where \mathbb{P}^1_K has a marked point ∞ such that all nonconstant rational functions f in H factor through ϕ via a degree one rational function on \mathbb{P}^1_K such that all poles of f map to ∞ in \mathbb{P}^1_K . Conversely, given a morphism $\phi: X \rightarrow \mathbb{P}^1_K$, where \mathbb{P}^1_K has a marked point ∞ , we know that the linear space $\mathcal{L}((\infty))$ associated with the divisor (∞) is a degree one two-dimensional linear space of rational functions on \mathbb{P}^1_K that pulls back to a two-dimensional linear space H of rational functions on X by ϕ and has the same degree as ϕ . Note that $\mathcal{L}((\infty))$ contains constant functions on \mathbb{P}^1_K and H contains constant functions on X .

Suppose $\mathcal{C}(T)$ is a genus zero-saturated metrized complex over κ with underlying metric tree T . Then any two distinct effective divisors on $\mathcal{C}(T)$ differ by a principal divisor associated with a degree one rational function on $\mathcal{C}(T)$. Moreover, by embedding T isometrically into the analytification of $\mathbb{P}^1_{\mathbb{K}}$ and using the lifting theorem (Theorem 3.4), we may consider T as a skeleton of $\mathbb{P}^1_{\mathbb{K}}$ whose associated saturated metrized complex is naturally isomorphic to $\mathcal{C}(T)$. In addition, since $\mathcal{C}(T)$ is of genus zero, all divisors and rational functions on $\mathcal{C}(T)$ are liftable to $\mathbb{P}^1_{\mathbb{K}}$ with linear equivalence respected.

Now let \mathcal{C} be a saturated metrized complex with underlying metric graph Γ .

Lemma 3.11 *A smoothable pre-limit g_d^r on \mathcal{C} is refined limit g_d^r .*

Proof A smoothable pre-limit g_d^r is a limit g_d^r by Theorem 3.7. We show that a smoothable limit g_d^r satisfies the two properties of refined limit g_d^r . For the first property of refined limit g_d^1 , we note that the constant function is contained in H where (D, H) is any smoothing of the limit g_d^r , and it follows from the Poincaré–Lelong Formula (Theorem A.5). Using the characterization of the image of the specialization map obtained in Lemma A.3, we deduce that a smoothable limit g_d^r satisfies the second property of a refined limit g_d^r . ■

Remark 3.12 For a smoothable pre-limit g_d^1 represented by $(\mathcal{D}, \mathcal{H})$ on \mathcal{C} , we must have \mathcal{D} and \mathcal{H} compatible with each other.

Theorem 3.13 *A pre-limit g_d^1 represented by $(\mathcal{D}, \mathcal{H})$ on \mathcal{C} is smoothable if and only if there exists a modification \mathcal{C}^{mod} of \mathcal{C} and a harmonic morphism*

$$\mathcal{C}\phi^{\text{mod}} = (\phi_{\Gamma^{\text{mod}}}, \{\phi_p\}_{p \in \Gamma^{\text{mod}}})$$

from \mathcal{C}^{mod} to a genus zero saturated metrized complex $\mathcal{C}(T)$ such that

- (i) \mathcal{D} is the retract onto \mathcal{C} of the pullback divisor on \mathcal{C}^{mod} by $\mathcal{C}\phi^{\text{mod}}$ over a degree one effective divisor (u') on $\mathcal{C}(T)$,
- (ii) for each $p \in \Gamma$, if g_p is the C_p -part of the pullback function of a rational function on $\mathcal{C}(T)$ of degree ≤ 1 with only one possible pole at u' , then $g_p \in H_p$.

Proof First, by Lemma 3.11 and Remark 3.12, we can assume that $(\mathcal{D}, \mathcal{H})$ is refined, and let us first assume $(\mathcal{D}, \mathcal{H})$ is base-point free.

If $(\mathcal{D}, \mathcal{H})$ is smoothable, there exists a smooth proper curve X over \mathbb{K} and a skeleton $\Sigma(X^{\text{an}}, V)$ of X such that the saturated metrized complex associated with

(X^{an}, V) is isomorphic to \mathcal{C} (we identify them in the following for simplicity of discussion and thus $\Gamma = \Sigma(X^{\text{an}}, V)$). Let (D, H) represent a g_d^1 on X corresponding to $(\mathcal{D}, \mathcal{H})$ that is also base-point free.

Consider a map $\phi: X \rightarrow \mathbb{P}_{\mathbb{K}}^1$ defined by H , where $\mathbb{P}_{\mathbb{K}}^1$ is marked with a point $\infty_{\mathbb{K}}$ and D is the pullback divisor of ϕ over the point $\infty_{\mathbb{K}}$ of $\mathbb{P}_{\mathbb{K}}^1$. Then H is the pullback of the two-dimensional linear space $\mathcal{L}((\infty_{\mathbb{K}}))$ associated with $(\infty_{\mathbb{K}}) \in \text{Div}(\mathbb{P}_{\mathbb{K}}^1)$ by ϕ . The analytification functor induces a map $\phi^{\text{an}}: X^{\text{an}} \rightarrow \mathbb{P}_{\text{Berk}}^1$, where $\mathbb{P}_{\text{Berk}}^1$ is the Berkovich analytification of $\mathbb{P}_{\mathbb{K}}^1$. We restrict the map ϕ^{an} to the skeleton Γ to obtain a surjective map $\phi_{\Gamma}: \Gamma \rightarrow T$, where T is a skeleton of $\mathbb{P}_{\text{Berk}}^1$. Let Γ^{mod} be $(\phi^{\text{an}})^{-1}(T)$, which is a skeleton of X such that $\Gamma^{\text{mod}} \supseteq \Gamma$. Restricting the map ϕ^{an} to the skeleton Γ^{mod} , we get a map $\phi_{\Gamma^{\text{mod}}}: \Gamma^{\text{mod}} \rightarrow T$. Let \mathcal{C}^{mod} be a saturated metrized complex associated with X with skeleton Γ^{mod} . Let $\mathcal{C}(T)$ be a saturated metrized complex associated to $\mathbb{P}_{\text{Berk}}^1$ with skeleton T .

The reduction to \mathcal{C}^{mod} of H and the reduction to $\mathcal{C}(T)$ of $\mathcal{L}((\infty_{\mathbb{K}}))$ define maps $\phi_p: C_p \rightarrow C'_{\phi_{\Gamma^{\text{mod}}}(p)}$ for all $p \in \Gamma^{\text{mod}}$ that respect ϕ . Then the data $(\phi_{\Gamma}, \{\phi_p\}_{p \in \Gamma})$ satisfies the compatibility conditions of a pseudo-harmonic morphism (Definition 3.1), and the data $(\phi_{\Gamma^{\text{mod}}}, \{\phi_p\}_{p \in \Gamma^{\text{mod}}})$ satisfies the conditions of a harmonic morphism (Definition 3.2). Furthermore, the specialization of D to \mathcal{C}^{mod} is the pullback divisor \mathcal{D}^{mod} by $(\phi_{\Gamma^{\text{mod}}}, \{\phi_p\}_{p \in \Gamma^{\text{mod}}})$ over a degree one effective divisor, denoted by (u') , that is the specialization of the divisor $(\infty_{\mathbb{K}}) \in \text{Div}(\mathbb{P}_{\mathbb{K}}^1)$ to $\mathcal{C}(T)$. In addition, \mathcal{D} is the specialization of D to \mathcal{C} that is also the retract of \mathcal{D}^{mod} to \mathcal{C} . On the other hand, since H is the pullback of $\mathcal{L}((\infty_{\mathbb{K}}))$ by ϕ and the space of rational functions f' on $\mathcal{C}(T)$ of degree ≤ 1 with the only possible pole at u' is exactly the reduction to $\mathcal{C}(T)$ of $\mathcal{L}((\infty_{\mathbb{K}}))$, the C_p -part of the pullback function of f' must be an element of H_p , the reduction of H at C_p .

Conversely, suppose that there is a harmonic morphism $\mathcal{C}\phi^{\text{mod}} = (\phi_{\Gamma}, \{\phi_p\}_{p \in \Gamma^{\text{mod}}})$ between saturated metrized complexes \mathcal{C}^{mod} that is a modification of \mathcal{C} and a genus zero metrized complex $\mathcal{C}(T)$. Let \mathcal{D} be the retract onto \mathcal{C} of the pullback divisor \mathcal{D}^{mod} over a point $u' \in C'_{p'}$ in $\mathcal{C}(T)$ by $\mathcal{C}\phi^{\text{mod}}$. For each $p' \in T$, restricted to the $C'_{p'}$ -parts, the rational functions f' on $\mathcal{C}(T)$ of degree ≤ 1 with the only possible pole at u' make up a two-dimensional linear space $H'_{p'}$ of rational functions on $C'_{p'}$. Pulling back $H'_{p'}$ for all $p' \in T$ by $\mathcal{C}\phi^{\text{mod}}$, we obtain two-dimensional linear spaces H_p for all $p \in \Gamma$. In this way, we get the data $(\mathcal{D}, \mathcal{H})$ where $\mathcal{H} = \{H_p\}$.

By the lifting theorem (Theorem 3.4), we can lift $\mathcal{C}\phi^{\text{mod}}: \mathcal{C}^{\text{mod}} \rightarrow \mathcal{C}(T)$ to a finite morphism $\phi: X \rightarrow \mathbb{P}_{\mathbb{K}}^1$ of \mathbb{K} -curves. Mark a point in $\mathbb{P}_{\mathbb{K}}^1$ by $\infty_{\mathbb{K}}$ whose reduction to $\mathcal{C}(T)$ is u' . Let D be the pullback divisor of ϕ over the point $\infty_{\mathbb{K}}$ and H be the pullback of the two-dimensional linear space $\mathcal{L}((\infty_{\mathbb{K}}))$ associated with the divisor $(\infty_{\mathbb{K}})$ by ϕ . Then the lifting theorem also guarantees that $(\mathcal{D}, \mathcal{H})$ can be smoothed to (D, H) .

Suppose that $(\mathcal{D}, \mathcal{H})$ has base points u_1, \dots, u_m with orders $\alpha_1, \dots, \alpha_m$, respectively. Then $\mathcal{D}' = \mathcal{D} - \sum_{i=1}^m \alpha_i(u_i)$ is base-point free. Since $(\mathcal{D}, \mathcal{H})$ is smoothable if and only if $(\mathcal{D}', \mathcal{H})$ is smoothable, the smoothing criterion on $(\mathcal{D}', \mathcal{H})$ can be extended to the smoothing criterion on $(\mathcal{D}, \mathcal{H})$. ■

4 Bifurcation Trees and Partition Trees

In this section, we investigate in detail the definitions and properties of bifurcation trees and partition trees, which are notions employed in the smoothing criterion. A solvable diagrammatic pre-limit g_d^1 has a solution ρ to its characteristic equation (unique up to addition by a constant function) from which we can canonically construct a rooted metric tree called the bifurcation tree \mathcal{B} and a projection $\pi_{\mathcal{B}}: \Gamma \rightarrow \mathcal{B}$ for ρ (§4.1). Partition trees are derived from the bifurcation tree \mathcal{B} by suitably gluing the branches of \mathcal{B} (§4.2). In particular, any metric tree T underlying the genus-0 metrized complex $\mathcal{C}(T)$ in the commutative diagram in Section 1 must be a partition tree. The treatment will be expanded in Sections 5 and 6 leading to the proof of the smoothing criterion.

4.1 Bifurcation Trees

Let ρ be a rational function on Γ with everywhere nonzero slopes and let $\widehat{\rho} := \rho - \min \rho$ be the normalized function of ρ with minimum value zero. For a real number c and $*$ $\in \{\geq, \leq, <, >, =\}$, the set S_{*c}^{ρ} is defined as $\{p \in \Gamma \mid \rho(p) * c\}$. Denote the set of connected components of S_{*c}^{ρ} by $\text{Comp}(S_{*c}^{\rho})$.

For each value $c \in \text{Im } \rho$, the connected components of $S_{\geq c}^{\rho}$ are called *closed superlevel components* at c , and the connected components of $S_{>c}^{\rho}$ are called *open superlevel components* at c .

Remark 4.1 Here we summarize some facts about closed and open superlevel components that are immediate from their definition.

- For $c \in \text{Im } \rho$, for any open superlevel component $\beta \in \text{Comp}(S_{>c}^{\rho})$, there exists a unique closed superlevel component $\alpha \in \text{Comp}(S_{\geq c}^{\rho})$ such that $\alpha \supseteq \beta$.
- For each $c, c' \in \text{Im } \rho$ such that $c' \leq c$ and $\alpha \in \text{Comp}(S_{\geq c}^{\rho})$, there exists a unique element $\alpha' \in \text{Comp}(S_{\geq c'}^{\rho})$ such that $\alpha' \supseteq \alpha$.
- $\text{Comp}(S_{\geq \min_{p \in \Gamma} \rho(p)}^{\rho})$ is a singleton whose element is the whole metric graph Γ .
- For $\alpha_1 \in \text{Comp}(S_{\geq c_1}^{\rho})$ and $\alpha_2 \in \text{Comp}(S_{\geq c_2}^{\rho})$, there exists a largest $c_3 \in \text{Im } \rho$ such that there exists $\alpha_3 \in \text{Comp}(S_{\geq c_3}^{\rho})$ with $\alpha_3 \supseteq \alpha_1 \cup \alpha_2$. In particular, $c_3 \leq \min(c_1, c_2)$ and α_3 is the unique smallest closed superlevel component containing $\alpha_1 \cup \alpha_2$.

We define the notion of bifurcation tree associated with ρ as follows (see also Example 4.8).

Definition 4.2 Consider a rational function ρ with everywhere nonzero slopes. The *bifurcation tree* \mathcal{B} with respect to ρ is a rooted metric tree constructed in the following way.

- By abuse of notation, we also use \mathcal{B} to represent the set of points of \mathcal{B} . We identify the set of points of \mathcal{B} with the set of all closed superlevel components of ρ by the bijection $\iota_{\mathcal{B}}: \mathcal{B} \rightarrow \coprod_{c \in \text{Im } \rho} \text{Comp}(S_{\geq c}^{\rho})$.
- We assign a metric structure $d_{\mathcal{B}}$ to \mathcal{B} . For $x_1, x_2 \in \mathcal{B}$, let $x_1 \vee x_2$ be the (unique) element in \mathcal{B} such that $\iota_{\mathcal{B}}(x_1 \vee x_2)$ is the smallest closed superlevel component that

contains $\iota_{\mathcal{B}}(x_1) \cup \iota_{\mathcal{B}}(x_2)$. Suppose $\iota_{\mathcal{B}}(x_1) \in \text{Comp}(S_{\geq c_1}^\rho)$, $\iota_{\mathcal{B}}(x_2) \in \text{Comp}(S_{\geq c_2}^\rho)$, and $\iota_{\mathcal{B}}(x_1 \vee x_2) \in \text{Comp}(S_{\geq c_3}^\rho)$. Then we let $d_{\mathcal{B}}(x_1, x_2) = c_1 + c_2 - 2c_3$.

- The root $r(\mathcal{B})$ of \mathcal{B} corresponds to the unique closed superlevel component at $\min_{p \in \Gamma} \rho(p)$, which is the whole metric graph Γ .

For $x \in \mathcal{B}$, if $\iota_{\mathcal{B}}(x) \in \text{Comp}(S_{\geq c}^\rho)$, where $c \in \text{Im } \rho$, we let $d_{\mathcal{B}}^\rho(x) = c$ and $d_{\mathcal{B}}^0(x) = c - \min_{p \in \Gamma} \rho(p)$. Note that $d_{\mathcal{B}}^\rho(r(\mathcal{B})) = \min_{p \in \Gamma} \rho(p)$, $d_{\mathcal{B}}^0(r(\mathcal{B})) = 0$, and $d_{\mathcal{B}}^0(x) = d_{\mathcal{B}}(r(\mathcal{B}), x)$. We now show that \mathcal{B} is well defined as a metric tree with an associated partial order.

Proposition 4.3 \mathcal{B} constructed in Definition 4.2 is a rooted metric tree.

Proof Here we give a proof following a general construction of parametrized rooted trees and rooted \mathbb{R} -trees as discussed in [7, Appendix B5]. We will show that \mathcal{B} can be constructed as a tree by gluing subsets of \mathcal{B} that are isometric to line segments and then $d_{\mathcal{B}}$ is a well-defined metric on \mathcal{B} .

We first note that a partial order can be associated with \mathcal{B} , i.e., for two points x and x' , we say $x \geq x'$ if $\iota_{\mathcal{B}}(x) \supseteq \iota_{\mathcal{B}}(x')$. Clearly $d_{\mathcal{B}}^\rho(x) \leq d_{\mathcal{B}}^\rho(x')$ if $x \geq x'$. By Remark 4.1, this partial order is well defined and it is easy to verify that \mathcal{B} is a join-semilattice under the join operation \vee where for any two elements x and x' in \mathcal{B} , $x \vee x'$ corresponds to the smallest closed superlevel component that contains $\iota_{\mathcal{B}}(x) \cup \iota_{\mathcal{B}}(x')$ as in Definition 4.2. Note that $x \geq x'$ whenever $x \vee x' = x$, and $r(\mathcal{B})$ is the unique maximal element. For $x_1, x_2 \in \mathcal{B}$, suppose that $x_1 \geq x_2$, which means $x_1 \vee x_2 = x_1$. Let $d_{\mathcal{B}}^\rho(x_1) = c_1$ and $d_{\mathcal{B}}^\rho(x_2) = c_2$. By definition, we have $d_{\mathcal{B}}(x_1, x_2) = c_1 + c_2 - 2c_1 = c_2 - c_1$. We claim that $X = \{x \in \mathcal{B} \mid x_1 \geq x \geq x_2\}$ is isometric to a closed segment of length $c_2 - c_1$. First, note that $d_{\mathcal{B}}^\rho(x) \in [c_1, c_2]$ for any $x \in X$. Now for each $c \in [c_1, c_2]$, there exists a unique $x \in \mathcal{B}$ such that $d_{\mathcal{B}}^\rho(x) = c$ and $x \geq x_2$, i.e., $\iota_{\mathcal{B}}(x)$ is the unique closed superlevel component in $\text{Comp}(S_{\geq c}^\rho)$ that contains $\iota_{\mathcal{B}}(x_2)$ (Remark 4.1). On the other hand, we must have $x_1 \geq x$ at the same time and thus $x \in X$. Therefore, by sending c to x , we can define a bijection $\phi: [c_1, c_2] \rightarrow X$. It remains to show that ϕ is an isometry. Actually, by an analogous argument as above, we see that for any $y_1, y_2 \in X$, we either have $y_1 \geq y_2$ or $y_2 \geq y_1$ and $d_{\mathcal{B}}(y_1, y_2) = d_2 - d_1$ if $y_1 \geq y_2$ and $d_{\mathcal{B}}^\rho(y_1) = d_1$ and $d_{\mathcal{B}}^\rho(y_2) = d_2$. It follows that ϕ is an isometry. Here, we write $[x_1, x_2]$ to represent X as a closed line segment, and let $(x_1, x_2) = [x_1, x_2] \setminus \{x_1\}$, $[x_1, x_2) = [x_1, x_2] \setminus \{x_2\}$, and $(x_1, x_2) = [x_1, x_2] \setminus \{x_1, x_2\}$.

Since ρ is a rational function with everywhere nonzero slopes, there are only finitely many points x_1, \dots, x_m in Γ at which ρ takes local maximum values. Let

$$X_i := \{x \in \mathcal{B} \mid x \geq x_i\} \quad \text{for } i = 1, \dots, m.$$

Then $X_i = [r(\mathcal{B}), x_i]$. Note that we must have $\mathcal{B} = \bigcup_{i=1}^m X_i$. Let us reconstruct \mathcal{B} by gluing X_i one by one. First let us glue X_1 and X_2 . Note that $x_1 \vee x_2 \geq x_1, x_2$, and thus $x_1 \vee x_2 \in X_1 \cap X_2$. This means that $X_1 \cap X_2 = [r(\mathcal{B}), x_1 \vee x_2]$ and $X_1 \cup X_2 = [r(\mathcal{B}), x_1 \vee x_2] \amalg (x_1 \vee x_2, x_1] \amalg (x_1 \vee x_2, x_2]$. Note that it follows that $X_1 \cup X_2$ is a (topological) tree. Let us do this construction in general. Suppose we have derived $X' = X_1 \cup \dots \cup X_i$ as a tree already. Consider $Y = \{x_{i+1} \vee x_j \mid j = 1, \dots, i\}$. Clearly Y is a subset of X_{i+1} , which means Y is totally ordered. Let y be the minimum element

of Y . Then $X' \cap X_{i+1} = [r(\mathcal{B}), y]$, and $X_1 \cup \dots \cup X_{i+1} = X' \cup X_{i+1} = X' \amalg (y, x_{i+1}]$, which is also a tree. Thus \mathcal{B} is a tree.

It remains to show that $d_{\mathcal{B}}$ is a metric on the whole tree \mathcal{B} . We verify from the definition that for each $x_1, x_2 \in \mathcal{B}$, $d_{\mathcal{B}}(x_1, x_2) = d_{\mathcal{B}}(x_1, x_1 \vee x_2) + d_{\mathcal{B}}(x_2, x_1 \vee x_2)$. For each $x_3 \in \mathcal{B}$, consider $x_1 \vee x_3$ and $x_2 \vee x_3$. Without loss of generality, we can assume: (1) $x_1 \vee x_3 > x_2 \vee x_3$ or (2) $x_1 \vee x_3 = x_2 \vee x_3$. For case (1), we have $x_1 \vee x_2 \vee x_3 = x_1 \vee x_2 = x_1 \vee x_3$, and thus

$$\begin{aligned} d_{\mathcal{B}}(x_1, x_2) &= d_{\mathcal{B}}(x_1, x_1 \vee x_2 \vee x_3) + d_{\mathcal{B}}(x_1 \vee x_2 \vee x_3, x_2 \vee x_3) + d_{\mathcal{B}}(x_2 \vee x_3, x_2) \\ &\leq (d_{\mathcal{B}}(x_1, x_1 \vee x_2 \vee x_3) + d_{\mathcal{B}}(x_1 \vee x_2 \vee x_3, x_2 \vee x_3) + d_{\mathcal{B}}(x_2 \vee x_3, x_3)) \\ &\quad + (d_{\mathcal{B}}(x_2 \vee x_3, x_2) + d_{\mathcal{B}}(x_2 \vee x_3, x_3)) \\ &= d_{\mathcal{B}}(x_1, x_3) + d_{\mathcal{B}}(x_2, x_3), \end{aligned}$$

where equality holds if and only if $x_3 \geq x_2$. For case (2), we have $x_1 \vee x_2 \vee x_3 = x_1 \vee x_3 = x_2 \vee x_3 \geq x_1 \vee x_2$, and thus

$$\begin{aligned} d_{\mathcal{B}}(x_1, x_2) &= d_{\mathcal{B}}(x_1, x_1 \vee x_2) + d_{\mathcal{B}}(x_2, x_1 \vee x_2) \\ &\leq (d_{\mathcal{B}}(x_1, x_1 \vee x_2) + d_{\mathcal{B}}(x_1 \vee x_2, x_1 \vee x_2 \vee x_3) + d_{\mathcal{B}}(x_3, x_1 \vee x_2 \vee x_3)) \\ &\quad + (d_{\mathcal{B}}(x_2, x_1 \vee x_2) + d_{\mathcal{B}}(x_1 \vee x_2, x_1 \vee x_2 \vee x_3) + d_{\mathcal{B}}(x_3, x_1 \vee x_2 \vee x_3)) \\ &= d_{\mathcal{B}}(x_1, x_3) + d_{\mathcal{B}}(x_2, x_3), \end{aligned}$$

where equality holds if and only if $x_3 = x_1 \vee x_2$. Therefore, the triangle equality is satisfied and $d_{\mathcal{B}}$ is a metric. ■

Remark 4.4 In the above proof, note that the leaves of \mathcal{B} other than $r(\mathcal{B})$ are in one-to-one correspondence with those closed superlevel sets which are singletons, and in one-to-one correspondence with local maximum points of ρ (which we may also call sink points of ρ). Denote the set of leaves of \mathcal{B} by $\text{Leaf}(\mathcal{B})$. We call a point x of \mathcal{B} with $|\text{Tan}_{\mathcal{B}}^+(x)| \geq 2$ a *bifurcation point* of \mathcal{B} and denote the set of bifurcation points by $\text{Bif}(\mathcal{B})$. Then $(\mathcal{B}) \cap \text{Bif}(\mathcal{B}) = \emptyset$, and $\text{Leaf}(\mathcal{B}) \cup \text{Bif}(\mathcal{B})$ is the set of points of valence other than 2 in \mathcal{B} together with $r(\mathcal{B})$. Note that we have either $r(\mathcal{B}) \in \text{Leaf}(\mathcal{B})$ or $r(\mathcal{B}) \in \text{Bif}(\mathcal{B})$. We call the image of $d_{\mathcal{B}}^{\rho}$ restricted to the minimal vertex set of \mathcal{B} the set of *bifurcation values*, denoted by Bif_{ρ} . Then Bif_{ρ} is finite and we have $\text{Bif}_{\rho} \subseteq \mathcal{E}_{\rho}$, where \mathcal{E}_{ρ} is the set of exceptional points of ρ .

For a point p in Γ , recall that $\text{Tan}_{\Gamma}^{\rho+}(p)$ is the set of tangent directions in $\text{Tan}_{\Gamma}(p)$ emanating from p where ρ locally increases. Similarly, we let $\text{Tan}_{\Gamma}^{\rho-}(p)$ be the set of tangent directions where ρ locally decreases. Then $\text{Tan}_{\Gamma}(p) = \text{Tan}_{\Gamma}^{\rho+}(p) \amalg \text{Tan}_{\Gamma}^{\rho-}(p)$.

Let \mathcal{T} be a metric tree rooted at $r(\mathcal{T})$. For a point x in \mathcal{T} , we say a tangent direction $t \in \text{Tan}_{\mathcal{T}}(x)$ is a *forward* (respectively, *backward*) tangent direction at x if the distance function from the root increases (respectively, decreases) along t . Denote by $\text{Tan}_{\mathcal{T}}^+(x)$ (respectively, $\text{Tan}_{\mathcal{T}}^-(x)$) the set of forward (respectively, backward) tangent directions at x . Note that $\text{Tan}_{\mathcal{T}}^-(x)$ is empty if x is the root of \mathcal{T} and a singleton otherwise.

We state without proofs of Lemma 4.5 and Lemma 4.6, which follow naturally from the construction of the bifurcation tree \mathcal{B} with respect to ρ . Lemma 4.5 states that ρ factors through $d_{\mathcal{B}}^{\rho}$ by the canonical projection $\pi_{\mathcal{B}}: \Gamma \rightarrow \mathcal{B}$. Lemma 4.6 states that

the set of forward tangent directions on \mathcal{B} can be identified with the set of all open superlevel components of ρ .

Lemma 4.5 For $p \in \Gamma$, there is a unique closed superlevel component α at $\rho(p)$ that contains p . Sending p to $\iota_{\mathcal{B}}^{-1}(\alpha)$, induces a canonical projection $\pi_{\mathcal{B}}: \Gamma \rightarrow \mathcal{B}$. Moreover, the map $\pi_{\mathcal{B}}$ is continuous, piecewise-linear, surjective, and satisfies $\rho = d_{\mathcal{B}}^{\rho} \circ \pi_{\mathcal{B}}$.

Lemma 4.6 There is a canonical bijection

$$\tilde{t}_{\mathcal{B}}: \coprod_{x \in \mathcal{B}} \text{Tan}_{\mathcal{B}}^{+}(x) \rightarrow \coprod_{c \in \text{Im } \rho} \text{Comp}(S_{>c}^{\rho}).$$

In particular, $\text{Tan}_{\mathcal{B}}^{+}(x)$ is in bijection with $\{\beta \in \text{Comp}(S_{>d_{\mathcal{B}}^{\rho}}^{\rho}(x)) \mid \beta \subseteq \iota_{\mathcal{B}}(x)\}$.

Remark 4.7 The projection $\pi_{\mathcal{B}}$ naturally induces a pushforward map

$$\pi_{\mathcal{B}*}: \coprod_{p \in \Gamma} \text{Tan}_{\Gamma}(p) \rightarrow \coprod_{x \in \mathcal{B}} \text{Tan}_{\mathcal{B}}(x).$$

In particular, if $t \in \text{Tan}_{\Gamma}^{\rho^{-}}(p)$, then $\pi_{\mathcal{B}*}(t)$ is the unique element in $\text{Tan}_{\mathcal{B}}^{-}(\pi(p))$; if $t \in \text{Tan}_{\Gamma}^{\rho^{+}}(p)$, then $\pi_{\mathcal{B}*}(t) \in \text{Tan}_{\mathcal{B}}^{+}(\pi_{\mathcal{B}}(p))$ and more precisely $\tilde{t}_{\mathcal{B}}(\pi_{\mathcal{B}*}(t))$ is the unique open superlevel component of ρ with p on its boundary and t pointing inwards. Note that $\pi_{\mathcal{B}*}$ is surjective.

Example 4.8 In Figure 2, suppose $\{o_1, o_2, p_1, p_2, p_3, q_1, q_2, q_3\}$ is a vertex set of Γ (upper panel) and all edges have length 1. Then a global diagram on Γ with all multiplicities being 1 along directions marked by the arrows is solvable and we suppose a solution is ρ with the corresponding bifurcation tree \mathcal{B} and the canonical projection $\pi_{\mathcal{B}}$. In particular, as shown by the vertical dashed lines, the point x is the root of the bifurcation tree corresponding to the unique closed superlevel set at $\min_{p \in \Gamma}(\rho)$ (the whole metric graph Γ), which is also the image of o_1 and o_2 under $\pi_{\mathcal{B}}$; $y_1 = \pi_{\mathcal{B}}(p_1)$, corresponds to the closed superlevel component $\{p_1\}$; $y_2 = \pi_{\mathcal{B}}(p_2) = \pi_{\mathcal{B}}(p_3)$ corresponds to the closed superlevel component that is the union of all closed edges connecting p_2, p_3, q_1, q_2 , and q_3 ; for $i = 1, 2, 3, z_i = \pi_{\mathcal{B}}(q_i)$ corresponds to closed superlevel component $\{q_i\}$. Note that $z_1 \vee z_2 = z_1 \vee z_3 = z_2 \vee z_3 = y_2$ and $y_1 \vee z_1 = y_1 \vee z_2 = y_1 \vee z_3 = x$. Then for $i, j = 1, 2, 3$ and $i \neq j$, we have $d_{\mathcal{B}}(z_i, z_j) = d_{\mathcal{B}}(z_i, y_2) + d_{\mathcal{B}}(z_j, y_2) = 1 + 1 = 2$ and $d_{\mathcal{B}}(z_i, y_1) = d_{\mathcal{B}}(z_i, x) + d_{\mathcal{B}}(y_1, x) = 2 + 1 = 3$. In addition, the tangent direction from y_2 to z_1 corresponds to the open superlevel component $(p_2, q_1] \cup (p_3, q_1]$, the tangent direction from y_2 to z_2 corresponds to the open superlevel component $(p_2, q_2] \cup (p_3, q_2]$, and the tangent direction from y_2 to z_3 corresponds to the open superlevel component $(p_2, q_3] \cup (p_3, q_3]$. Moreover, $\text{Leaf}(\mathcal{B}) = \{y_1, z_1, z_2, z_3\}$ and $\text{Bif}(\mathcal{B}) = \{x, y_2\}$.

4.2 Partition Trees

We now generalize the notion of bifurcation trees to objects called partition trees. Partition trees are the elements in the spaces $\Lambda_{\mathcal{D}, \mathcal{H}}^{(1)}, \Lambda_{\mathcal{D}, \mathcal{H}}^{(2)}, \Lambda_{\mathcal{D}, \mathcal{H}}^{(3)}$, and $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$ treated in Section 5.

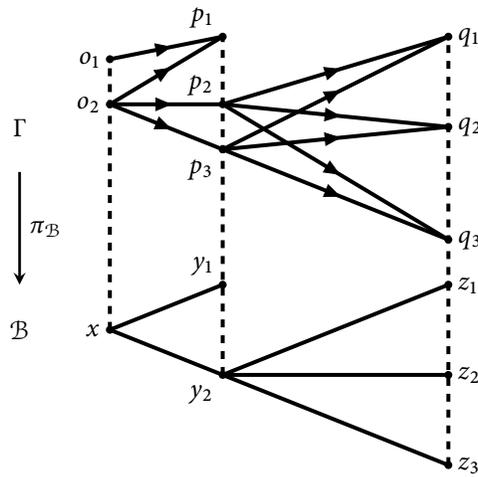


Figure 2: An illustration of a bifurcation tree \mathcal{B} of Γ and the canonical projection $\pi_{\mathcal{B}}$.

For each rooted metric tree \mathcal{T} with root $r(\mathcal{T})$, we can define a distance function $d_{\mathcal{T}}^0: \mathcal{T} \rightarrow \mathbb{R}$ that takes a point $x \in \mathcal{T}$ to the distance between $r(\mathcal{T})$ and x . Note that this is consistent with the definition of $d_{\mathcal{B}}^0$, where \mathcal{B} is some bifurcation tree \mathcal{B} with respect to ρ introduced in the previous subsection.

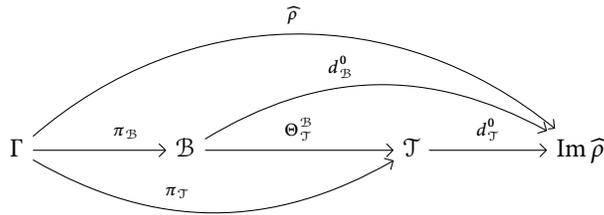
Definition 4.9 Let $\rho: \Gamma \rightarrow \mathbb{R}$ be a rational function on Γ with everywhere nonzero slopes. We call $(\mathcal{T}, \pi_{\mathcal{T}})$, or simply \mathcal{T} , a *partition tree* with respect to ρ if \mathcal{T} is a rooted metric tree and $\pi_{\mathcal{T}}: \Gamma \rightarrow \mathcal{T}$ is a continuous finite surjection (finite means all fibers are finite) such that $\widehat{\rho} = d_{\mathcal{T}}^0 \circ \pi_{\mathcal{T}}$, where $\widehat{\rho} = \rho - \min_{p \in \Gamma} \rho(p)$. We let Λ_{ρ} be the set of all partition trees with respect to ρ .

Example 4.10 Let \mathcal{B} be the bifurcation tree with respect to ρ and $\pi_{\mathcal{B}}$ the canonical projection from Γ onto \mathcal{B} . Then $(\mathcal{B}, \pi_{\mathcal{B}}) \in \Lambda_{\rho}$, following from Lemma 4.5 directly.

Example 4.11 The segment $\text{Im } \rho$ can be considered as a metric graph with root $\min_{p \in \Gamma} \rho(p)$. Clearly $(\text{Im } \rho, \rho) \in \Lambda_{\rho}$.

The following proposition says that all partition trees can be constructed by gluing the bifurcation tree properly.

Proposition 4.12 Let $\rho: \Gamma \rightarrow \mathbb{R}$ be a rational function on a metric graph Γ with everywhere nonzero slopes and let \mathcal{B} be the bifurcation tree with respect to ρ . Let $\widehat{\rho} = \rho - \min_{p \in \Gamma} \rho(p)$. Let \mathcal{T} be a rooted metric tree and let $\pi_{\mathcal{T}}: \Gamma \rightarrow \mathcal{T}$ be a continuous finite surjection. Then $(\mathcal{T}, \pi_{\mathcal{T}}) \in \Lambda_{\rho}$ if and only if there exists a continuous surjection $\Theta_{\mathcal{T}}^{\mathcal{B}}: \mathcal{B} \rightarrow \mathcal{T}$ such that the following diagram commutes.



Proof If the diagram commutes, then $(\mathcal{T}, \pi_{\mathcal{T}}) \in \Lambda_{\rho}$ by the definition of partition trees. Now suppose $(\mathcal{T}, \pi_{\mathcal{T}}) \in \Lambda_{\rho}$. Then $\widehat{\rho} = d_{\mathcal{T}}^0 \circ \pi_{\mathcal{T}}$ and we just need to find the canonical projection $\Theta_{\mathcal{T}}^{\mathcal{B}}$ from \mathcal{B} to \mathcal{T} such that $\pi_{\mathcal{T}} = \Theta_{\mathcal{T}}^{\mathcal{B}} \circ \pi_{\mathcal{B}}$ and $d_{\mathcal{B}}^0 = d_{\mathcal{T}}^0 \circ \Theta_{\mathcal{T}}^{\mathcal{B}}$.

For a point $x \in \mathcal{B}$, let $c = d_{\mathcal{B}}^0(x)$. Note that $\iota_{\mathcal{B}}(x) \in \text{Comp}(S_{\geq c}^{\widehat{\rho}})$ and by definition of $\pi_{\mathcal{B}}$, we have $\pi_{\mathcal{B}}^{-1}(x) = \partial\iota_{\mathcal{B}}(x)$, where $\partial\iota_{\mathcal{B}}(x)$ is the set of boundary points of $\iota_{\mathcal{B}}(x)$. We claim that for any two points $p, q \in \partial\iota_{\mathcal{B}}(x)$, we have $\pi_{\mathcal{T}}(p) = \pi_{\mathcal{T}}(q)$. Note that $d_{\mathcal{T}}^0(\pi_{\mathcal{T}}(p)) = d_{\mathcal{T}}^0(\pi_{\mathcal{T}}(q)) = \widehat{\rho}(p) = \widehat{\rho}(q) = c$, since $\widehat{\rho} = d_{\mathcal{B}}^0 \circ \pi_{\mathcal{B}} = d_{\mathcal{T}}^0 \circ \pi_{\mathcal{T}}$. Let $Y = \{y \in \mathcal{T} \mid d_{\mathcal{T}}^0(y) \geq c\}$ and $\partial Y = \{y \in \mathcal{T} \mid d_{\mathcal{T}}^0(y) = c\}$. Note that each connected component of Y has exactly one boundary point in ∂Y , since \mathcal{T} is a metric tree. Clearly $\pi_{\mathcal{T}}(p), \pi_{\mathcal{T}}(q) \in \partial Y$ and $\pi_{\mathcal{T}}(\iota_{\mathcal{B}}(x)) \subseteq Y$. Since $\iota_{\mathcal{B}}(x)$ is connected and $\pi_{\mathcal{T}}$ is continuous, $\pi_{\mathcal{T}}(\iota_{\mathcal{B}}(x))$ must be connected, which is only possible when $\pi_{\mathcal{T}}(\iota_{\mathcal{B}}(x))$ is contained in one connected component of Y . This implies $\pi_{\mathcal{T}}(p) = \pi_{\mathcal{T}}(q)$.

Define $\Theta_{\mathcal{T}}^{\mathcal{B}}(x)$ to be this point $\pi_{\mathcal{T}}(p) = \pi_{\mathcal{T}}(q) \in \mathcal{T}$. The above argument shows that $\Theta_{\mathcal{T}}^{\mathcal{B}}$ is well defined. The continuity of $\Theta_{\mathcal{T}}^{\mathcal{B}}$ and commutativity of the diagram also naturally follow. ■

Remark 4.13 Consider the bifurcation tree \mathcal{B} and a partition tree \mathcal{T} with respect to ρ . The canonical projection $\Theta_{\mathcal{T}}^{\mathcal{B}}$ induces a partition P_c of $(d_{\mathcal{B}}^{\rho})^{-1}(c)$ for any $c \in \text{Im } \rho$ as follows: $x_1 \sim x_2$ in P_c if and only if $\Theta_{\mathcal{T}}^{\mathcal{B}}(x_1) = \Theta_{\mathcal{T}}^{\mathcal{B}}(x_2)$. (See Appendix B for further discussions.)

Remark 4.14 Like the pushforward $\pi_{\mathcal{B}*}$ induced by the canonical projection $\pi_{\mathcal{B}}$, we also have the pushforward map $\pi_{\mathcal{T}*}: \coprod_{p \in \Gamma} \text{Tan}_{\Gamma}(p) \rightarrow \coprod_{x \in \mathcal{T}} \text{Tan}_{\mathcal{T}}(x)$ such that (i) if $t \in \text{Tan}_{\Gamma}^{\rho-}(p)$, then $\pi_{\mathcal{T}*}(t)$ is the unique element in $\text{Tan}_{\mathcal{T}}^{\rho-}(\pi_{\mathcal{T}}(p))$; (ii) if $t \in \text{Tan}_{\Gamma}^{\rho+}(p)$, then $\pi_{\mathcal{T}*}(t) \in \text{Tan}_{\mathcal{T}}^{\rho+}(\pi_{\mathcal{T}}(p))$.

Example 4.15 In Figure 3, we show an example of partition tree \mathcal{T} based on the bifurcation tree \mathcal{B} constructed in Example 4.8. In particular, \mathcal{T} is constructed by gluing edges y_2z_1 and y_2z_2 of \mathcal{B} (the grey edges) isometrically into edge $y'_2y'_{12}$ of \mathcal{T} . As a result, $\pi_{\mathcal{T}}$ maps all the edges of Γ connecting p_2 and p_3 with q_1 and q_2 to the edge $y'_2y'_{12}$.

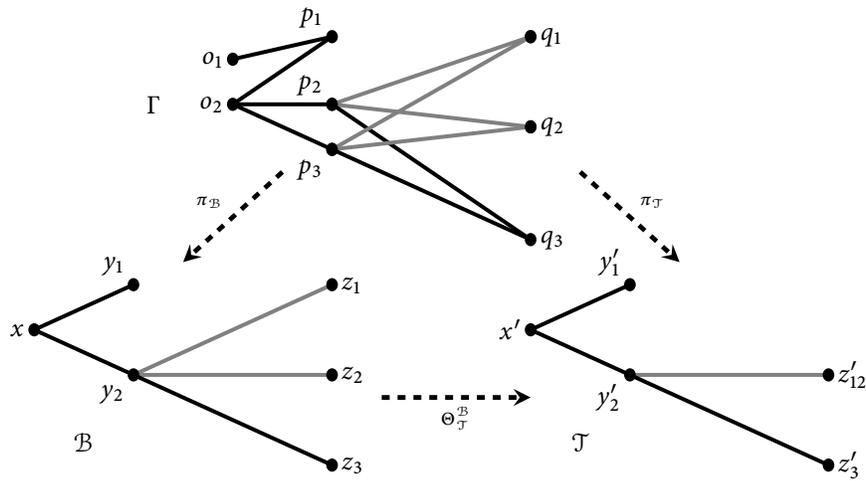


Figure 3: An illustration of a projection $\pi_{\mathcal{T}}$ from Γ to a partition tree \mathcal{T} .

5 Obstructions of Smoothability and Spaces $\Lambda_{\mathcal{D}, \mathcal{H}}^{(1)}$, $\Lambda_{\mathcal{D}, \mathcal{H}}^{(2)}$, $\Lambda_{\mathcal{D}, \mathcal{H}}^{(3)}$ and $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$ of Partition Trees

5.1 An Example of a Non-Solvable Limit g_d^1

In this subsection, we will show that the additional restriction on limit g_d^1 over pre-limit g_d^1 does not guarantee solvability by presenting an example of non-solvable limit g_d^1 .

Consider the global diagram on a cycle as shown in Figure 4 with multiplicity 1 on each edge. The metric graph Γ has the lengths: $\ell_{v_2, v_3} = \ell_{v_2, v_8}$, $\ell_{v_4, v_3} = \ell_{v_4, v_5}$, $\ell_{v_1, v_5} = \ell_{v_1, v_7}$, and $\ell_{v_1, v_6} = \ell_{v_1, v_8}$. At every point $p \in \Gamma$ the algebraic curve C_p is a projective line over κ . Let the two outgoing tangent directions $t_{1,1}^o$ and $t_{1,2}^o$ at v_1 be locally equivalent, let the two outgoing tangent directions $t_{3,1}^o$ and $t_{3,2}^o$ at v_3 be in different local equivalence classes, let the two incoming tangent directions $t_{2,1}^i$ and $t_{2,2}^i$ at v_2 be locally equivalent, and let the two incoming tangent directions $t_{4,1}^i$ and $t_{4,2}^i$ at v_4 be locally equivalent. For all $p \in \Gamma \setminus \{v_1, v_2, v_3, v_4\}$, there is only one incoming tangent direction t_p^i and one outgoing tangent direction t_p^o at p , while t_p^i and t_p^o are in different local equivalence classes. We verify that this global diagram is not solvable.

On the other hand, from this global diagram, we can construct a diagrammatic pre-limit g_d^1 represented by $(\mathcal{D}, \mathcal{H})$ in the following way. Let $D_{\Gamma} = 2(v_1) + (v_3)$. Let $D_{v_1} = (x_{1,1}) + (x_{1,2})$, where $x_{1,1}$ and $x_{1,2}$ are two distinct non-marked points in C_{v_1} , $D_{v_3} = (x_3)$, where x_3 is a non-marked point on C_{v_3} and $D_p = 0$ for all $p \in \Gamma \setminus \{v_1, v_3\}$. Using the approach shown in Remark 2.5, we can construct H_p conversely using the local diagrams at p induced from the global diagram. More precisely, let f_{v_1} be a rational function on C_{v_1} whose associated divisor is $(\text{red}_{v_1}(t_{1,1}^o)) + (\text{red}_{v_1}(t_{1,2}^o)) - (x_{1,1}) - (x_{1,2})$,

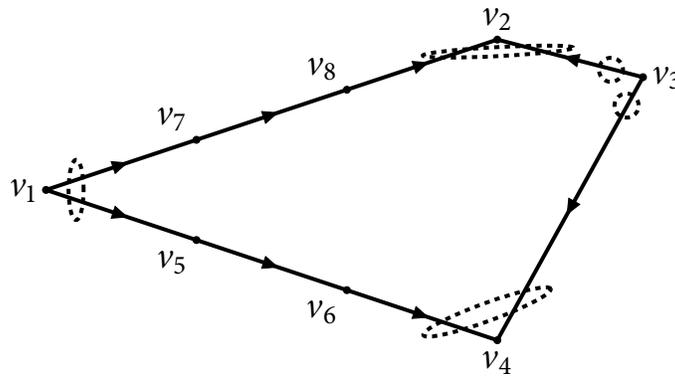


Figure 4: An example of a diagrammatic limit g_d^1 such that the characteristic equation associated with the global diagram does not have a solution.

let $f_{v_3}^{(1)}$ be a rational function on C_{v_3} whose associated divisor is $(\text{red}_{v_3}(t_{3,1}^o)) - (x_3)$, let $f_{v_3}^{(2)}$ be a rational function on C_{v_3} whose associated divisor is $(\text{red}_{v_3}(t_{3,2}^o)) - (x_3)$, let f_{v_2} be a rational function on C_{v_2} whose associated divisor is

$$(x_{2,1}) + (x_{2,2}) - (\text{red}_{v_2}(t_{2,1}^i)) - (\text{red}_{v_2}(t_{2,2}^i)),$$

where $x_{2,1}$ and $x_{2,2}$ are two non-marked points in C_{v_2} , and let f_{v_4} be a rational function on C_{v_4} whose associated divisor is $(x_{4,1}) + (x_{4,2}) - (\text{red}_{v_4}(t_{4,1}^i)) - (\text{red}_{v_4}(t_{4,2}^i))$, where $x_{4,1}$ and $x_{4,2}$ are two non-marked points in C_{v_4} . For all $p \in \Gamma \setminus \{v_1, v_2, v_3, v_4\}$, let f_p be a rational function on C_p whose associated divisor is $(\text{red}_p(t_p^o)) - (\text{red}_p(t_p^i))$. Then we let H_p be a linear space of rational functions on C_p with a basis $\{1, f_p\}$ for all $p \in \Gamma$ (for $p = v_3$, we choose f_{v_3} to be either $f_{v_3}^{(1)}$ or $f_{v_3}^{(2)}$, noting that the rational functions $1, f_{v_3}^{(1)}$, and $f_{v_3}^{(2)}$ are linear dependent).

We claim that $(\mathcal{D}, \mathcal{H})$ constructed this way represents a limit g_d^1 . To this end, we must show that for every effective divisor $\mathcal{E} = (u, z_u)$ of degree one on the saturated metrized complex where $u \in \Gamma$ and $z_u \in C_u$, there exists a rational function $g = (g_\Gamma, \{g_p\}_{p \in \Gamma})$ such that $g_p \in H_p$ and $\mathcal{D} + \text{div}(g) - \mathcal{E} \geq 0$.

We first specify g_Γ . We describe g_Γ in terms of a series of chip-firing moves: (1) if u lies in $[v_1, v_8]$ or $[v_1, v_6]$, we can fire both chips from v_1 until one of the chips hits u ; (2) if u lies in $[v_2, v_8]$ or $[v_2, v_3]$, we first fire both chips from v_1 until one of the chips hits v_8 and then fire the chip at v_3 and v_8 simultaneously until one of the chips hits u ; (3) if u lies in $[v_4, v_5]$ or $[v_4, v_3]$, we fire first both chips from v_1 until a chip hits v_5 and then fire the chips at v_3 and v_5 simultaneously until one of the chips hits u .

Now let us specify g_p . (1) First, let $g_u = f_u - f_u(z_u)$. Then g_u has the same poles as f_u and has a zero at z_u . (2) For all the points $p \in \Gamma$ such that g_Γ is locally constant, we let g_p be a constant. (3) If the slope of g_Γ along $t_{3,1}^o$ is nonzero, let $g_{v_3} = f_{v_3}^{(1)}$. If the slope of g_Γ along $t_{3,2}^o$ is nonzero, let $g_{v_3} = f_{v_3}^{(2)}$. (4) For all remaining points $p \in \Gamma$, we let $g_p = f_p$.

For this choice of \mathfrak{g} , we have $\mathcal{D} - \mathcal{E} + \text{div}(\mathfrak{g}) \geq 0$. Therefore, $(\mathcal{D}, \mathcal{H})$ represents a non-solvable limit g_d^1 and we have the following proposition.

Proposition 5.1 *There exists a non-smoothable diagrammatic limit g_d^1 . In particular, there exists a diagrammatic limit g_d^1 that is not solvable.*

5.2 Four Levels of Obstructions to Pre-limit g_d^1 Being Smoothable

The following two subsections are a preparation for the proof of the smoothing criterion in Section 6.

We say a diagrammatic pre-limit g_d^1 represented by $(\mathcal{D}, \mathcal{H})$ satisfies a Level-I restriction if it is solvable, and in the following we will introduce Level-II, Level-III, and Level-IV restrictions that form additional obstructions of $(\mathcal{D}, \mathcal{H})$ to being smoothable.

Now assume that $(\mathcal{D}, \mathcal{H})$ is solvable with a solution ρ and corresponding bifurcation tree \mathcal{B} . Recall that $(\mathcal{D}, \mathcal{H})$ satisfies the intrinsic global compatibility conditions if and only if \mathcal{H} contains an admissible collection $\{g_p\}_{p \in \Gamma}$ of non-constant rational functions $g_p \in H_p$.

Definition 5.2 *A bifurcation partition system $\{\tilde{P}_x\}_{x \in \mathcal{B}}$ on the bifurcation tree \mathcal{B} is a collection of partitions \tilde{P}_x of $\text{Tan}_{\mathcal{B}}^+(x)$ for all points $x \in \mathcal{B}$.*

Note that there are only finitely many possible bifurcation partition systems on \mathcal{B} , since $\text{Tan}_{\mathcal{B}}^+(x)$ is a singleton for all but finitely many points $x \in \mathcal{B}$, and for the exceptions, \tilde{P}_x is a partition of a finite set.

Remark 5.3 Suppose \mathcal{H} contains an admissible collection $G = \{g_p\}_{p \in \Gamma}$ of non-constant rational functions $g_p \in H_p$ (Definition 2.11). This means that for each $x \in \mathcal{B}$, if $\text{Tan}_{\mathcal{B}}^+(x) = \{t_1, \dots, t_n\}$, then we can assign values $c_1, \dots, c_n \in \kappa$ to t_1, \dots, t_n , respectively, such that $G \circ \text{Red}(t) = c_i$ for $i = 1, \dots, n$ and each $t \in \pi_{\mathcal{B}^*}^{-1}(t_i)$. Canonically, we can associate a bifurcation partition system $\{\tilde{P}_x\}_{x \in \mathcal{B}}$ with G by letting t_i and t_j be equivalent in \tilde{P}_x if and only if $c_i = c_j$. Moreover, we say $\{\tilde{P}_x\}_{x \in \mathcal{B}}$ is *globally compatible* with \mathcal{H} (or $(\mathcal{D}, \mathcal{H})$ even if this compatibility does not depend on \mathcal{D}) if \mathcal{H} contains an admissible $G = \{g_p\}_{p \in \Gamma}$ such that $\{\tilde{P}_x\}_{x \in \mathcal{B}}$ is exactly the bifurcation partition system associated with G . (In this case, to be more specific, we sometimes say $\{\tilde{P}_x\}_{x \in \mathcal{B}}$ is globally compatible with \mathcal{H} via G .) In addition, the intrinsic global compatibility conditions on $(\mathcal{D}, \mathcal{H})$ can be restated equivalently as that there exists some bifurcation partition system globally compatible with $(\mathcal{D}, \mathcal{H})$.

We define Level-II, Level-III, and Level-IV restrictions as follows.

- (1) We say that a bifurcation partition system $\{\tilde{P}_x\}_{x \in \mathcal{B}}$ is *Level-II compatible* (respectively, *Level-III compatible*) with $(\mathcal{D}, \mathcal{H})$ if it satisfies the following property: for each point $p \in \Gamma$, the tangent directions t_1 and t_2 in $\text{Tan}_{\Gamma}^+(p)$ are equivalent in the local diagram at p (we also say t_1 and t_2 are locally equivalent), if (respectively, if and only if) $\pi_{\mathcal{B}^*}(t_1)$ and $\pi_{\mathcal{B}^*}(t_2)$ are equivalent in $\tilde{P}_{\pi_{\mathcal{B}}(p)}$.

(2) We also call a bifurcation partition system Level-IV compatible with $(\mathcal{D}, \mathcal{H})$ if it is globally compatible with $(\mathcal{D}, \mathcal{H})$.

(3) We say that $(\mathcal{D}, \mathcal{H})$ satisfies Level-II (respectively, Level-III and Level-IV) restrictions if there exists a bifurcation partition system which is Level-II (respectively, Level-III and Level-IV) compatible with $(\mathcal{D}, \mathcal{H})$. Note that by the smoothing criterion, Theorem 2.6 actually means that $(\mathcal{D}, \mathcal{H})$ is smoothable if and only if Level-IV compatibility is satisfied.

The motivation for introducing the two intermediate compatibility levels (Level-II and Level-III) will be addressed in more detail in our subsequent work. Note that Level-II and Level-III compatibilities can be determined purely combinatorially given the local diagrams induced by \mathcal{H} .

We denote the set of all bifurcation partition systems on \mathcal{B} by $\text{BP}_{\mathcal{D}, \mathcal{H}}^{(1)}$, and the sets of bifurcation partition systems Level-II, Level-III, and Level-IV compatible with $(\mathcal{D}, \mathcal{H})$ by $\text{BP}_{\mathcal{D}, \mathcal{H}}^{(2)}$, $\text{BP}_{\mathcal{D}, \mathcal{H}}^{(3)}$ and $\text{BP}_{\mathcal{D}, \mathcal{H}}^{(4)}$, respectively.

Lemma 5.4 Level-IV \Rightarrow Level-III \Rightarrow Level-II \Rightarrow Level-I and $\text{BP}_{\mathcal{D}, \mathcal{H}}^{(1)} \supseteq \text{BP}_{\mathcal{D}, \mathcal{H}}^{(2)} \supseteq \text{BP}_{\mathcal{D}, \mathcal{H}}^{(3)} \supseteq \text{BP}_{\mathcal{D}, \mathcal{H}}^{(4)}$.

Proof To see this, note that we only need to verify that Level-IV implies Level-III. Note that an equivalent way to say that a bifurcation partition system $\{\vec{P}_x\}_{x \in \mathcal{B}}$ is globally (Level-IV) compatible with \mathcal{H} via $\{g_p\}_{p \in \Gamma} \in H$ is as follows: for each pair of tangent directions $t_1 \in \text{Tan}_{\Gamma}^{p_1}$ and $t_2 \in \text{Tan}_{\Gamma}^{p_2}$ such that $\pi_{\mathcal{B}}(p_1) = \pi_{\mathcal{B}}(p_2)$, we have the equivalence of the statements (1) $g_{p_1}(\text{red}_{p_1}(t_1)) = g_{p_2}(\text{red}_{p_2}(t_2))$, and (2) $\pi_{\mathcal{B}*}(t_1)$ and $\pi_{\mathcal{B}*}(t_2)$ are in the same equivalence class in \vec{P}_x , where $x = \pi_{\mathcal{B}}(p_1) = \pi_{\mathcal{B}}(p_2)$. Then, by specializing to cases $p_1 = p_2$, we conclude that Level-IV implies Level-III. ■

Example 5.5 Consider a saturated metrized complex $\mathcal{C} = (\Gamma, \{C_p\}_{p \in \Gamma})$, where the underlying metric graph Γ is as shown in Figure 2 and all C_p are projective lines over \mathbb{C} . Suppose that $(\mathcal{D}, \mathcal{H})$ is a solvable diagrammatic limit g_d^1 with solution ρ , where the corresponding bifurcation tree \mathcal{B} and the canonical projection $\pi_{\mathcal{B}}: \Gamma \rightarrow \mathcal{B}$ are as in Figure 2 (see Example 4.8). For each $p \in \Gamma$, fix a basis $\{1, f_p\}$ of H_p where f_p is a non-constant rational function.

Here we apply Algorithm 2.13 to check the smoothability of $(\mathcal{D}, \mathcal{H})$, i.e., whether \mathcal{H} contains an admissible $G = \{g_p\}_{p \in \Gamma}$. Note that by the algorithm, we only need to check the finite set of exceptional points (see definition in §2.7), which in this case is the vertex set, i.e., $\mathcal{E}_{\rho} = \{o_1, o_2, p_1, p_2, p_3, q_1, q_2, q_3\}$. For adjacent $p, q \in \mathcal{E}_{\rho}$, we let t_{pq} represent the tangent direction in $\text{Tan}_{\Gamma}(p)$ emanating from p towards q and then $\text{red}_p(t_{pq})$ is the marked point on C_p associated with t_{pq} . Note that

$$\begin{aligned} x = \pi_{\mathcal{B}}(o_1) = \pi_{\mathcal{B}}(o_2), \quad y_1 = \pi_{\mathcal{B}}(p_1), \quad y_2 = \pi_{\mathcal{B}}(p_2) = \pi_{\mathcal{B}}(p_3), \\ z_1 = \pi_{\mathcal{B}}(q_1), \quad z_2 = \pi_{\mathcal{B}}(q_2), \quad z_3 = \pi_{\mathcal{B}}(q_3), \end{aligned}$$

and

$$\begin{aligned} t_{xy_1} &= \pi_{\mathcal{B}^*}(t_{o_1p_1}) = \pi_{\mathcal{B}^*}(t_{o_2p_1}), & t_{xy_2} &= \pi_{\mathcal{B}^*}(t_{o_2p_2}) = \pi_{\mathcal{B}^*}(t_{o_2p_3}), \\ t_{y_2z_1} &= \pi_{\mathcal{B}^*}(t_{p_2q_1}) = \pi_{\mathcal{B}^*}(t_{p_3q_1}), & t_{y_2z_2} &= \pi_{\mathcal{B}^*}(t_{p_2q_2}) = \pi_{\mathcal{B}^*}(t_{p_3q_2}), \\ t_{y_2z_3} &= \pi_{\mathcal{B}^*}(t_{p_2q_3}) = \pi_{\mathcal{B}^*}(t_{p_3q_3}). \end{aligned}$$

Consider a collection of finitely many variables $\{\alpha_p, \beta_p\}_{p \in \mathcal{E}_\rho}$. Then the algorithm reduces the intrinsic global compatibility conditions to the solvability of the following linear equations of variables $\{\alpha_p, \beta_p\}_{p \in \mathcal{E}_\rho}$.

$$\begin{aligned} \alpha_{o_1} + f_{o_1}(\text{red}_{o_1}(t_{o_1p_1}))\beta_{o_1} &= \alpha_{o_2} + f_{o_2}(\text{red}_{o_2}(t_{o_2p_1}))\beta_{o_2}, \\ f_{o_2}(\text{red}_{o_2}(t_{o_2p_2})) &= f_{o_2}(\text{red}_{o_2}(t_{o_2p_3})), \\ \alpha_{p_2} + f_{p_2}(\text{red}_{p_2}(t_{p_2q_1}))\beta_{p_2} &= \alpha_{p_3} + f_{p_3}(\text{red}_{p_3}(t_{p_3q_1}))\beta_{p_3}, \\ \alpha_{p_2} + f_{p_2}(\text{red}_{p_2}(t_{p_2q_2}))\beta_{p_2} &= \alpha_{p_3} + f_{p_3}(\text{red}_{p_3}(t_{p_3q_2}))\beta_{p_3}, \\ \alpha_{p_2} + f_{p_2}(\text{red}_{p_2}(t_{p_2q_3}))\beta_{p_2} &= \alpha_{p_3} + f_{p_3}(\text{red}_{p_3}(t_{p_3q_3}))\beta_{p_3}. \end{aligned}$$

Now suppose

$$\begin{aligned} f_{o_1}(\text{red}_{o_1}(t_{o_1p_1})) &= 1, \\ f_{o_2}(\text{red}_{o_2}(t_{o_2p_1})) &= 2, \quad f_{o_2}(\text{red}_{o_2}(t_{o_2p_2})) = f_{o_2}(\text{red}_{o_2}(t_{o_2p_3})) = 1, \\ f_{p_2}(\text{red}_{p_2}(t_{p_2q_1})) &= f_{p_2}(\text{red}_{p_2}(t_{p_2q_2})) = 1, \quad f_{p_2}(\text{red}_{p_2}(t_{p_2q_3})) = -2, \\ f_{p_3}(\text{red}_{p_3}(t_{p_3q_1})) &= f_{p_3}(\text{red}_{p_3}(t_{p_3q_2})) = 2, \quad f_{p_3}(\text{red}_{p_3}(t_{p_3q_3})) = -1. \end{aligned}$$

Then the above system of linear equations is solvable, and thus $(\mathcal{D}, \mathcal{H})$ is smoothable by Algorithm 2.13. In particular, we have a solution $\alpha_{o_1} = 0, \beta_{o_1} = 2, \alpha_{o_2} = 0, \beta_{o_2} = 1, \alpha_{p_2} = 2, \beta_{p_2} = 1, \alpha_{p_3} = 1, \beta_{p_3} = 1$. Therefore, if we let $g_{o_1} = 2f_{o_1}, g_{o_2} = f_{o_2}, g_{p_1} = f_{p_1}, g_{p_2} = 2 + f_{p_2}, g_{p_3} = 1 + f_{p_3}, g_{q_1} = f_{q_1}, g_{q_2} = f_{q_2}, g_{q_3} = f_{q_3}$, and $g_p = f_p$ for all $p \notin \mathcal{E}_\rho$, then $G = \{g_p\}_{p \in \Gamma}$ is admissible.

Moreover, let \mathcal{P} be the unique bifurcation partition system that has a partition $\{\{t_{xy_1}\}, \{t_{xy_2}\}\}$ at x and $\{\{t_{y_2z_1}, t_{y_2z_2}\}, \{t_{y_2z_3}\}\}$ at y_2 , and let \mathcal{P}' be the unique bifurcation partition system that has a partition $\{\{t_{xy_1}\}, \{t_{xy_2}\}\}$ at x and

$$\{\{t_{y_2z_1}\}, \{t_{y_2z_2}\}, \{t_{y_2z_3}\}\}$$

at y_2 . Then \mathcal{P} is the bifurcation partition system associated with $G, \text{BP}_{\mathcal{D}, \mathcal{H}}^{(3)} = \text{BP}_{\mathcal{D}, \mathcal{H}}^{(4)} = \{\mathcal{P}\}$, and $\text{BP}_{\mathcal{D}, \mathcal{H}}^{(2)} = \{\mathcal{P}, \mathcal{Q}\}$.

5.3 The Spaces $\Lambda_{\mathcal{D}, \mathcal{H}}^{(1)}, \Lambda_{\mathcal{D}, \mathcal{H}}^{(2)}, \Lambda_{\mathcal{D}, \mathcal{H}}^{(3)}$ and $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$ of Partition Trees

Suppose that $(\mathcal{D}, \mathcal{H})$ represents a solvable diagrammatic pre-limit g_d^1 with a solution ρ and the corresponding bifurcation tree \mathcal{B} . As we have defined four levels of compatibilities between bifurcation partition systems and $(\mathcal{D}, \mathcal{H})$ in the previous subsection, here we define four levels of compatibilities between bifurcation trees and $(\mathcal{D}, \mathcal{H})$, and construct the spaces $\Lambda_{\mathcal{D}, \mathcal{H}}^{(1)}, \Lambda_{\mathcal{D}, \mathcal{H}}^{(2)}, \Lambda_{\mathcal{D}, \mathcal{H}}^{(3)}$ and $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$ of partition trees as follows.

- (1) Let $\Lambda_{\mathcal{D}, \mathcal{H}}^{(1)} := \Lambda_\rho$. (We let $\Lambda_{\mathcal{D}, \mathcal{H}}^{(1)} = \emptyset$ when $(\mathcal{D}, \mathcal{H})$ is non-solvable.) In other words, $\Lambda_{\mathcal{D}, \mathcal{H}}^{(1)}$ is the space of all possible partition trees with respect to a solution ρ of the global diagram defined by $(\mathcal{D}, \mathcal{H})$.
- (2) We say that a partition tree \mathcal{T} is Level-II (respectively, Level-III) compatible with $(\mathcal{D}, \mathcal{H})$ if for every point $p \in \Gamma$ and each pair of tangent directions $t_1, t_2 \in \text{Tan}_\Gamma^+(p)$, we have that t_1 is locally equivalent to t_2 if (respectively, if and only if) $\pi_{\mathcal{T}*}(t_1) = \pi_{\mathcal{T}*}(t_2)$.
- (3) We say that a partition tree \mathcal{T} is globally (or Level-IV) compatible with $(\mathcal{D}, \mathcal{H})$ if there exists a collection $G = \{g_p\}_{p \in \Gamma}$ of non-constant functions $g_p \in H_p$ such that one of the following equivalent statements are satisfied.
 - (a) There is a function $\xi: \coprod_{x \in \mathcal{T}} \text{Tan}_\mathcal{T}^+(x) \rightarrow \kappa$ such that ξ , restricted to $\text{Tan}_\mathcal{T}^+(x)$, is injective for each $x \in \mathcal{T}$, and $G \circ \text{Red}(t) = \xi \circ \pi_{\mathcal{T}*}(t)$ for all $t \in \coprod_{p \in \Gamma} \text{Tan}_\Gamma^+(p)$.
 - (b) Whenever $t_1 \in \text{Tan}_\Gamma^+(p_1)$ and $t_2 \in \text{Tan}_\Gamma^+(p_2)$, where $\pi_\mathcal{T}(p_1) = \pi_\mathcal{T}(p_2)$, we have $g_{p_1}(\text{red}_{p_1}(t_1)) = g_{p_2}(\text{red}_{p_1}(t_2))$ if and only if $\pi_{\mathcal{T}*}(t_1) = \pi_{\mathcal{T}*}(t_2)$.
 In this sense, to be more specific, we also say that \mathcal{T} and $(\mathcal{D}, \mathcal{H})$ are globally compatible via $\{g_p\}_{p \in \Gamma}$.
- (4) Denote by $\Lambda_{\mathcal{D}, \mathcal{H}}^{(2)}$, $\Lambda_{\mathcal{D}, \mathcal{H}}^{(3)}$ and $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$ the spaces of partition trees Level-II, Level-III, and Level-IV compatible with $(\mathcal{D}, \mathcal{H})$, respectively.
- (5) As in Lemma 5.4, one can easily see that $\Lambda_{\mathcal{D}, \mathcal{H}}^{(1)} \supseteq \Lambda_{\mathcal{D}, \mathcal{H}}^{(2)} \supseteq \Lambda_{\mathcal{D}, \mathcal{H}}^{(3)} \supseteq \Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$. An example of $\Lambda_{\mathcal{D}, \mathcal{H}}^{(1)}$, $\Lambda_{\mathcal{D}, \mathcal{H}}^{(2)}$, $\Lambda_{\mathcal{D}, \mathcal{H}}^{(3)}$, and $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$ for simple metric graphs is shown in Example B.4 in the Appendix.

There is a natural map $\phi^\Lambda: \Lambda_{\mathcal{D}, \mathcal{H}}^{(1)} \rightarrow \text{BP}_{\mathcal{D}, \mathcal{H}}^{(1)}$ such that for every $x \in \mathcal{B}$, every two forward tangent directions t_1 and t_2 in $\text{Tan}_\mathcal{B}^+(x)$ are equivalent in $\phi^\Lambda(\mathcal{T})$ if and only if they are pushed forward to the same tangent direction in $\text{Tan}_\mathcal{T}^+(\Theta_\mathcal{T}^\mathcal{B}(x))$ by $\Theta_\mathcal{T}^\mathcal{B}$.

On the other hand, given a bifurcation partition system $\{\vec{P}_x\}_{x \in \mathcal{B}}$, we want to construct a partition tree. For a small enough δ (precisely, we can let δ be less than the minimal distance between two distinct exceptional values of ρ), we derive a metric in the following way: for each point $x \in \text{Bif}(\mathcal{B})$ and each equivalence class $E \subseteq \text{Tan}_\mathcal{B}^+(x)$ in \vec{P}_x of $\text{Tan}_\mathcal{B}^+(x)$, we isometrically glue all the segments of length δ with one endpoint being x and the other being in a forward tangent direction of \mathcal{B} in E . Then it can easily be seen that \mathcal{T} is a partition tree induced by the gluing and $\phi^\Lambda(\mathcal{T}) = \{\vec{P}_x\}_{x \in \mathcal{B}}$. We call the partition tree constructed in this way the δ -glued partition tree with respect to $\{\vec{P}_x\}_{x \in \mathcal{B}}$. An example is shown in Figure 5, where a δ -glued partition tree in the right panel is derived from the bifurcation tree in Figure 2 and a bifurcation partition system in the left panel.

Proposition 5.6 ϕ^Λ is a surjection from $\Lambda_{\mathcal{D}, \mathcal{H}}^{(1)}$ to $\text{BP}_{\mathcal{D}, \mathcal{H}}^{(1)}$. In addition, the images of ϕ^Λ restricted to $\Lambda_{\mathcal{D}, \mathcal{H}}^{(2)}$, $\Lambda_{\mathcal{D}, \mathcal{H}}^{(3)}$, and $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$ are $\text{BP}_{\mathcal{D}, \mathcal{H}}^{(2)}$, $\text{BP}_{\mathcal{D}, \mathcal{H}}^{(3)}$, and $\text{BP}_{\mathcal{D}, \mathcal{H}}^{(4)}$, respectively.

Proof The surjectivity of ϕ^Λ follows from the construction of the δ -glued partition tree with respect to any bifurcation partition system.

Now let us show that the image of ϕ^Λ restricted to $\Lambda_{\mathcal{D}, \mathcal{H}}^{(2)}$ is contained in $\text{BP}_{\mathcal{D}, \mathcal{H}}^{(2)}$, and the image of ϕ^Λ restricted to $\Lambda_{\mathcal{D}, \mathcal{H}}^{(3)}$ is contained in $\text{BP}_{\mathcal{D}, \mathcal{H}}^{(3)}$. Suppose \mathcal{T} is Level-II

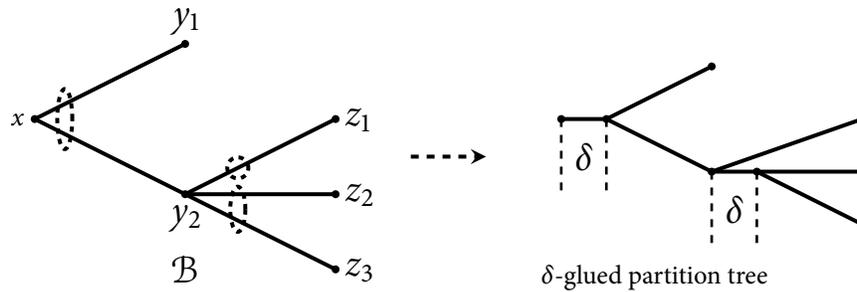


Figure 5: An illustration of a δ -glued partition tree.

(respectively, Level-III) compatible with $(\mathcal{D}, \mathcal{H})$. For all $p \in \Gamma$ and each pair $t_1, t_2 \in \text{Tan}_\Gamma^{\rho+}(p)$, we have $\pi_{\mathcal{B}*}(t_1), \pi_{\mathcal{B}*}(t_2) \in \text{Tan}_{\mathcal{B}}^+(\pi_{\mathcal{B}}(p))$. Also, $\pi_{\mathcal{B}*}(t_1) \sim \pi_{\mathcal{B}*}(t_2)$ in $\phi^\Lambda(\mathcal{T})$ if and only if $\pi_{\mathcal{T}*}(t_1) = \pi_{\mathcal{T}*}(t_2)$, since $\pi_{\mathcal{T}}$ factors through $\pi_{\mathcal{B}}$. Therefore, we derive that t_1 and t_2 in $\text{Tan}_\Gamma^{\rho+}(p)$ are locally equivalent if (respectively, if and only if) $\pi_{\mathcal{B}*}(t_1) \sim \pi_{\mathcal{B}*}(t_2)$ in $\phi^\Lambda(\mathcal{T})$. Therefore, $\phi^\Lambda(\mathcal{T})$ is a bifurcation partition system Level-II (respectively, Level-III) compatible with $(\mathcal{D}, \mathcal{H})$.

Next we will show that the image of ϕ^Λ restricted to $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$ is contained in $\text{BP}_{\mathcal{D}, \mathcal{H}}^{(4)}$. Let \mathcal{T} be a partition tree globally compatible with $(\mathcal{D}, \mathcal{H})$ via $\{g_p\}_{p \in \Gamma} \in \mathcal{H}$. Then one can see that $\phi^\Lambda(\mathcal{T})$ will be a bifurcation partition system globally compatible with $(\mathcal{D}, \mathcal{H})$ via $\{g_p\}_{p \in \Gamma}$.

Conversely, we will show that the images of ϕ^Λ restricted to $\Lambda_{\mathcal{D}, \mathcal{H}}^{(2)}$, $\Lambda_{\mathcal{D}, \mathcal{H}}^{(3)}$, and $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$ contain $\text{BP}_{\mathcal{D}, \mathcal{H}}^{(2)}$, $\text{BP}_{\mathcal{D}, \mathcal{H}}^{(3)}$, and $\text{BP}_{\mathcal{D}, \mathcal{H}}^{(4)}$, respectively. To this end, we show that a δ -glued partition tree with respect to a bifurcation partition system in $\text{BP}_{\mathcal{D}, \mathcal{H}}^{(2)}$, $(\text{BP}_{\mathcal{D}, \mathcal{H}}^{(3)}$, and $\text{BP}_{\mathcal{D}, \mathcal{H}}^{(4)}$, respectively), is an element in $\Lambda_{\mathcal{D}, \mathcal{H}}^{(2)}$, $(\Lambda_{\mathcal{D}, \mathcal{H}}^{(3)}$, and $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$, respectively). The first two cases for $\text{BP}_{\mathcal{D}, \mathcal{H}}^{(2)}$ and $\text{BP}_{\mathcal{D}, \mathcal{H}}^{(3)}$ are straightforward by definitions.

Now let $\{\vec{P}_x\}_{x \in \mathcal{B}} \in \text{BP}_{\mathcal{D}, \mathcal{H}}^{(4)}$ be the bifurcation partition system associated with an admissible $\{g_p\}_{p \in \Gamma} \in \mathcal{H}$ and let \mathcal{T} be its corresponding δ -glued partition tree. We claim that $\mathcal{T} \in \Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$. To show this, we want to turn $\{g_p\}_{p \in \Gamma}$ into $\{g'_p\}_{p \in \Gamma} \in \mathcal{H}$ and define a function $\xi: \coprod_{x \in \mathcal{T}} \text{Tan}_{\mathcal{T}}^+(x) \rightarrow \kappa$ such that \mathcal{T} and $(\mathcal{D}, \mathcal{H})$ are compatible via $\{g_p\}_{p \in \Gamma}$.

- 1 Assign values $g_p(\text{red}_p(t))$ to $\pi_{\mathcal{T}*}(t)$ for all $p \in \mathcal{E}_\rho$ and $t \in \text{Tan}_\Gamma^{\rho+}(p)$
- 2 Assign values in κ for the remaining elements in $\coprod_{x \in \mathcal{T}} \text{Tan}_{\mathcal{T}}^+(x)$ such that for each $x \in \mathcal{T}$, ξ restricted to $\text{Tan}_{\mathcal{T}}^+(x)$ is injective.
- 3 Let $g'_p = g_p$ for all $p \in \mathcal{E}_\rho$.
- 4 For an ordinary point $p \in \mathcal{O}_\rho$, there is a unique forward tangent direction (denoted by t) at p . Therefore, we can always find a non-constant rational function $g'_p \in H_p$ such that $g'_p(\text{red}_p(t)) = \xi(\pi_{\mathcal{T}*}(t))$ (Lemma 2.12).

By the above construction, \mathcal{T} is globally compatible with $(\mathcal{D}, \mathcal{H})$ via $\{g'_p\}_{p \in \Gamma}$, which means $\mathcal{T} \in \Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$. ■

Remark 5.7 We will employ the surjectivity of the map $\phi^\Lambda|_{\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}} : \Lambda_{\mathcal{D}, \mathcal{H}}^{(4)} \rightarrow \text{BP}_{\mathcal{D}, \mathcal{H}}^{(4)}$ in the proof of the smoothing criterion in the next section.

6 Proof of the Smoothing Criterion

We restate the smoothing criterion combining the two versions as follows.

Theorem 6.1 Given a saturated metrized complex \mathcal{C} and a pre-limit g_d^1 represented by $(\mathcal{D}, \mathcal{H})$, the following statements are equivalent.

- (i) $(\mathcal{D}, \mathcal{H})$ is smoothable.
- (ii) $(\mathcal{D}, \mathcal{H})$ is solvable and $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$ is nonempty.
- (iii) $(\mathcal{D}, \mathcal{H})$ is solvable and satisfies the intrinsic global compatibility conditions.

Proof (ii) is equivalent to saying that there exists a partition tree globally compatible with $(\mathcal{D}, \mathcal{H})$, and (iii) is equivalent to saying that there exists a bifurcation partition system globally compatible with $(\mathcal{D}, \mathcal{H})$. Then the equivalence of (ii) and (iii) follows from Proposition 5.6.

(i) \Rightarrow (ii). Let $(\mathcal{D}, \mathcal{H})$ represent a smoothable pre-limit g_d^1 on \mathcal{C} . Then by Theorem 3.13, we know that there exists a harmonic morphism

$$\mathcal{C}\phi^{\text{mod}} = (\phi_{\Gamma^{\text{mod}}}, \{\phi_p\}_{p \in \Gamma^{\text{mod}}})$$

of degree $\text{deg}(\mathcal{D})$ from a modification \mathcal{C}^{mod} of \mathcal{C} to a genus zero saturated metrized complex $\mathcal{C}(T)$ such that (1) \mathcal{D} is the retraction to \mathcal{C} of a divisor \mathcal{D}^{mod} on $\mathcal{C}\phi^{\text{mod}}$, which is a pullback divisor by $\mathcal{C}\phi^{\text{mod}}$ of an effective degree one divisor on $\mathcal{C}(T)$, and (2) ϕ_p coincides with the morphism from C_p to \mathbb{P}^1_κ , defined by H_p . Here the underlying metric graphs of \mathcal{C} , \mathcal{C}^{mod} , and $\mathcal{C}(T)$ are denoted by Γ , Γ^{mod} , and T , respectively. Now it remains to show that T must be an element in $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$.

Denote by $r(T)$ the root of T which is the image of $D_{\Gamma^{\text{mod}}}^{\text{mod}}$ (the tropical part of \mathcal{D}^{mod}) under $\phi_{\Gamma^{\text{mod}}}$. Then $r(T)$ and the map $\phi_\Gamma : \Gamma \rightarrow T$ that is the restriction to Γ of the harmonic morphism $\phi_{\Gamma^{\text{mod}}} : \Gamma^{\text{mod}} \rightarrow T$ induces a global diagram on Γ in the following way. For any point $p \in \Gamma$ and any tangent direction $t \in \text{Tan}_\Gamma(p)$, the multiplicity $m_1(p, t)$ is the expansion factor with sign “−” if the pushforward of t by ϕ_Γ coincides with the tangent direction on T along the unique path from $\phi_\Gamma(p)$ to $r(T)$, and with sign “+” otherwise.

On the other hand, we also construct local diagrams from $(\mathcal{D}, \mathcal{H})$ (Remark 2.5). In particular, the multiplicity $m_2(p, t)$ in the local diagram at point p associated with $(\mathcal{D}, \mathcal{H})$ equals the ramification index of ϕ_p at $\text{red}_p(t)$ with an appropriate sign. By the compatibility property of the harmonic morphisms, we know that the ramification index of ϕ_p at the marked point $\text{red}_p(t)$ on C_p corresponding to t equals the expansion factor of ϕ_Γ at t . Therefore $m_1(p, t) = m_2(p, t)$, which means $(\mathcal{D}, \mathcal{H})$ is diagrammatic and solvable with a solution $d_T^0 \circ \phi_\Gamma$, where d_T^0 is the distance from

points on T to the root point $r(T)$. Hence, (T, ϕ_Γ) is an element of $\Lambda_{\mathcal{D}, \mathcal{H}}^{(1)}$. In addition, the compatibility property of the harmonic morphisms also guarantees that (T, ϕ_Γ) must be in $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$.

(ii) \Rightarrow (i). Let $(T, \pi_T) \in \Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$ be globally compatible with $(\mathcal{D}, \mathcal{H})$. To show that $(\mathcal{D}, \mathcal{H})$ is smoothable, using Theorem 3.13, we need to find some harmonic morphism compatible with $(\mathcal{D}, \mathcal{H})$. More precisely, we want to construct a modification \mathcal{C}^{mod} of \mathcal{C} with underlying metric graph Γ^{mod} such that (1) $\phi_{\Gamma^{\text{mod}}}: \Gamma^{\text{mod}} \rightarrow T$ is a harmonic morphism (between metric graphs) whose restriction to Γ is π_T , and (2) $\phi_{\Gamma^{\text{mod}}}$ can be lifted to a harmonic morphism $\mathcal{C}\phi^{\text{mod}}$ from \mathcal{C}^{mod} to a genus zero saturated metrized complex $\mathcal{C}(T)$ whose underlying metric tree is T . In Section 3.1, we also introduced the notion of a pseudo-harmonic morphism, which does not require the balancing condition as for harmonic morphisms. In the rest of the proof, we will show that we can first find a compatible pseudoharmonic morphism from \mathcal{C} to a genus zero saturated metrized complex $\mathcal{C}(T)$. Then we will show that we can always extend this pseudo-harmonic morphism to a desired harmonic morphism by generating a suitable modification \mathcal{C}^{mod} of \mathcal{C} , while we single out the statement and proof in Proposition 6.2 together with Example 6.3 to aid our exposition.

Assume $\{g_p\}_{p \in \Gamma}$ is a collection of rational functions $g_p \in H_p$ on C_p that makes $(\mathcal{D}, \mathcal{H})$ and (T, π_T) compatible. There is a function $\xi: \coprod_{x \in T} \text{Tan}_T^+(x) \rightarrow \kappa$ such that ξ is injective restricted to $\text{Tan}_T^+(x)$ for each $x \in T$, and $g_p \circ \text{red}_p = \xi \circ \pi_{T*}$ for all $p \in \Gamma$. Let $\bar{g}_p: C_p \rightarrow \kappa_\infty$ be the function on C_p extending g_p to its poles, and let $\bar{\xi}: \coprod_{x \in T} \text{Tan}_T(x) \rightarrow \kappa_\infty$ be the extension of ξ such that for each $x \in T$, $\bar{\xi}$ maps the incoming tangent direction at x to ∞ in κ_∞ . Then we also have $\bar{g}_p \circ \text{red}_p = \bar{\xi} \circ \pi_{T*}$.

Since κ_∞ is isomorphic to a projective line over κ , we can build a genus zero metrized complex $\mathcal{C}(T)$ from T (for all $x \in T$, the curve C'_x associated with x is a projective line) by letting $\gamma_x \circ \bar{\xi}_x$ be the reduction map at $x \in T$, where $\bar{\xi}_x$ is the function $\bar{\xi}$ restricted to $\text{Tan}_T(x)$ and $\gamma_x: \kappa_\infty \xrightarrow{\sim} C'_x$ is the isomorphism between κ_∞ and C'_x .

Now let $\phi_p = \gamma_{\pi_T(p)} \circ \bar{g}_p$. Then $(\pi_T, \{\phi_p\}_{p \in \Gamma})$ is a pseudo-harmonic morphism from \mathcal{C} to $\mathcal{C}(T)$, since the compatibility conditions of a pseudo-harmonic morphism (Definition 3.1) are guaranteed by the solvability of $(\mathcal{D}, \mathcal{H})$ and the relation

$$\phi_p \circ \text{red}_p = \gamma_{\pi_T(p)} \circ \bar{g}_p \circ \text{red}_p = \gamma_{\pi_T(p)} \circ \bar{\xi}_{\pi_T(p)} \circ \pi_{T*},$$

where $\gamma_{\pi_T(p)} \circ \bar{\xi}_{\pi_T(p)}$ is the reduction map at $\pi_T(p)$ by the construction of $\mathcal{C}(T)$.

By Proposition 6.2, we can extend the pseudo-harmonic morphism $(\pi_T, \{\phi_p\}_{p \in \Gamma})$ to a harmonic morphism $(\phi^{\text{mod}}, \{\phi_p\}_{p \in \Gamma^{\text{mod}}})$ from a modification \mathcal{C}^{mod} of \mathcal{C} to $\mathcal{C}(T)$. ■

Proposition 6.2 *Let $\mathcal{C}\phi$ be a pseudo-harmonic morphism from a saturated metrized complex \mathcal{C} to a saturated metrized complex $\mathcal{C}(T)$ of genus zero. If $\mathcal{C}\phi$ is harmonic at all but finitely many points in its underlying metric graph, then there is a modification \mathcal{C}^{mod} of \mathcal{C} and a harmonic morphism $\mathcal{C}\phi^{\text{mod}}$ from \mathcal{C}^{mod} to $\mathcal{C}(T)$ such that $\mathcal{C}\phi$ is the restriction of $\mathcal{C}\phi^{\text{mod}}$ to \mathcal{C} .*

Proof Assume that the underlying metric graphs of \mathfrak{C} and $\mathfrak{C}(T)$ are Γ and T , respectively, and the associated curves of \mathfrak{C} and $\mathfrak{C}(T)$ are $\{C_p\}_{p \in \Gamma}$ and $\{C'_x\}_{x \in T}$, respectively. Let $\mathfrak{C}\phi = (\phi_\Gamma, \{\phi_p\}_{p \in \Gamma})$, where $\phi_\Gamma: \Gamma \rightarrow T$ is the associated pseudo-harmonic morphism of metric graphs and $\phi_p: C_p \rightarrow C'_{\phi_\Gamma(p)}$ is the associated finite morphism of curves at p . We will derive a modification $\mathfrak{C}^{\text{mod}}$ of \mathfrak{C} in the following way.

Consider a point $q \in \Gamma$. For each tangent direction $t' \in \text{Tan}_T(\phi_\Gamma(q))$ at $\phi_\Gamma(q)$ on T , let $u \in C_q$ be a non-marked point of C_q that is an element in the fiber $\phi_q^{-1}(\text{red}_{\phi_\Gamma(q)}(t'))$. Suppose the ramification index of ϕ_q at u is m .

Let T' be the connected component of $T \setminus \{\phi_\Gamma(q)\}$ corresponding to the tangent direction t at $\phi_\Gamma(q)$. Let T'_1, \dots, T'_m be m copies of T' . For $i = 1, \dots, m$, let x_i be the open end of T'_i corresponding to the open end $\phi_\Gamma(q)$ of T' and let y_i be the point in T'_i with a small distance l to x_i (l is less than the minimum distance of branching points of T' to $\phi_\Gamma(q)$).

Now we want to attach to Γ an extra branch Γ'_u with respect to u . Then by equipping Γ'_u with projective lines, we will get a modification of \mathfrak{C} with respect to u .

We construct Γ'_u from T'_1, \dots, T'_m by first identifying the segments $(x_i, y_i]$ and then shrinking the glued segment by a factor of m . Denote by $(x, y]$ the corresponding segment in Γ'_u with x being its open end. Then by this construction, the length of $(x, y]$ is l/m and $\Gamma'_u \setminus (x, y]$ is a disjoint union of $T'_i \setminus (x_i, y_i]$. Forgetting the compactness restriction of a metric graph, we also call T', T'_i , and Γ'_u metric graphs. Then there is a natural harmonic morphism $\phi_{\Gamma'_u}$ from Γ'_u to T' , where the balancing condition (Definition 3.2) is automatically satisfied by the construction of Γ'_u .

Let $\mathfrak{C}(T')$ be $\mathfrak{C}(T)$ restricted to T' . We can construct a saturated metrized complex $\mathfrak{C}(\Gamma'_u)$ with underlying metric graph Γ'_u by associating each point $p \in \Gamma'_u$ with a projective line C_p . Let $x' = \phi_{\Gamma'_u}(p)$. The reduction map red_p at p is derived as follows.

- If $p \in (x, y)$, then there are two tangent directions t_1 and t_2 in $\text{Tan}_{\Gamma'_u}(p)$ and two tangent directions t'_1 and t'_2 in $\text{Tan}_{T'}(x')$ where t_1 and t_2 are pullbacks of t'_1 and t'_2 by $\phi_{\Gamma'_u}$. Let $\phi_p: C_p \rightarrow C'_{x'}$ be a degree m morphism from C_p to $C'_{x'}$ (the curve associated with $\phi_{\Gamma'_u}(p)$ in $\mathfrak{C}(T')$) such that there are two points v_1 and v_2 in C_p with ramification index m over the marked points $\text{red}_{x'}(t'_1)$ and $\text{red}_{x'}(t'_2)$ in $C'_{x'}$, respectively. Let the marked point $\text{red}_p(t_1)$ associated with t_1 be v_1 and the marked point $\text{red}_p(t_2)$ associated with t_2 be v_2 .

- If $p = y$, then there are $m + 1$ tangent directions t_1, \dots, t_{m+1} in $\text{Tan}_{\Gamma'_u}(p)$ and two tangent directions t'_1 and t'_2 in $\text{Tan}_{T'}(x')$. We can assume that t'_1 is the tangent direction corresponding to the edge between x' and the open end of T' , t_1 is the pullback of t'_1 by $\phi_{\Gamma'_u}$, and $\{t_2, \dots, t_{m+1}\}$ is the pullback of t'_2 by $\phi_{\Gamma'_u}$. Let $\phi_p: C_p \rightarrow C'_{x'}$ be a degree m morphism from C_p to $C'_{x'}$ such that there is a point $v_1 \in C_p$ with ramification index m over the marked point $\text{red}_{x'}(t'_1)$ and there are distinct points $v_2, \dots, v_{m+1} \in C_p$ with ramification index 1 over the marked point $\text{red}_{x'}(t'_2)$. Then we let the marked point $\text{red}_p(t_i)$ associated with t_i be v_i for $i = 1, \dots, m + 1$.

- If $p \in \Gamma'_u \setminus (x, y]$, then $\text{Tan}_{T'}(x')$ pulls back bijectively to $\text{Tan}_{\Gamma'_u}(p)$ by $\phi_{\Gamma'_u}$. We let $\phi_p: C_p \rightarrow C'_{x'}$ be an isomorphism. For every pair of corresponding tangent directions $t_p \in \text{Tan}_{\Gamma'_u}(p)$ and $t_{x'} \in \text{Tan}_{T'}(x')$, we let the marked point $\text{red}_p(t_p)$ associated with t_p be $\phi_p^{-1}(\text{red}_{x'}(t_{x'}))$.

Note that in all cases above, the morphism $\phi_p: C_p \rightarrow C'_{x'}$ always exists, since C_p and $C'_{x'}$ are projective lines. We conclude that $(\phi_{\Gamma'_u}, \{\phi_p\}_{p \in \Gamma'_u})$ is a harmonic morphism from $\mathcal{C}(\Gamma'_u)$ to $\mathcal{C}(T')$, since $\phi_{\Gamma'_u}$ is a harmonic morphism of metric graphs and the compatibility conditions of Definition 3.1 are automatically satisfied by the above construction of $\mathcal{C}(\Gamma'_u)$.

Now we get a modification of Γ with respect to u by attaching the open end of the extra branch Γ'_u to Γ at q , and a modification of \mathcal{C} with respect to u by adding u as a marked point of C_q and attaching $\mathcal{C}(\Gamma'_u)$ to \mathcal{C} . Moreover, the pseudo-harmonic morphisms ϕ_Γ and $\mathcal{C}\phi$ also naturally extend to these modifications of Γ and \mathcal{C} , respectively, with respect to u , which are harmonic at all the points in the extra branch Γ'_u (not necessarily at q).

Recall that u is a non-marked point of C_q that at the same time is an element of $\phi_q^{-1}(\text{red}_{\phi_\Gamma(q)}(t'))$, where t' is a tangent direction at $\phi_\Gamma(q)$ on T . Therefore, we can get modifications of Γ and \mathcal{C} with respect to q by performing modifications of Γ and \mathcal{C} on all possible u in this sense at the same time. Moreover, the pseudo-harmonic morphisms ϕ_Γ and $\mathcal{C}\phi$ also naturally extend respectively to these modifications of Γ and \mathcal{C} with respect to q , which are harmonic at the point q and all the points in the extra branches. Note that if $\mathcal{C}\phi$ is already harmonic at q , then no modification is performed.

The final modifications of Γ and \mathcal{C} , denoted by Γ^{mod} and \mathcal{C}^{mod} , respectively, are derived by performing modifications of Γ and \mathcal{C} at the same time to all $q \in \Gamma$ at which $\mathcal{C}\phi$ is not harmonic. In this way, we get a harmonic morphism $\mathcal{C}\phi^{\text{mod}}: \mathcal{C}^{\text{mod}} \rightarrow \mathcal{C}(T)$ as required. ■

Example 6.3 In Figure 6, we show how a modification is performed at point o_1 in Figure 3 of Example 4.15. The image of o_1 under $\pi_{\mathcal{J}}$ is x' . First note that there is only one outgoing edge o_1p_1 from o_1 with expansion factor 1 of the map $\pi_{\mathcal{J}}$. Suppose that the degree of the nonconstant rational function $g_{o_1} \in H_{o_1}$ is 3. Suppose that the forward tangent direction from x' to y'_1 corresponds to $c_1 \in \kappa$ and the forward tangent direction from x' to y'_2 corresponds to $c_2 \in \kappa$. Suppose that $g_{o_1}^{-1}(c_1) = \{u_1, u_2, u_3\}$ and u_1 is the reduction of the tangent direction from o_1 to p_1 . Then two copies of $x'y'_1$ will be attached to o_1 as extra branches corresponding to u_2 and u_3 , respectively. Suppose that $g_{o_1}^{-1}(c_2) = \{v_1, v_2\}$, while the ramification index of v_1 is 1 and the ramification index of v_2 is 2. Let T' be the subgraph of \mathcal{J} connecting x', z'_{12} , and z'_3 . Then one copy of T' is attached to o_1 as the extra branch corresponding to v_1 . Accordingly, the extra branch corresponding to v_2 is made from two copies T'_1 and T'_2 of T' by first gluing from the open ends of T'_1 and T'_2 along small segments of the same length and then shrinking the glued segment by a factor of 2.

7 Applications

We apply the smoothing criterion to the saturated metrized complex versions of certain types of curves: curves of compact type studied by Eisenbud and Harris, nodal curves with dual graphs made of separate loops and curves considered by Harris and Mumford to characterize gonality stratification. We also extend the smoothing criterion to metrized complexes by showing a concrete example.

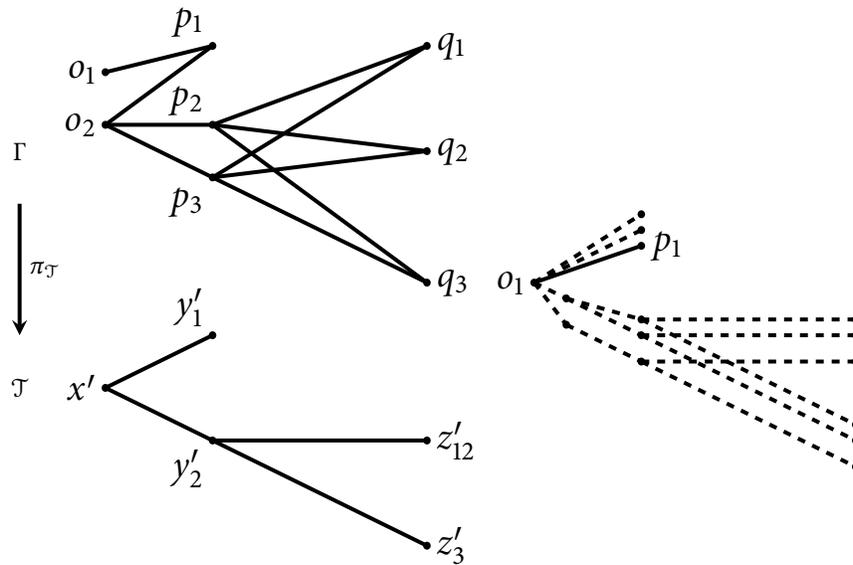


Figure 6: A modification performed at point o_1 based on the projection map $\pi_{\mathcal{T}}$ in Example 4.15 and the local data in H_{o_1} .

7.1 Saturated Metrized Complexes of Compact Type

We show that every diagrammatic pre-limit g_d^1 on a saturated metrized complex of compact type, *i.e.*, whose underlying metric graph is a metric tree, is smoothable. This theorem is an analogue of [16, Proposition 3.1] by Eisenbud and Harris for curves of compact type.

Theorem 7.1 *Every diagrammatic pre-limit g_d^1 represented by $(\mathcal{D}, \mathcal{H})$ on a saturated metrized complex \mathcal{C} of compact type is smoothable.*

Proof The underlying metric graph Γ of \mathcal{C} is a tree. Therefore $(\mathcal{D}, \mathcal{H})$ must be solvable. Let ρ be a solution to $(\mathcal{D}, \mathcal{H})$ and \mathcal{E}_ρ be the set of exceptional points of ρ (§2.7). Then we can subdivide Γ into segments L_1, \dots, L_n by \mathcal{E}_ρ (the end points of L_i are exceptional points). The restrictions of $(\mathcal{D}, \mathcal{H})$ to each L_i must be smoothable since the intrinsic global compatibility conditions will be trivial. This theorem then follows directly from Proposition 7.2. ■

Consider a diagrammatic pre-limit g_d^1 represented by $(\mathcal{D}, \mathcal{H})$ on a saturated metrized complex \mathcal{C} whose underlying metric graph is Γ . Let Γ' be a connected closed metric subgraph of Γ . A saturated metrized complex \mathcal{C}' is said to be the restriction of \mathcal{C} to Γ' if the underlying metric graph of \mathcal{C}' is Γ' , the associated curves of \mathcal{C}' and \mathcal{C} at p are identical for all $p \in \Gamma'$, and the reduction maps of \mathcal{C}' and \mathcal{C} at p restricted to $\text{Tan}_{\Gamma'}(p)$ are identical for all $p \in \Gamma'$ (note that $\text{Tan}_{\Gamma'}(p) \subsetneq \text{Tan}_\Gamma(p)$ when p is a

boundary point of Γ' in Γ). Moreover, a diagrammatic pre-limit g_d^1 represented by $(\mathcal{D}', \mathcal{H}')$ on \mathcal{C}' is said to be the restriction of $(\mathcal{D}, \mathcal{H})$ to Γ' (or \mathcal{C}') if the following are satisfied.

- H'_p (the C_p -part of \mathcal{H}') is identical to H_p (the C_p -part of \mathcal{H}) for all $p \in \Gamma'$.
- D'_p (the C_p -part of \mathcal{D}') is identical to D_p (the C_p -part of \mathcal{D}) for all $p \in \Gamma' \setminus \partial\Gamma'$ (here $\partial\Gamma'$ stands for the boundary of Γ' in Γ).
- For all $p \in \partial\Gamma'$, D'_p is modified from D_p as

$$D_p = D'_p + \sum_{t \in \text{In}_\Gamma(p) \setminus \text{In}_{\Gamma'}(p)} (-m(p, t))(\text{red}_p(t)),$$

where $m(p, t)$ is the multiplicity of t (which is negative if $t \in \text{In}_\Gamma(p)$) in the local diagram induced by \mathcal{H} at p . Note that this modification guarantees the compatibility between D_p and H_p which further implies that $(\mathcal{D}', \mathcal{H}')$ is diagrammatic (Definition 2.10).

Proposition 7.2 *Let Γ_1 and Γ_2 be connected metric subgraphs of a metric graph Γ such that $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2$ is a singleton. For a saturated metrized complex \mathcal{C} whose underlying metric graph is Γ , let $(\mathcal{D}, \mathcal{H})$ represent a diagrammatic pre-limit g_d^1 on \mathcal{C} . Let $(\mathcal{D}_1, \mathcal{H}_1)$ and $(\mathcal{D}_2, \mathcal{H}_2)$ be the restrictions of $(\mathcal{D}, \mathcal{H})$ to Γ_1 and Γ_2 , respectively. Then $(\mathcal{D}, \mathcal{H})$ is smoothable if and only if $(\mathcal{D}_1, \mathcal{H}_1)$ and $(\mathcal{D}_2, \mathcal{H}_2)$ are both smoothable.*

Proof Let $\Gamma_1 \cap \Gamma_2 = \{q\}$. Let \mathcal{C}_1 and \mathcal{C}_2 be the restrictions of \mathcal{C} to Γ_1 and Γ_2 , respectively. Then $(\mathcal{D}_1, \mathcal{H}_1)$ is on \mathcal{C}_1 and $(\mathcal{D}_2, \mathcal{H}_2)$ is on \mathcal{C}_2 . It follows easily from the smoothing criterion that $(\mathcal{D}, \mathcal{H})$ being smoothable implies that $(\mathcal{D}_1, \mathcal{H}_1)$ and $(\mathcal{D}_2, \mathcal{H}_2)$ are both smoothable.

Now suppose that $(\mathcal{D}_1, \mathcal{H}_1)$ and $(\mathcal{D}_2, \mathcal{H}_2)$ are both smoothable and we claim that $(\mathcal{D}, \mathcal{H})$ is smoothable. This means $(\mathcal{D}_1, \mathcal{H}_1)$ and $(\mathcal{D}_2, \mathcal{H}_2)$ are solvable, and since $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \{q\}$, $(\mathcal{D}, \mathcal{H})$ must be solvable. Thus we may assume ρ is a solution to the global diagram of $(\mathcal{D}, \mathcal{H})$, while the restriction of ρ to Γ_1 (respectively, Γ_2), denoted by ρ_1 (respectively, ρ_2), is a solution to the global diagram of $(\mathcal{D}_1, \mathcal{H}_1)$ (respectively, $(\mathcal{D}_2, \mathcal{H}_2)$). Let $\mathcal{B}, \mathcal{B}_1$, and \mathcal{B}_2 be the bifurcation trees with respect to ρ, ρ_1 , and ρ_2 , respectively.

By the smoothing criterion, \mathcal{H}_1 contains an admissible collection $\{f_p^{(1)}\}_{p \in \Gamma_1}$ of rational functions $f_p^{(1)} \in H_p$ and \mathcal{H}_2 contains an admissible collection $\{f_p^{(2)}\}_{p \in \Gamma_2}$ of rational functions $f_p^{(2)} \in H_p$.

Note that \mathcal{B}_1 and \mathcal{B}_2 are subtrees of \mathcal{B} and $\mathcal{B}_1 \cup \mathcal{B}_2 = \mathcal{B}$. Let $r(\mathcal{B}), r(\mathcal{B}_1)$, and $r(\mathcal{B}_2)$ be the roots of $\mathcal{B}, \mathcal{B}_1$, and \mathcal{B}_2 , respectively. Clearly $r(\mathcal{B})$ must be either $r(\mathcal{B}_1)$ or $r(\mathcal{B}_2)$. Without loss of generality, we assume $r(\mathcal{B}) = r(\mathcal{B}_1)$. Let $y = \pi_{\mathcal{B}}(q)$ and L be the closed segment connecting $r(\mathcal{B}_2)$ and y in \mathcal{B} . Then one can observe that $\mathcal{B}_1 \cap \mathcal{B}_2 = L$. We construct a desirable admissible $\{f_p\}_{p \in \Gamma} \in \mathcal{H}$ by clutching $\{f_p^{(1)}\}_{p \in \Gamma_1}$ and $\{f_p^{(2)}\}_{p \in \Gamma_2}$ as follows.

- (1) For $x \in \mathcal{B}_1 \setminus L$, let $\tilde{P}_x = \tilde{P}_x^{(1)}$.
- (2) For $x \in \mathcal{B}_2 \setminus L$, let $\tilde{P}_x = \tilde{P}_x^{(2)}$.
- (3) For $p \in \Gamma_1 \setminus \pi_{\mathcal{B}}^{-1}(L)$, let $f_p = f_p^{(1)}$.

- (4) For $p \in \Gamma_2 \setminus \pi_{\mathbb{B}}^{-1}(L)$, let $f_p = f_p^{(2)}$.
- (5) For $x \in L$, consider all the forward tangent directions in

$$\text{Tan}_{\mathbb{B}_1}^+(x) = \{t_1^{(1)}, \dots, t_k^{(1)}\}$$

and all the forward tangent directions in $\text{Tan}_{\mathbb{B}_2}^+(x) = \{t_1^{(2)}, \dots, t_l^{(2)}\}$. Then by the smoothing criterion on $(\mathcal{D}_1, \mathcal{H}_1)$ and $(\mathcal{D}_2, \mathcal{H}_2)$, we can assign values

$$c_1^{(1)}, \dots, c_k^{(1)} \in \kappa$$

to $t_1^{(1)}, \dots, t_k^{(1)}$, respectively, and values $c_1^{(2)}, \dots, c_l^{(2)} \in \kappa$ to $t_1^{(2)}, \dots, t_l^{(2)}$ respectively such that

- (a) $f_p^{(1)}(\text{red}_p(t)) = c_i^{(1)}$ for all $i = 1, \dots, k$ whenever $p \in \pi_{\mathbb{B}_1}^{-1}(x)$ and

$$t \in \text{Tan}_{\Gamma_1}^{\rho_1^+}(p) \cap \pi_{\mathbb{B}_1^*}^{-1}(t_i^{(1)}),$$

- (b) $f_p^{(2)}(\text{red}_p(t)) = c_j^{(2)}$ for all $j = 1, \dots, l$ whenever $p \in \pi_{\mathbb{B}_2}^{-1}(x)$ and $t \in \text{Tan}_{\Gamma_2}^{\rho_2^+}(p) \cap \pi_{\mathbb{B}_2^*}^{-1}(t_j^{(2)})$.

When $x \in L \setminus \{y\}$, we have

$$\pi_{\mathbb{B}}^{-1}(x) = \pi_{\mathbb{B}_1}^{-1}(x) \cup \pi_{\mathbb{B}_2}^{-1}(x),$$

$$\pi_{\mathbb{B}_1}^{-1}(x) \cap \pi_{\mathbb{B}_2}^{-1}(x) = \emptyset,$$

$$\text{Tan}_{\mathbb{B}}^+(x) = \text{Tan}_{\mathbb{B}_1}^+(x) \cup \text{Tan}_{\mathbb{B}_2}^+(x),$$

and $\text{Tan}_{\mathbb{B}_1}^+(x) \cap \text{Tan}_{\mathbb{B}_2}^+(x)$ is a singleton. Without loss of generality, we let $t_1 = t_1^{(1)} = t_1^{(2)}$ be the forward tangent direction common to both $\text{Tan}_{\mathbb{B}_1}^+(x)$ and $\text{Tan}_{\mathbb{B}_2}^+(x)$, which means that $\bar{t}_{\mathbb{B}_1}(t_1)$ and $\bar{t}_{\mathbb{B}_2}(t_1)$ are the open superlevel components (for Γ_1 and Γ_2 , respectively) containing q and $\bar{t}_{\mathbb{B}}(t_1) = \bar{t}_{\mathbb{B}_1}(t_1) \cup \bar{t}_{\mathbb{B}_2}(t_1)$. So we can let $f_p = f_p^{(1)} - c_1^{(1)}$ for all $p \in \pi_{\mathbb{B}_1}^{-1}(x)$ and $f_p = f_p^{(2)} - c_1^{(2)}$ for all $p \in \pi_{\mathbb{B}_2}^{-1}(x)$. In this way of clutching, we conclude the following.

- (a) $f_p(\text{red}_p(t)) = c_i^{(1)} - c_1^{(1)}$ for all $i = 1, \dots, k$ whenever $p \in \pi_{\mathbb{B}_1}^{-1}(x)$ and $t \in \text{Tan}_{\Gamma_1}^{\rho_1^+}(p) \cap \pi_{\mathbb{B}_1^*}^{-1}(t_i^{(1)})$.
- (b) $f_p(\text{red}_p(t)) = c_j^{(2)} - c_1^{(2)}$ for all $j = 1, \dots, l$ whenever $p \in \pi_{\mathbb{B}_2}^{-1}(x)$ and $t \in \text{Tan}_{\Gamma_2}^{\rho_2^+}(p) \cap \pi_{\mathbb{B}_2^*}^{-1}(t_j^{(2)})$.
- (c) In particular, by (a) and (b), $f_p(\text{red}_p(t)) = 0$ whenever $p \in \pi_{\mathbb{B}}^{-1}(x)$ and $t \in \text{Tan}_{\Gamma}^{\rho^+}(p) \cap \pi_{\mathbb{B}^*}^{-1}(t_1)$. (Note that $\pi_{\mathbb{B}^*}^{-1}(t_1) = \pi_{\mathbb{B}_1^*}^{-1}(t_1^{(1)}) \cup \pi_{\mathbb{B}_2^*}^{-1}(t_1^{(2)})$.)

When $x = y$, we have $\pi_{\mathbb{B}}^{-1}(x) = \pi_{\mathbb{B}_1}^{-1}(x) \cup \pi_{\mathbb{B}_2}^{-1}(x)$, $\pi_{\mathbb{B}_1}^{-1}(x) \cap \pi_{\mathbb{B}_2}^{-1}(x) = \{q\}$, $\text{Tan}_{\mathbb{B}}^+(x) = \text{Tan}_{\mathbb{B}_1}^+(x) \cup \text{Tan}_{\mathbb{B}_2}^+(x)$, and $\text{Tan}_{\mathbb{B}_1}^+(x) \cap \text{Tan}_{\mathbb{B}_2}^+(x) = \emptyset$. Since

$$f_q^{(1)}, f_q^{(2)} \in H_q,$$

we must have $f_q^{(2)} = \alpha + \beta f_q^{(1)}$ for some $\alpha, \beta \in \kappa$. So we can let $f_p = \alpha + \beta f_p^{(1)}$ for all $p \in \pi_{\mathbb{B}_1}^{-1}(x)$ and $f_p = f_p^{(2)}$ for all $p \in \pi_{\mathbb{B}_2}^{-1}(x)$. In this way of clutching, we

conclude the following.

- (a) $f_p(\text{red}_p(t)) = \alpha + \beta c_i^{(1)}$ for all $i = 1, \dots, k$ whenever $p \in \pi_{\mathcal{B}_1}^{-1}(x)$ and $t \in \text{Tan}_{\Gamma_1}^{\rho_1+}(p) \cap \pi_{\mathcal{B}_1^*}^{-1}(t_i^{(1)})$,
- (b) $f_p(\text{red}_p(t)) = c_j^{(2)}$ for all $j = 1, \dots, l$ whenever $p \in \pi_{\mathcal{B}_2}^{-1}(x)$ and

$$t \in \text{Tan}_{\Gamma_2}^{\rho_2+}(p) \cap \pi_{\mathcal{B}_2^*}^{-1}(t_j^{(2)}),$$

- (c) in particular, (a) and (b) coincide at $p = q$ by our assumption as $f_q(\text{red}_q(t)) = \alpha + \beta f_q^{(1)}(\text{red}_q(t)) = f_q^{(2)}(\text{red}_q(t))$ for all $t \in \text{Tan}_{\Gamma}^{\rho+}(q) = \text{Tan}_{\Gamma_1}^{\rho_1+}(q) \cup \text{Tan}_{\Gamma_2}^{\rho_2+}(q)$.

The above construction guarantees that $\{f_p\}_{p \in \Gamma}$ is admissible in \mathcal{H} . Therefore, $(\mathcal{D}, \mathcal{H})$ is smoothable. ■

Example 7.3 For a smoothable pre-limit g_d^1 represented by $(\mathcal{D}, \mathcal{H})$, it is possible that we can construct different smoothings, or equivalently different pseudo-harmonic morphisms (see §3.1 for a precise definition) from the saturated metrized complex to a genus 0 metrized complex. Figure 7 is an example for a case of a saturated metrized complex of compact type whose underlying metric graph is a segment. Fix a rational function $f_1 \in H_{p_1}$. Suppose the value of f_1 on the marked point corresponding to p_1p_0 is c_0 and on the marked point corresponding to p_1p_3 is c_3 . Then the rational functions in H_{p_2} that take the value c_0 at the marked point corresponding to p_2p_0 form a one-dimensional subspace H of H_{p_2} . Let $f_2 \in H$ take the value c_3 on the marked point corresponding to p_2p_4 and let $f'_2 \in H$ take a value other than c_3 on the marked point corresponding to p_2p_4 . Then f_1 and f_2 can be used to construct a pseudo-harmonic morphism as in Figure 7(a), while f_1 and f'_2 can be used to construct a pseudo-harmonic morphism as in Figure 7(b).

7.2 Saturated Metrized Complexes with Genus- g Underlying Metric Graphs Containing g Separate Loops

For a generalization of saturated metrized complexes of compact type, we consider a saturated metrized complex \mathcal{C} whose underlying metric graph Γ has genus g and contains g separate loops $\Omega_1, \dots, \Omega_g$ (see Figure 8 for such a metric graph of genus 6). Here Ω_i and Ω_j are separate if the intersection of Ω_i and Ω_j is either empty or just a singleton. By the smoothing criterion, one prerequisite for being smoothable is solvability, *i.e.*, the integration along each Ω_i for $i = 1, \dots, g$ with respect to the global diagram induced by $(\mathcal{D}, \mathcal{H})$ is 0. We let ρ be a solution to the global diagram and \mathcal{B} be the corresponding bifurcation tree. Consider a loop $\Omega \in \{\Omega_1, \dots, \Omega_g\}$. We let $\Omega_{\min}(\rho)$ be the set of points where ρ restricted to Ω_i achieves minimum. Then $\Omega_{\min}(\rho)$ is a finite set with at least one element, and for each point $p \in \Omega_{\min}(\rho)$, there are exactly two forward tangent directions in $\text{Tan}_{\Omega}^{\rho+}(p)$ (forward tangent directions restricted to Ω). We say p is a closing point if these two tangent directions are locally equivalent in the local diagram at p induced by \mathcal{H} , and say p is an opening point otherwise. Denote the set of closing points in $\Omega_{\min}(\rho)$ by $\Omega_{\min}^c(\rho)$, and the

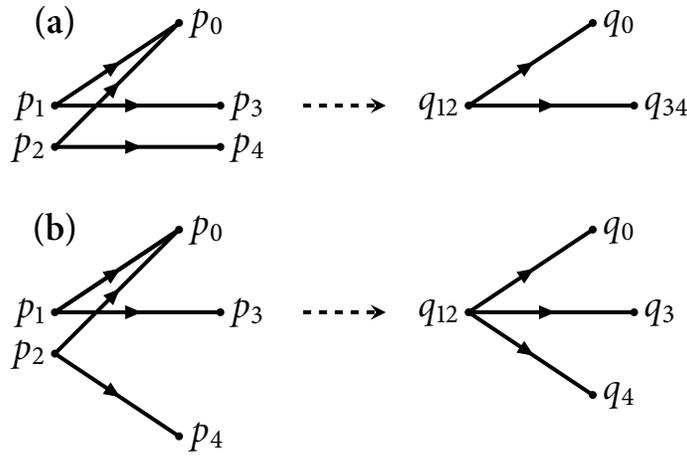


Figure 7: Examples of pseudo-harmonic morphisms (see §3.1 for a precise definition) that can be derived from the same $(\mathcal{D}, \mathcal{H})$ on a saturated metrized complex of compact type.

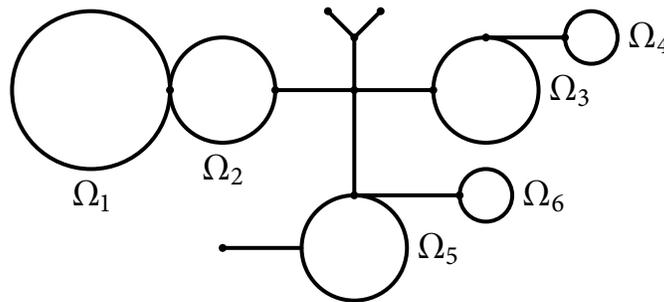


Figure 8: A genus-6 metric graph containing 6 separate loops $\Omega_1, \dots, \Omega_6$. All the loops are disjoint with each other except that Ω_1 and Ω_2 intersect at a single point.

set of opening points in $\Omega_{i,\min}(\rho)$ by $\Omega_{\min}^o(\rho)$. The following theorem says that determining whether $(\mathcal{D}, \mathcal{H})$ is smoothable can be reduced to a purely combinatorial point-counting problem in $\Omega_{\min}^c(\rho)$ and $\Omega_{\min}^o(\rho)$.

Theorem 7.4 *Let \mathfrak{C} be a saturated metrized complex whose underlying metric graph Γ has genus g and contains g separate loops $\Omega_1, \dots, \Omega_g$. Let $(\mathcal{D}, \mathcal{H})$ represent a solvable pre-limit g_d^1 on \mathfrak{C} with a solution ρ . Then $(\mathcal{D}, \mathcal{H})$ is smoothable if and only if the following are satisfied on each loop $\Omega \in \{\Omega_1, \dots, \Omega_g\}$.*

- (i) *If $\Omega_{\min}(\rho)$ is a singleton, the unique point $p \in \Omega_{\min}(\rho)$ is a closing point.*
- (ii) *If the cardinality of $\Omega_{\min}(\rho)$ is at least 2, then either $\Omega_{\min}^c(\rho) = \emptyset$ or $\Omega_{\min}^c(\rho)$ has the same parity as $\Omega_{\min}(\rho)$.*

Proof By Proposition 7.2, we can subdivide Γ into segments and loops and test whether $(\mathcal{D}, \mathcal{H})$ restricted to each segment or loop is smoothable. The case of segments can be dealt with analogously as for compact type saturated metrized complexes, and it only needs to show the case for $(\mathcal{D}, \mathcal{H})$ restricted to a single loop Ω . Without loss of generality, we can assume $\Gamma = \Omega$ and apply intrinsic global compatibility conditions to this specific graph.

Let ρ be a solution to $(\mathcal{D}, \mathcal{H})$ and \mathcal{B} be the corresponding bifurcation tree. We need to test the possibility of constructing an admissible $\{f_p\}_{p \in \Gamma} \in \mathcal{H}$. Actually, we will consider cases for all $p \in \pi_{\mathcal{B}}^{-1}(x)$ for each $x \in \mathcal{B}$ separately.

First we consider the root y of Ω . Note that the closed superlevel component $\iota_{\mathcal{B}}(y)$ corresponds to y and $\Omega_{\min}(\rho) = \pi_{\mathcal{B}}^{-1}(y)$.

- (1) When $\Omega_{\min}(\rho) = \{p\}$, there is only one open superlevel component with the boundary point p . Therefore, to pass the compatibility test, the two tangent directions in $\text{Tan}_{\Omega}^{\rho+}(p)$ must be locally equivalent, which means p must be a closing point.
- (2) When $\Omega_{\min}(\rho) = \{p_1, \dots, p_k\}$ with $k \geq 2$, there are exactly k open superlevel components β_1, \dots, β_k with the boundary points in $\Omega_{\min}(\rho)$ (the k open edges in Ω with end points in $\Omega_{\min}(\rho)$). We let the end points of β_i be p_i and p_{i+1} for $i = 1, \dots, k - 1$ and the end points of β_k be p_k and p_1 .

First note that for any point $\Omega_{\min}(\rho)$, if the two tangent directions in $\text{Tan}_{\Omega}^{\rho+}(p)$ are not locally equivalent, then for any two arbitrarily chosen distinct values, we can always find a rational function $f_p \in H_p$ taking these values, respectively, at the two reduction points in C_p corresponding to these two tangent directions (Lemma 2.12). Assigning values $c_1, \dots, c_k \in \kappa$ to β_1, \dots, β_k , respectively, we have the following cases.

- (a) If c_1, \dots, c_k are all distinct, then to pass the compatibility test, it is equivalent to say that the two tangent directions in $\text{Tan}_{\Omega}^{\rho+}(p_i)$ for each $i = 1, \dots, k$ must not be locally equivalent, i.e., p_1, \dots, p_k are all opening points.
- (b) If at least one point in $\Omega_{\min}(\rho)$ is a closing point, we can assume p_k is a closing point without loss of generality. Then to pass the compatibility test, c_k must be equal to c_{k-1} and the whole case reduces to dropping p_k from $\Omega_{\min}(\rho)$ and assigning values c_1, \dots, c_{k-1} to $\beta_1, \dots, \beta_{k-1}$.

It is straightforward to verify that the above arguments afford the conditions (i) and (ii) stated in the theorem.

It remains to show that the compatibility test can always be passed for those points x in $\mathcal{B} \setminus \{y\}$. Note that each forward tangent direction $t \in \text{Tan}_{\mathcal{B}}^+(x)$ corresponds to an open superlevel component $\tilde{\iota}_{\mathcal{B}}(t)$ of ρ . If $\iota_{\mathcal{B}}(x)$ is a singleton $\{p\}$, then $\text{Tan}_{\mathcal{B}}^{\rho+}(p) = \text{Tan}_{\mathcal{B}}^+(x) = \emptyset$ and we simply let f_p be any non-constant function in H_p . Otherwise, if $\partial\tilde{\iota}_{\mathcal{B}}(t)$ is the set of boundary points of $\tilde{\iota}_{\mathcal{B}}(t)$, we must have $\bigcup_{t \in \text{Tan}_{\mathcal{B}}^+(x)} \partial\tilde{\iota}_{\mathcal{B}}(t) = \pi_{\mathcal{B}}^{-1}(x)$, which is assumed in the following discussion.

Note that for $x \in \mathcal{B} \setminus \{y\}$, $\text{Tan}_{\mathcal{B}}^+(x)$ is either empty or satisfies the ordering property: starting from an arbitrary $t_1 \in \text{Tan}_{\mathcal{B}}^+(x)$, there exists an ordering t_1, t_2, \dots, t_k of all the tangent directions in $\text{Tan}_{\mathcal{B}}^+(x)$ such that $(\bigcup_{j=1}^{i-1} \partial\tilde{\iota}_{\mathcal{B}}(t_j)) \cap \partial\tilde{\iota}_{\mathcal{B}}(t_i)$ is a singleton $\{q_i\}$ for $i = 2, \dots, k$. In other words, we can rebuild the closed superlevel component

$\iota_{\mathcal{B}}(x)$ from the open superlevel components $\bar{\iota}_{\mathcal{B}}(t)$ with $t \in \text{Tan}_{\mathcal{B}}^+(x)$ one by one by choosing the attaching point from $\pi_{\mathcal{B}}^{-1}(x)$ properly.

Note that at each q_i , there exists two forward tangent directions in $\text{Tan}_{\Gamma}^{\rho+}(q_i)$. On the other hand, if a point p in $\pi_{\mathcal{B}}^{-1}(x)$ is not any of the q_i , then there is exactly one forward tangent direction in $\text{Tan}_{\Gamma}^{\rho+}(p)$.

We use the following procedure to assign values to the tangent directions in $\text{Tan}_{\mathcal{B}}^+(x)$ and find compatible rational functions $f_p \in H_p$ for all points $p \in \pi_{\mathcal{B}}^{-1}(x)$.

(1) We assign an arbitrary value $c_1 \in \kappa$ to t_1 . For each $p \in \partial \bar{\iota}_{\mathcal{B}}(t_1)$, we are able to find a nonconstant rational function $f_p \in H_p$ such that $f_p(\text{red}_p(t_{1,p})) = c_1$, where $t_{p,1}$ is the unique tangent in $\text{Tan}_{\Gamma}^{\rho+}(p)$ such that $\pi_{\mathcal{B}*}(t_{1,p}) = t_1$.

(2) Now suppose we have already assigned values $c_1, \dots, c_i \in \kappa$ to t_1, \dots, t_i , respectively, and found rational functions $f_p \in H_p$ for all $p \in \bigcup_{j=1}^i \partial \bar{\iota}_{\mathcal{B}}(t_j)$ such that for $j = 1, \dots, i$, we have $f_p(\text{red}_p(t)) = c_j$ as long as $t \in \text{Tan}_{\Gamma}^{\rho+}(p) \cap \pi_{\mathcal{B}}^{-1}(t_j)$. Note that q_{i+1} is the unique element in both $\bigcup_{j=1}^i \partial \bar{\iota}_{\mathcal{B}}(t_j)$ and $\partial \bar{\iota}_{\mathcal{B}}(t_{i+1})$. Let $t_{q_{i+1}}$ be the unique tangent direction in $\text{Tan}_{\Gamma}^{\rho+}(q_{i+1})$ such that $\pi_{\mathcal{B}*}(t_{q_{i+1}}) = t_{i+1}$. We let $c_{i+1} = f_{q_{i+1}}(\text{red}_{q_{i+1}}(t_{q_{i+1}}))$, and for each $p \in \partial \bar{\iota}_{\mathcal{B}}(t_{i+1}) \setminus \{t_{q_{i+1}}\}$, we let f_p be a nonconstant rational function in H_p such that $f_p(\text{red}_p(t)) = c_{i+1}$ where t is the unique tangent direction in $\text{Tan}_{\Gamma}^{\rho+}(p) \cap \pi_{\mathcal{B}}^{-1}(t_{i+1})$.

In this way, we derive a sequence of $c_1, \dots, c_k \in \kappa$ and a family of rational functions $\{f_p\}_{p \in \pi_{\mathcal{B}}^{-1}(x)}$. By our construction, for $j = 1, \dots, k$ and all $p \in \pi_{\mathcal{B}}^{-1}(x)$, we have $f_p(\text{red}_p(t)) = c_j$ as long as $t \in \text{Tan}_{\Gamma}^{\rho+}(p) \cap \pi_{\mathcal{B}}^{-1}(t_j)$. Hence, the compatibility test get passed at all $x \in \mathcal{B} \setminus \{y\}$. ■

Remark 7.5 Here are a few cases of the conditions in the above theorem.

- If $\#\Omega_{\min}(\rho) = 1$, then $\#\Omega_{\min}^c(\rho)$ can only be 1.
- If $\#\Omega_{\min}(\rho) = 2$, then $\#\Omega_{\min}^c(\rho)$ can only be 0 or 2.
- If $\#\Omega_{\min}(\rho) = 3$, then $\#\Omega_{\min}^c(\rho)$ can only be 0, 1, or 3.
- If $\#\Omega_{\min}(\rho) = 4$, then $\#\Omega_{\min}^c(\rho)$ can only be 0, 2, or 4.

7.3 Saturated Metrized Complexes of the Harris–Mumford Type

Here we study saturated metrized complexes arising from the construction of Harris and Mumford [24, Theorem 5]. These correspond to two types of saturated metrized complexes.

A *Harris–Mumford saturated metrized complex of type I* is a saturated metrized complex \mathcal{C} with underlying metric graph Γ homeomorphic to a topological bouquet graph obtained by gluing together finitely many (0 is allowed) circles along a single vertex o (Figure 9) while $g(\mathcal{C}) = g(C_o) + g(\Gamma)$. We call this central vertex o the *eye vertex* and C_o the *eye* of \mathcal{C} . We also call the middle points of the attached circles *petal vertices*. In particular, we have $g(\Gamma)$ petal vertices. We call the pair of marked points on the eye corresponding to the two edges connecting o and p_i the p_i -*marked points* on the eye.

A *Harris–Mumford saturated metrized complex of type II* is a saturated metrized complex \mathcal{C} with underlying metric graph Γ homeomorphic to a topological graph obtained by gluing together finitely many (0 is allowed) circles along the endpoints

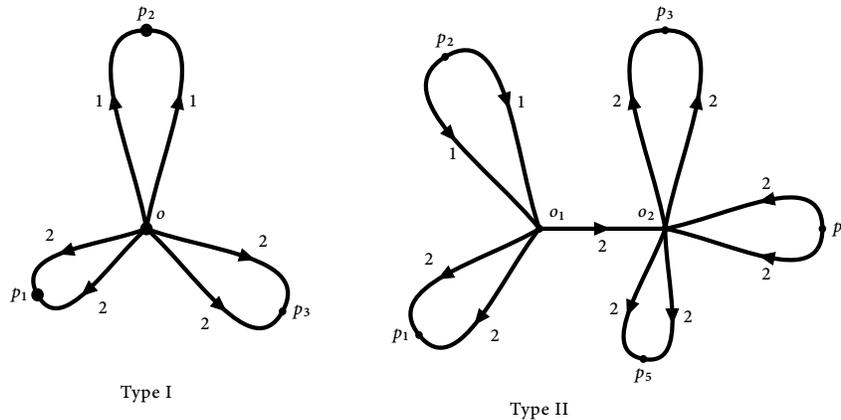


Figure 9: Example of the underlying metric graphs of Harris–Mumford saturated metrized complex of type I and II. For type I, o is the eye vertex, p_1 , p_2 , and p_3 are the three petal vertices. For type II, o_1 and o_2 are the eye vertices; p_1 , p_2 , p_3 , p_4 , and p_5 are the three petal vertices. Examples of regulated global diagrams are also shown with edge multiplicities marked.

o_1 and o_2 of a line segment (Figure 9), while $g(\mathcal{C}) = g(C_{o_1}) + g(C_{o_2}) + g(\Gamma)$. We call the two vertices o_1 and o_2 the *eye vertices* and the associated curves C_{o_1} and C_{o_2} the *eyes* of \mathcal{C} . Analogously, the middle points of the attached circles are called *petal vertices*. We also call the pair of marked points on the eyes corresponding to the two edges connecting the segment o_1o_2 and p_i the p_i -marked points.

For both types, we say a global diagram on Γ is *regulated* if edge multiplicities only possibly change across the eye vertices and the petal vertices (Figure 9). The following theorem is an analogue for Harris–Mumford [24, Theorem 5] saturated metrized complexes.

Theorem 7.6 (i) A Harris–Mumford saturated metrized complex \mathcal{C} of type I has a base-point-free smoothable limit g_d^1 whose global diagram is regulated if and only if there exists a rational function f of degree d on the eye which has the same value and ramification indices on each pair of the p_i -marked points for all petal vertices p_i .

(ii) A Harris–Mumford saturated metrized complex \mathcal{C} of type II has a base-point-free smoothable limit g_d^1 whose global diagram is regulated if and only if there exist rational functions f_1 and f_2 on the two eyes, respectively, such that

- (a) for all petal vertices p_i , the pair of p_i -marked points have the same value and ramification indices for the corresponding f_j ($j = 1$ or 2), and
- (b) $d = d_1 + d_2 - l$, where d_1 and d_2 are the degrees of f_1 and f_2 , respectively, and both ramification indices of f_1 and f_2 at respective marked points corresponding to the central segment take the same value l .

Proof For (i), first let $(\mathcal{D}, \mathcal{H})$ represent a base-point-free smoothable limit g_d^1 with regulated global diagram. Then the only possible global diagrams are those with the

same multiplicity along each pair of edges connecting the eye vertex o and a petal vertex. Consider a nonconstant function f in H_o where o is the eye vertex. By the smoothing criterion (version II, Theorem 2.6), f has the same value and ramification indices on each pair of the p_i -marked points for all petal vertices p_i . In addition, since $(\mathcal{D}, \mathcal{H})$ is base-point free, f must also have degree d .

On the other hand, if f is a rational function of degree d on the eye C_o that has the same value and ramification indices on each pair of the p_i -marked points for all petal vertices p_i , then we can construct a base-point-free smoothable limit g_d^1 represented by $(\mathcal{D}, \mathcal{H})$ with regulated global diagram in the following way. First we let H_o be the space of rational functions spanned by a constant function and f . Since for each petal vertex p_i , the distances of the two edges connecting p_i and o are the same and f has the same multiplicities on the pair of p_i -marked points, we can construct a regulated global diagram compatible with the local diagram for H_p (Remark 2.5). Associate projective lines with the points p other than o and we can always construct a two-dimensional space H_p of rational functions on C_p whose local diagram is compatible with the global diagram. Now for all $p \in \Gamma$, let $D_p = \sum_{t \in \text{In}(p)} m(p, t)(\text{red}_p(t)) + D_{f_p}^-$ based on the notations in Remark 2.5. By the smoothing criterion, the $(\{D_p\}_{p \in \Gamma}, \{H\}_{p \in \Gamma})$ constructed in this way represents a smoothable g_d^1 .

For (ii), suppose $(\mathcal{D}, \mathcal{H})$ represents a base-point-free smoothable limit g_d^1 with regulated global diagram. Then the only possible global diagrams are those with the same multiplicity along each pair of edges connecting a petal vertex to its corresponding eye vertex o_1 or o_2 and with unchanged multiplicity along o_1o_2 . Consider non-constant functions $f_1 \in H_{o_1}$ and $f_2 \in H_{o_2}$. Therefore condition (a) is satisfied using a similar argument as for (i) and the ramification indices of f_1 and f_2 at respective marked points u_1 and u_2 corresponding with the segment o_1o_2 take the same value, say l . Moreover, if u_1 is a pole of f_1 , then u_2 is not a pole of f_2 , and vice versa. Then the total degree d of $(\mathcal{D}, \mathcal{H})$ must be $d = d_1 + d_2 - l$, where d_1 and d_2 are the degrees of f_1 and f_2 , respectively.

On the other hand, suppose we have rational functions f'_1 on C_{o_1} and f'_2 on C_{o_2} satisfying conditions (a) and (b). If none of u_1 and u_2 are respectively poles of f_1 and f_2 , we let $f_1 = f'_1$ and $f_2 = 1/(f'_2 - f'_2(u_2))$. If both u_1 and u_2 are respectively poles of f_1 and f_2 , we let $f_1 = 1/f'_1$ and $f_2 = f'_2$. Now u_1 is not a pole of f_1 and u_2 is a pole of f_2 , while both f_1 and f_2 satisfy conditions (a) and (b). Moreover, the ramification index of f_1 at u_1 is the same as the ramification index of f_2 at u_2 . For $i = 1, 2$, let H_{o_i} be spanned by a constant function and f_i . Then we can construct a regulated global diagram compatible with the local diagrams associated with H_{o_1} and H_{o_2} and build a base-point-free smoothable limit g_d^1 on this global diagram following a similar approach as in (i). ■

7.4 The Smoothing Criterion on Metrized Complexes

Recall that in Remark 2.2, we compared the notion of metrized complex introduced by Amini and Baker [3] and the notion of saturated metrized complex. Consider a metric graph Γ and a vertex set A of Γ . Suppose \mathfrak{C}_A is a metrized complex with its underlying metric graph being Γ and each point $p \in A$ is associated with a curve C_p . As for

pre-limit g_d^r defined on saturated metrized complexes (Definition 2.5), we say that a pre-limit g_d^r on the metrized complex \mathcal{C}_A is represented by the data $(\mathcal{D}_A, \mathcal{H}_A)$, where $\mathcal{D}_A = (D_\Gamma, \{D_p\}_{p \in A})$ is an effective divisor of degree d on \mathcal{C}_A and $\mathcal{H}_A = \{H_p\}_{p \in A}$, where H_p is an $(r + 1)$ -dimensional subspace of the function field of C_p . (See [3] for more details about the divisor theory on metrized complexes.) In addition, as in Definition 3.8, we say $(\mathcal{D}_A, \mathcal{H}_A)$ is smoothable if there exists a g_d^r represented by (D, H) on a smooth proper curve X/\mathbb{K} specialized to $(\mathcal{D}_A, \mathcal{H}_A)$ on \mathcal{C}_A .

Consider a saturated metrized complex \mathcal{C} which is a saturation of \mathcal{C}_A (Remark 2.2). We say a divisor $\mathcal{D} = (D'_\Gamma, \{D'_p\}_{p \in \Gamma})$ on \mathcal{C} is a saturation of $\mathcal{D}_A = (D_\Gamma, \{D_p\}_{p \in A})$ on \mathcal{C}_A if $D_\Gamma = D'_\Gamma$ and $D_p = D'_p$ for all $p \in A$, and we say $\mathcal{H} = \{H'_p\}_{p \in \Gamma}$ is a saturation of $\mathcal{H}_A = \{H_p\}_{p \in A}$ if $H'_p = H_p$ for all $p \in A$. Then naturally we have the following statement.

Lemma 7.7 *A pre-limit g_d^r represented by $(\mathcal{D}_A, \mathcal{H}_A)$ on a metrized complex \mathcal{C}_A is smoothable if and only if there exists a saturation $(\mathcal{D}, \mathcal{H})$ of $(\mathcal{D}_A, \mathcal{H}_A)$ on a saturation \mathcal{C} of \mathcal{C}_A such that $(\mathcal{D}, \mathcal{H})$ is smoothable.*

Our smoothing criterion for the rank one case on saturated metrized complexes can be extended to the case for metrized complexes. The subtlety here is that we need to consider all possible saturations of $(\mathcal{D}_A, \mathcal{H}_A)$ and each saturation $(\mathcal{D}, \mathcal{H})$ affords its own global diagram that might either be solvable or not solvable. So to say $(\mathcal{D}_A, \mathcal{H}_A)$ is smoothable, we should be able to single out a solvable saturation $(\mathcal{D}, \mathcal{H})$ that satisfies the intrinsic global compatibility conditions. Fortunately, even though the process of determining rank-one smoothability on metrized complexes is usually more complicated, it is still finitely verifiable, as in the following example.

Example 7.8 Consider a metric banana graph Γ of genus 3 as shown in Figure 10 (a). Let $A = \{p, q\}$ be a vertex set of Γ and then suppose the four edges $L_1, L_2, L_3,$ and L_4 connecting p and q have the same length 1. Let o be the middle point of L_1 . Let \mathcal{C}_A be a metrized complex with underlying metric graph being Γ and associated curves being C_p and C_q with respect to p and q , respectively. Let $(\mathcal{D}_A, \mathcal{H}_A)$, where $\mathcal{D}_A = (D_\Gamma, \{D_p, D_q\})$ has degree d and $\mathcal{H}_A = \{H_p, H_q\}$ represent a pre-limit g_d^1 on \mathcal{C}_A . More specifically, we assume that $D_\Gamma(o) = 3$, D_p is compatible with H_p , D_q is compatible with H_q , and $D_\Gamma(p') = 0$ for every $p' \in \Gamma \setminus \{o, p, q\}$. The local diagrams at p and q induced by H_p and H_q , respectively, are also shown in Figure 10 (a), i.e., the tangent directions at p corresponding to $L_1, L_2, L_3,$ and L_4 have multiplicities $-1, 2, 1,$ and 1 , respectively, and the tangent directions at q corresponding to $L_1, L_2, L_3,$ and L_4 have multiplicities $-1, 1, 1,$ and 2 , respectively. Let \mathcal{C} be a saturation of \mathcal{C}_A . Then $(\mathcal{D}_A, \mathcal{H}_A)$ is smoothable if and only if there exists a saturation of $(\mathcal{D}_A, \mathcal{H}_A)$ on \mathcal{C} which is smoothable. Let $(\mathcal{D}, \mathcal{H})$ be a diagrammatic saturation of $(\mathcal{D}_A, \mathcal{H}_A)$ on \mathcal{C} . First we need to determine whether $(\mathcal{D}, \mathcal{H})$ is solvable. Figure 10 (b) shows the allowable cases of the discrete 1-form ω in the global diagram of $(\mathcal{D}, \mathcal{H})$ restricted to the four edges $L_1, L_2, L_3,$ and L_4 . Note that the variation of the multiplicity along $L_2, L_3,$ or L_4 from p to q must be non-increasing since $D_\Gamma(p') = 0$ for p' in the interior of these edges. In addition, we have the following restrictions for the lengths of segments with uniform multiplicities: $x_2 + y_2 + z_2 = x_3 + y_3 = x_4 + y_4 + z_4 = 1$,

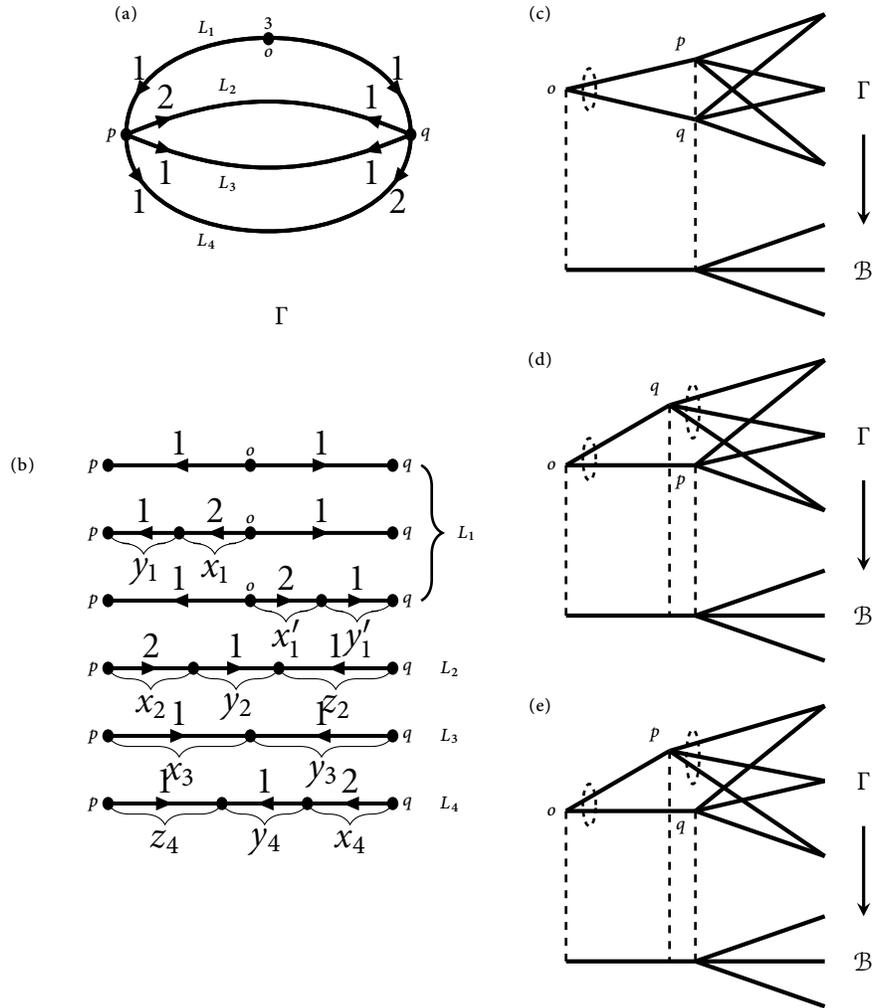


Figure 10: An example of the smoothability test on a metrized complex with its underlying metric graph being a banana graph of genus 3 and its vertex set being the two valence-4 points.

$x_2, z_2, x_3, y_3, x_4, z_4 > 0$, and $y_2, y_4 \geq 0$. The case for L_1 is a little bit special, since D_Γ has value 3 at the middle point o of L_1 . Again, the variation of multiplicity from o to p or q must be non-increasing. Thus the two tangent directions at o (one from o to p and the other from o to q) must be outgoing tangent directions in the local diagram at o induced by $(\mathcal{D}, \mathcal{H})$. In addition, these two tangent directions must be locally equivalent if we want $(\mathcal{D}, \mathcal{H})$ to be smoothable since o must map to the root of the corresponding bifurcation tree and there is only one forward tangent direction from

the root (see Figure 10 (c), (d), and (e)). The three possible cases on L_1 are shown in Figure 10 (b) and the restrictions are $x_1 + y_1 = x'_1 + y'_1 = 1/2$, $x_1, x'_1 \geq 0$, and $y_1, y'_1 > 0$.

Let $\int_{L_i} \omega$ be the integral of ω along L_i from p to q . Then $(\mathcal{D}, \mathcal{H})$ is solvable if and only if $\int_{L_1} \omega = \int_{L_2} \omega = \int_{L_3} \omega = \int_{L_4} \omega$. Depending on the cases, $\int_{L_1} \omega = 0$ or $1/2 - 2x_1 - y_1$ or $2x'_1 + y'_1 - 1/2$. Therefore, $-1/2 < \int_{L_1} \omega < 1/2$. Similarly, $-1 < \int_{L_2} \omega = 2x_2 + y_2 - z_2 < 2$, $-1 < \int_{L_3} \omega = x_3 - y_3 < 1$, and $-2 < \int_{L_4} \omega = -2x_4 - y_4 + z_4 < 1$. Therefore, by adjusting the values of $x_2, y_2, z_2, x_3, y_3, x_4, y_4, z_4$, for the three cases of the global diagram restricted to L_1 in Figure 10 (b), we can always find a solvable global diagram and the corresponding $(\mathcal{D}, \mathcal{H})$. Moreover, the projection from Γ to a bifurcation tree corresponding to each of the three solvable cases are sketched in Figure 10 (c), (d), and (e), respectively.

For the case in Figure 10 (c), p and q map to the same point that has three forward tangent directions in the bifurcation tree. Therefore, to determine the smoothability of a solvable $(\mathcal{D}, \mathcal{H})$ whose projection to its bifurcation tree is as in Figure 10 (c), the intrinsic global compatibility conditions are trivially satisfied and only need to be tested for H_p and H_q . More precisely, suppose H_p has a basis $\{1, f_p\}$, H_q has a basis $\{1, f_q\}$, u_2, u_3 , and u_4 are the marked points on C_p that are the reductions of the tangent directions at p corresponding to the edges L_2, L_3 , and L_4 , respectively, and v_2, v_3 , and v_4 are the marked points on C_q that are the reduction of the tangent directions at q corresponding with the edges L_2, L_3 , and L_4 , respectively. Then by Algorithm 2.13, $(\mathcal{D}, \mathcal{H})$ is smoothable if and only if the linear equations $\alpha_p + f_p(u_i)\beta_p = \alpha_q + f_q(v_i)\beta_q$ have a solution for the unknowns $\alpha_p, \beta_p, \alpha_q, \beta_q$ such that $\beta_p \neq 0$ and $\beta_q \neq 0$.

To determine the smoothability of a solvable $(\mathcal{D}, \mathcal{H})$ whose projection to its bifurcation tree is as in Figure 10 (d) (respectively, Figure 10 (e)), one only needs to test the intrinsic global compatibility conditions at p (respectively, at q), which reduces to saying that $(\mathcal{D}, \mathcal{H})$ is smoothable if and only if the three forward tangent directions at p (respectively, at q) are locally equivalent.

Finally, let us sum up the smoothability test for $(\mathcal{D}_A, \mathcal{H}_A)$ based on the above discussion on all cases of possible saturations of $(\mathcal{D}_A, \mathcal{H}_A)$ as follows: $(\mathcal{D}_A, \mathcal{H}_A)$ is smoothable if and only if either at one of p and q , the three forward tangent directions are locally equivalent, or the linear equations $\alpha_p + f_p(u_i)\beta_p = \alpha_q + f_q(v_i)\beta_q$ have a solution for the unknowns $\alpha_p, \beta_p, \alpha_q, \beta_q$ such that $\beta_p \neq 0$ and $\beta_q \neq 0$.

A Berkovich Skeleta and Saturated Metrized Complexes

A.1 Saturated Metrized Complex associated with a Berkovich Skeleton

We begin by briefly recalling the concept of the skeleton of the Berkovich analytic curve. A semistable vertex set V of X^{an} is a finite set of type II points of X^{an} such that the complement of V in X^{an} is a disjoint union of a finite number of open annuli and an infinite number of open balls. Let $\Sigma(X^{\text{an}}, V)$ be the skeleton of X^{an} with respect to a semistable vertex set V .

In order to associate a saturated metrized complex $\mathfrak{C}(\Sigma)$ with $\Sigma(X^{\text{an}}, V)$, we must associate the following data with it: a metric graph Γ , a smooth algebraic curve C_p for each point $p \in \Gamma$, and for each C_p , we must specify a set A_p of marked points that are in bijection with the set of tangent directions at p . The metric graph Γ underlying the

saturated metrized complex is defined as being isometric to $\Sigma(X^{\text{an}}, V)$. We associate the algebraic curve C_p with each point $p \in \Gamma$ as follows: since the value group of \mathbb{K} is \mathbb{R} , every point in $\Sigma(X^{\text{an}}, V)$ is a type II point [8]. Hence, the double residue field has transcendence degree one over κ and is isomorphic to the function field of a smooth curve over κ . This smooth curve is well defined up to isomorphism and we associate this curve C_p with the point $p \in \Gamma$. We define marked points associated with the algebraic curve C_p as follows: let x be the type II point corresponding to p , the set of tangent directions at any type II point in X^{an} has a canonical bijection with the set of discrete valuations of the double residue field at that point [8, Chapter 1]. The set of discrete valuations of the double residue field is in turn in bijection with the set of closed points of C_p [8, Chapter 1]. For each tangent direction $t \in \text{Tan}_\Gamma(p)$, we define its marked point as the point in C_p associated with the corresponding tangent direction in the skeleton $\Sigma(X^{\text{an}}, V)$. Note that the marked point associated with each tangent direction is distinct.

Lemma A.1 For any skeleton $\Sigma(X^{\text{an}}, V)$ of X^{an} , the data $\mathfrak{C}(\Sigma)$ define a saturated metrized complex. In particular, for all but a finite number of points in Γ , the curve C_p is a projective line over κ .

Proof To show that $\mathfrak{C}(\Sigma)$ is a saturated metrized complex, we must verify that the curve C_p has genus zero for all but finitely many points of Γ . Using [6, (5.45.1)], we have $g(X) = g(\Gamma) + \sum_{p \in \Gamma} g(C_p)$. Hence, $g(C_p) = 0$ for all but finitely many p . ■

Remark A.2 The semistable vertex sets of X are in one-to-one correspondence with the semistable models of X . (We refer to [6, §5.14 and §5.29] for a detailed treatment of the topic.) Via this correspondence, we can associate a saturated metrized complex with a semistable model of X . This saturated metrized complex is the "limit" of the metrized complexes associated with semistable models obtained by successively blowing up the special fiber at its nodes.

We define a morphism from $\tau_*: \text{Div}(X) \rightarrow \text{Div}(\mathfrak{C}(\Sigma))$ called the *specialization map* and a map that takes a rational function on X to a rational function on $\mathfrak{C}(\Sigma)$ is called the *reduction map*. We follow the analogous construction for metrized complexes by Amini and Baker [3, §4].

A.2 Specialization Map

Suppose that $r_V: X^{\text{an}} \rightarrow \Sigma(X^{\text{an}}, V)$ is the retraction map and let $\{r_{V,s}\}_{s \in [0,1]}$ be the family of retraction maps associated with the deformation retraction from X^{an} to $\Sigma(X^{\text{an}}, V)$. In particular, $r_{V,1} = r_V$. For a closed point $z \in X$, the point $r_V(z)$ has a unique tangent direction $t_V^{\text{an}}(z)$ in X^{an} that lies in the image of the retraction map $r_{V,s}$, where s is in an open neighborhood of 1. The map τ_* takes z to the point $(r_V(z), \text{red}_p(t_V^{\text{an}}(z)))$ on $\mathfrak{C}(\Sigma)$, where $\text{red}_p(t_V^{\text{an}}(z))$ is the marked point in C_p corresponding to the tangent direction $t_V^{\text{an}}(z)$. We extend this map linearly to define a *specialization map* from $\text{Div}(X)$ to $\text{Div}(\mathfrak{C}(\Sigma))$.

Lemma A.3 *The specialization map τ_* is a homomorphism from $\text{Div}(X)$ to $\text{Div}(\mathcal{C}(\Sigma))$*

that takes effective divisors on $\text{Div}(X)$ to effective divisors on $\text{Div}(\mathcal{C}(\Sigma))$. The image of τ_ is the set of all divisors $(D_\Gamma, \{D_p\}_{p \in \Gamma}) \in \text{Div}(\mathcal{C}(\Sigma))$ such that the support of D_p is contained in the set $C_p \setminus A_p$ for all $p \in \Gamma$.*

A.3 Reduction of Rational Functions

Consider a point $p \in \Gamma$ and let x be the corresponding type (2) point in $\Sigma(X^{\text{an}}, V)$. By $f(x)$, we denote the multiplicative semi-norm defined by x evaluated at f and let $c \in \mathbb{K}^*$ such that $|c| = |f(x)|$. Let $\tilde{H}(x)$ be the double residue field of x and note that the field $\tilde{H}(x)$ is isomorphic to the function field of C_p . Suppose that f maps to f_x in $\tilde{H}(x)$. The reduction map takes f to $(c^{-1}f)_x$; we denote $(c^{-1}f)_x$ by \tilde{f}_x and the corresponding rational function in C_p by \tilde{f}_p . Note that f_x is only defined up to multiplication by κ^* and hence, its divisor is well defined.

Lemma A.4 ([3, Lemma 4.3]) *The dimension of any finite-dimensional subspace of $\kappa(X)$ is preserved by reduction.*

Given a rational function f on X , we let f_Γ be a rational function on Γ given by the restriction to the skeleton $\Gamma = \Sigma(X^{\text{an}}, V)$ of the function $\log|f|: X^{\text{an}} \rightarrow \mathbb{R} \cup \{\pm\infty\}$. Hence, given a rational function f on X , we associate a rational function

$$\mathfrak{f} = (f_\Gamma, \{\tilde{f}_p\}_{p \in \Gamma})$$

on $\mathcal{C}(\Sigma)$. The following version of the Poincaré–Lelong Formula for saturated metrized complexes establishes a compatibility between the specialization and the reduction maps.

Theorem A.5 (Poincaré–Lelong Formula) *For any non-zero rational function f on X , suppose that \mathfrak{f} is the reduction of f on $\mathcal{C}(\Sigma)$, so we have $\tau_*(\text{div}(f)) = \text{div}(\mathfrak{f})$. Hence, the map τ_* takes principal divisors in X to principal divisors in $\mathcal{C}(\Sigma)$.*

Proof For a point x in the skeleton $\Sigma(X^{\text{an}}, V)$, we partition the set of Tan_x of tangent directions at x into the tangent directions in $\Sigma(X^{\text{an}}, V)$ and its complement and denote them by $\text{Tan}_{i,x}$ and $\text{Tan}_{r,x}$, respectively. By parts (2) and (5) of the slope formula [6, Theorem 5.69], we note that $\text{ord}_t(\tilde{f}_x) = 0$ for all but points $x \in \Sigma(X^{\text{an}}, V)$ and $t \in \text{Tan}_{r,x}$ except those that lie in the image (under the retraction map) of the support of $\text{div}(f)$. By part (2) of the slope formula, $\text{sl}_t(f_\Gamma) = \text{ord}_t(f_x)$. Hence, $\text{div}(\mathfrak{f})$ has support at a finite number of points and its support coincides with the support of $\tau_*(\text{div}(f))$. Hence, $\text{div}(\mathfrak{f})$ is a divisor (not just a pseudo-divisor). Let S be the union of the support of $\text{div}(f_\Gamma)$ and the points of Γ with valence at least three. Thus, $\tau_*(\text{div}(f))$ and $\text{div}(\mathfrak{f})$ coincide on points in $\Gamma \setminus S$. Consider the metrized complex $\mathcal{C}(\Sigma)|_S$ obtained by restricting \mathcal{C} to S . More precisely, $\mathcal{C}(\Sigma)|_S$ is a metrized complex whose metric graph is Γ with the model given by the set S and the algebraic curves C_v for every point in $v \in S$ and the marked points exactly as in $\mathcal{C}(\Sigma)$. By the Poincaré–Lelong formula [3], we have $\tau_*(\text{div}(f))$ and $\text{div}(\mathfrak{f})$ coincide on $\mathcal{C}(\Sigma)|_S$. ■

B $\Lambda_{\mathcal{D},\mathcal{H}}^{(1)}$, $\Lambda_{\mathcal{D},\mathcal{H}}^{(2)}$, $\Lambda_{\mathcal{D},\mathcal{H}}^{(3)}$ and $\Lambda_{\mathcal{D},\mathcal{H}}^{(4)}$ as Partially Ordered Sets

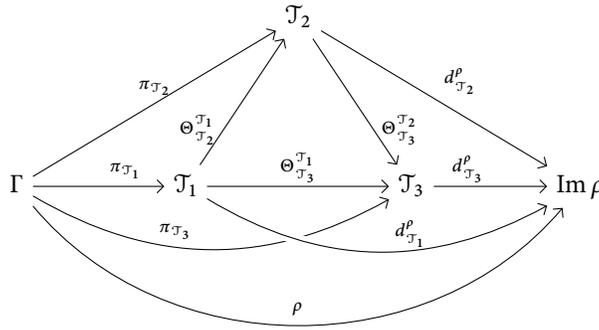
In this section, we show that a partial order can be naturally imposed on the spaces $\Lambda_{\mathcal{D},\mathcal{H}}^{(1)}$, $\Lambda_{\mathcal{D},\mathcal{H}}^{(2)}$, $\Lambda_{\mathcal{D},\mathcal{H}}^{(3)}$ and $\Lambda_{\mathcal{D},\mathcal{H}}^{(4)}$ of partition trees.

Let ρ be a solution to $(\mathcal{D}, \mathcal{H})$ and \mathcal{B} be the corresponding partition tree. Recall that for a partition tree $(\mathcal{T}, \pi_{\mathcal{T}})$ in $\Lambda_{\mathcal{D},\mathcal{H}}^{(1)} = \Lambda_{\rho}$, we have a canonical projection $\Theta_{\mathcal{T}}^{\mathcal{B}}$ from \mathcal{B} to \mathcal{T} (Proposition 4.12) which induces a partition P_c of $(d_{\mathcal{B}}^{\rho})^{-1}(c)$ for any $c \in \text{Im } \rho$ (Remark 4.13).

For two partition trees $(\mathcal{T}_1, \pi_{\mathcal{T}_1})$ and $(\mathcal{T}_2, \pi_{\mathcal{T}_2})$, we say $(\mathcal{T}_1, \pi_{\mathcal{T}_1}) \leq (\mathcal{T}_2, \pi_{\mathcal{T}_2})$, or simply $\mathcal{T}_1 \leq \mathcal{T}_2$, if the partition of $(d_{\mathcal{B}}^{\rho})^{-1}(c)$ induced by \mathcal{T}_1 is a refinement of the partition induced by \mathcal{T}_2 for each $c \in \text{Im } \rho$. Moreover, if $\mathcal{T}_1 \leq \mathcal{T}_2$, there is a natural map $\Theta_{\mathcal{T}_2}^{\mathcal{T}_1}: \mathcal{T}_1 \rightarrow \mathcal{T}_2$ with $x \mapsto y$ if $(\Theta_{\mathcal{T}}^{\mathcal{B}})^{-1}(x) \subseteq (\Theta_{\mathcal{T}}^{\mathcal{B}})^{-1}(y)$. Clearly, in this sense, the coarsest partition tree $(\text{Im } \rho, \rho)$ and the finest partition tree $(\mathcal{B}, \pi_{\mathcal{B}})$ are the maximum and minimum of Λ_{ρ} , respectively, (recall that as a rooted metric tree, $\text{Im } \rho = [\min_{p \in \Gamma} \rho(p), \max_{p \in \Gamma} \rho(p)]$ has its root at $\min_{p \in \Gamma} \rho(p)$).

The following lemma is a natural consequence of the definitions of partition trees and the maps between them.

Lemma B.1 *If $\mathcal{T}_1 \leq \mathcal{T}_2 \leq \mathcal{T}_3$ as partition trees, the following diagram commutes.*



■

The following lemma says that $\Lambda_{\mathcal{D},\mathcal{H}}^{(2)}$ is lower closed.

Lemma B.2 *If $\mathcal{T} \in \Lambda_{\mathcal{D},\mathcal{H}}^{(2)}$, then any element $\mathcal{T}' \in \Lambda_{\mathcal{D},\mathcal{H}}^{(1)}$ with $\mathcal{T}' \leq \mathcal{T}$ is also in $\Lambda_{\mathcal{D},\mathcal{H}}^{(2)}$.*

Proof Recall that, by definition, to say $\mathcal{T} \in \Lambda_{\mathcal{D},\mathcal{H}}^{(2)}$ is equivalent to saying that for every point $p \in \Gamma$ and each pair of tangent directions $t_1, t_2 \in \text{Tan}_{\Gamma}^+(p)$, t_1 is locally equivalent to t_2 if $\pi_{\mathcal{T}*}(t_1) = \pi_{\mathcal{T}*}(t_2)$.

On the other hand, $\mathcal{T}' \leq \mathcal{T}$ means that \mathcal{T}' induces finer partitions on forward tangent directions of \mathcal{B} than \mathcal{T} . Thus if $\pi_{\mathcal{T}'*}(t_1) = \pi_{\mathcal{T}'*}(t_2)$, then $\pi_{\mathcal{T}*}(t_1) = \pi_{\mathcal{T}*}(t_2)$ and we conclude $\mathcal{T}' \in \Lambda_{\mathcal{D},\mathcal{H}}^{(2)}$. ■

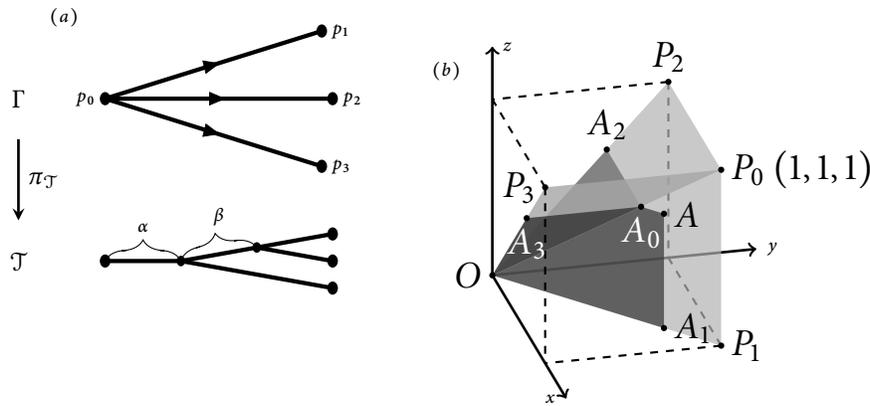


Figure 11: (a) A metric tree Γ with the projection onto a partition tree. (b) $\Lambda_{\mathcal{D}, \mathcal{H}}^{(1)}$ is a union of three triangles $\triangle OP_0P_1$, $\triangle OP_0P_2$, and $\triangle OP_0P_3$, and the space $\{T \in \Lambda_{\mathcal{D}, \mathcal{H}}^{(1)} \mid T \leq A\}$ is the union of triangles $\triangle OA_0A_2$, $\triangle OA_0A_3$, and trapezoid OA_0AA_1 .

Since \mathcal{B} is the minimum element of $\Lambda_{\mathcal{D}, \mathcal{H}}^{(1)}$, we have the following corollary.

Corollary B.3 *If $\Lambda_{\mathcal{D}, \mathcal{H}}^{(2)}$ is nonempty, then \mathcal{B} is an element of $\Lambda_{\mathcal{D}, \mathcal{H}}^{(2)}$.*

Example B.4 We consider a simple metric graph Γ that is a metric tree with root p_0 and leaves $p_1, p_2,$ and p_3 and all edge lengths being 1 as shown in Figure 11 (a). Let \mathcal{C} be a saturated metrized complex of genus 0 with underlying metric graph Γ . Suppose the global diagram of a base-point-free diagrammatic pre-limit $g_d^1(\mathcal{D}, \mathcal{H})$ on \mathcal{C} has its differential form part with multiplicity 1 on each direct edge from p_0 to p_i ($i = 1, 2, 3$). Then each partition tree \mathcal{T} in $\Lambda_{\mathcal{D}, \mathcal{H}}^{(1)}$ can be derived by the following procedure: (1) glue the three edges $p_0p_1, p_0p_2,$ and p_0p_3 continuously from p_0 of length α , and then (2) choose two edges and continue the gluing on the selected edges for length β (as shown in Figure 11 (a)). Note that we have three cases for step (2) based on which two edges are selected: Case 1 for p_0p_1 and p_0p_2 being selected, Case 2 for p_0p_2 and p_0p_3 being selected, and Case 3 for p_0p_1 and p_0p_3 being selected. Let x be the total length glued for p_0p_1 , y the total length glued for p_0p_2 , and z the total length glued for p_0p_3 . Then we can represent \mathcal{T} uniquely by a point with coordinates (x, y, z) . In particular, the coordinates are $(\alpha + \beta, \alpha + \beta, \alpha)$ for Case 1, $(\alpha, \alpha + \beta, \alpha + \beta)$ for Case 2, and $(\alpha + \beta, \alpha, \alpha + \beta)$ for Case 3. Therefore $\Lambda_{\mathcal{D}, \mathcal{H}}^{(1)}$ is a union of three triangles inside a unit cube. As shown in Figure 11 (b), $\Lambda_{\mathcal{D}, \mathcal{H}}^{(1)} = \triangle OP_0P_1 \cup \triangle OP_0P_2 \cup \triangle OP_0P_3$, where $\triangle OP_0P_1$ corresponds to Case 1, $\triangle OP_0P_2$ corresponds to Case 2 and $\triangle OP_0P_3$ corresponds to Case 3. Moreover, let point A with coordinates $(\alpha_A + \beta_A, \alpha_A + \beta_A, \alpha_A)$ be a point in $\triangle OP_0P_1$. In Figure 11 (b), the space $\{T \in \Lambda_{\mathcal{D}, \mathcal{H}}^{(1)} \mid T \leq A\}$ is shown as the darker region (polyhedral complex with vertices $O, A, A_0, A_1, A_2,$ and A_3).

For this simple metric graph, to derive $\Lambda_{\mathcal{D}, \mathcal{H}}^{(2)}$, $\Lambda_{\mathcal{D}, \mathcal{H}}^{(3)}$, and $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$, the only local data needing to be examined are for the curve at p_0 . Note that this also means $\Lambda_{\mathcal{D}, \mathcal{H}}^{(3)} = \Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$. Denote the outgoing tangent directions at p_0 by t_1 , t_2 , and t_3 , where t_i is the tangent direction from p_0 to p_i . We have the following cases under the assumption that $(\mathcal{D}, \mathcal{H})$ is base-point free (the conventional notations of open, closed and half-open-half-closed intervals are used for those of segments).

Case 1: $(\mathcal{D}, \mathcal{H})$ has degree 1. Then the local partition at p_0 is the finest partition $\{\{t_1\}, \{t_2\}, \{t_3\}\}$. We have $\Lambda_{\mathcal{D}, \mathcal{H}}^{(2)} = \Lambda_{\mathcal{D}, \mathcal{H}}^{(3)} = \Lambda_{\mathcal{D}, \mathcal{H}}^{(4)} = \{O\}$.

Case 2: $(\mathcal{D}, \mathcal{H})$ has degree 2. Then the local partition at p_0 is made of a singleton and a set of two elements.

Subcase 2.1: The local partition at p_0 is $\{\{t_1, t_2\}, \{t_3\}\}$. Then $\Lambda_{\mathcal{D}, \mathcal{H}}^{(2)} = [O, P_1]$ and $\Lambda_{\mathcal{D}, \mathcal{H}}^{(3)} = \Lambda_{\mathcal{D}, \mathcal{H}}^{(4)} = (O, P_1]$.

Subcase 2.2: The local partition at p_0 is $\{\{t_2, t_3\}, \{t_1\}\}$. Then $\Lambda_{\mathcal{D}, \mathcal{H}}^{(2)} = [O, P_2]$ and $\Lambda_{\mathcal{D}, \mathcal{H}}^{(3)} = \Lambda_{\mathcal{D}, \mathcal{H}}^{(4)} = (O, P_2]$.

Subcase 2.3: The local partition at p_0 is $\{\{t_1, t_3\}, \{t_2\}\}$. Then $\Lambda_{\mathcal{D}, \mathcal{H}}^{(2)} = [O, P_3]$ and $\Lambda_{\mathcal{D}, \mathcal{H}}^{(3)} = \Lambda_{\mathcal{D}, \mathcal{H}}^{(4)} = (O, P_3]$.

Case 3: $(\mathcal{D}, \mathcal{H})$ has degree 3. Then all tangent directions at p_0 are locally equivalent. We have $\Lambda_{\mathcal{D}, \mathcal{H}}^{(2)} = \Lambda_{\mathcal{D}, \mathcal{H}}^{(1)}$ and $\Lambda_{\mathcal{D}, \mathcal{H}}^{(3)} = \Lambda_{\mathcal{D}, \mathcal{H}}^{(4)} = \Lambda_{\mathcal{D}, \mathcal{H}}^{(1)} \setminus ([O, P_1] \cup [O, P_2] \cup [O, P_3])$. One can consider $\Lambda_{\mathcal{D}, \mathcal{H}}^{(1)}$ as the compactification of $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$'s in case (3) by $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$ in case (1) and (2).

Remark B.5 The spaces $\Lambda_{\mathcal{D}, \mathcal{H}}^{(1)}$, $\Lambda_{\mathcal{D}, \mathcal{H}}^{(2)}$, $\Lambda_{\mathcal{D}, \mathcal{H}}^{(3)}$, and $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$ have much richer structures, e.g., lattice structure, metric, and convexity, which will be presented in our follow-up work. Moreover, we expect that the space $\Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$ has an interpretation in terms of a skeleton of the analytification of the following moduli space of maps: the moduli space of all maps of the form $X \rightarrow \mathbb{P}_{\mathbb{K}}^1$, where X is the smooth curve in the commutative diagram 1.2. Results of this flavor have been obtained by Cavalieri *et al.* [13] for spaces of admissible covers and for moduli spaces of curves by Abramovich *et al.* [1]. As we vary over all limits g_d^1 on the saturated metrized complex \mathcal{C} , the space $\cup_{(\mathcal{D}, \mathcal{H})} \Lambda_{\mathcal{D}, \mathcal{H}}^{(4)}$ parameterizes the space of metric trees underlying all smoothings of limit g_d^1 in \mathcal{C} .

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