J. Aust. Math. Soc. **84** (2008), 233–245
doi:10.1017/S1446788708000190

TRANSVERSAL HARMONIC THEORY FOR TRANSVERSALLY SYMPLECTIC FLOWS

HONG KYUNG PAK

(Received 3 November 2005; accepted 4 October 2006)

Communicated by C. D. Hodgson

Abstract

We develop the transversal harmonic theory for a transversally symplectic flow on a manifold and establish the transversal hard Lefschetz theorem. Our main results extend the cases for a contact manifold (H. Kitahara and H. K. Pak, 'A note on harmonic forms on a compact manifold', *Kyungpook Math. J.* **43** (2003), 1–10) and for an almost cosymplectic manifold (R. Ibanez, 'Harmonic cohomology classes of almost cosymplectic manifolds', *Michigan Math. J.* **44** (1997), 183–199). For the point foliation these are the results obtained by Brylinski ('A differential complex for Poisson manifold', *J. Differential Geom.* **28** (1988), 93–114), Haller ('Harmonic cohomology of symplectic manifolds', *Adv. Math.* **180** (2003), 87–103), Mathieu ('Harmonic cohomology classes of symplectic manifolds', *Comment. Math. Helv.* **70** (1995), 1–9) and Yan ('Hodge structure on symplectic manifolds', *Adv. Math.* **120** (1996), 143–154).

2000 *Mathematics subject classification*: 53C20, 57R30.

Keywords and phrases: transversally symplectic foliation, basic cohomology, contact manifold, symplectic manifold, hard Lefschetz theorem.

1. Introduction

Let $(N, \bar{\Phi})$ be a symplectic manifold of dimension $2n$. On the graded algebra $\Omega^*(N) = \sum_k \Omega^k(N)$ of all differential forms on N an operator $\bar{L}$ defined by

$$\bar{L}\alpha := \bar{\Phi} \wedge \alpha, \quad \forall \alpha \in \Omega^k(N),$$

induces a homomorphism

$$\bar{L} : H^k(N) \longrightarrow H^{k+2}(N),$$

in the de Rham cohomology $H^*(N)$ of N.

Brylinski [5] introduced the notion of symplectic harmonic forms on a symplectic manifold. He conjectured that on a closed symplectic manifold any de Rham cohomology class has a symplectic harmonic representative. This conjecture is true for a closed Kähler manifold. However, it is not true in general. For example, Mathieu [12] gave a counter example of a four-dimensional closed nilmanifold (Kodaira–Thurston surface). Furthermore, he showed the following result.

THEOREM A. *Let $(N, \bar{\Phi})$ be a symplectic manifold of dimension $2n$. Then the following are equivalent:*

(1) *any de Rham cohomology class has a symplectic harmonic representative;*
(2) *for any $k \leq n$, the homomorphism*

$$\bar{L}^k : H^{n-k}(N) \longrightarrow H^{n+k}(N),$$

is surjective.

It should be noted that in view of the Poincaré duality, when N is closed $\bar{L}^k$ is an isomorphism. Theorem A is a generalization of the hard Lefschetz theorem for a closed Kähler manifold (refer to [8, 27] for another proof). On the other hand, there are known many closed symplectic manifolds which do not satisfy the hard Lefschetz theorem [2, 13, 27].

Let (M, ω) be a contact manifold of dimension $2n + 1$. Then we have the Reeb vector field T, that is, a nowhere vanishing vector field on M such that

$$\iota(T)\omega = 1, \quad \mathcal{L}_T\omega = 0, \tag{1.1}$$

which defines a flow $\mathcal{F}$, called the contact flow. Here and hereafter, $\iota(\cdot)$ and $\mathcal{L}_{(\cdot)}$ denote the interior product and the Lie derivative with respect to the vector field $(\cdot)$ respectively. The harmonic theory on a closed Sasakian manifold has been extensively studied by many geometers (say, [4, 7, 15, 22, 23]). Some of these results have been extended to closed contact metric manifolds (say [19, 20]). The harmonic theory developed before is usually founded on an adapted metric g_ω defined by

$$g_\omega = \omega \otimes \omega + d\omega \circ J, \tag{1.2}$$

where J is the complex structure on the contact distribution $\mathcal{D} := \ker \omega$ [26].

We observe that the contact flow $\mathcal{F}$ on a contact manifold is geodesible and transversally symplectic with exact transversally symplectic form $d\omega$. From the viewpoint of transversal geometry for foliations, in [11] an analogy was established for Theorem A for the contact flow $\mathcal{F}$.

In the present paper, we are interested in a tense, transversally symplectic flow $\mathcal{F}$ with a transversally symplectic form Φ on a manifold M. The contact flow on a contact manifold is a typical example. Another example is the (contact) flow generated by the Reeb vector field on an almost cosymplectic manifold. Our main purpose is to develop a transversal harmonic theory for such a flow $(\mathcal{F}, \Phi)$.

In this situation, we consider an operator L on $\Omega^*(M)$ defined by

$$L\alpha := \Phi \wedge \alpha, \quad \forall \alpha \in \Omega^k(M), \tag{1.3}$$

which induces a homomorphism

https://doi.org/10.1017/S1446788708000190 Published online by Cambridge University Press

$$L : H_B^k(\mathcal{F}) \longrightarrow H_B^{k+2}(\mathcal{F}),$$

in the basic cohomology $H_B^*(\mathcal{F})$ for $\mathcal{F}$. Moreover, we introduce the notion of transversally symplectic harmonic forms and then establish the transversal hard Lefschetz theorem for $\mathcal{F}$.

THEOREM B. *Let $(\mathcal{F}, \Phi)$ be a tense, transversally symplectic flow on a manifold M of dimension $2n + 1$. Then the following are equivalent:*

(1) *any basic cohomology class for $\mathcal{F}$ has a transversally symplectic harmonic representative;*

(2) *for any $k \leq n$, the homomorphism*

$$L^k : H_B^{n-k}(\mathcal{F}) \longrightarrow H_B^{n+k}(\mathcal{F}),$$

is surjective.

For the point foliation, Theorem B reduces to Theorem A. Theorem B extends the results in [11] for a contact manifold and in [9] for an almost cosymplectic manifold.

2. Transversally symplectic flows

Let $\mathcal{F}$ be the flow generated by a nonsingular vector field T on a manifold M of dimension $2n + 1$. Let $(\Omega_B^*(\mathcal{F}), d_B)$ be the basic complex for $\mathcal{F}$ given by

$$\Omega_B^*(\mathcal{F}) := \{\alpha \in \Omega^*(M) \mid \iota(T)\alpha = \mathcal{L}_T\alpha = 0\},$$

which is a subcomplex of the de Rham complex $(\Omega^*(M), d)$ on M. Denote its basic cohomology by $H_B^*(\mathcal{F}) := H(\Omega_B^*(\mathcal{F}), d_B)$, which plays the role of the de Rham cohomology of the leaf space $M/\mathcal{F}$.

The flow $\mathcal{F}$ generated by a nonsingular vector field T on M is said to be transversally symplectic if it admits a transversally symplectic form Φ, that is, $\Phi \in \Omega_B^2(\mathcal{F})$ is closed and has rank $2n$ on $\Omega^*(M)$. Then, by definition, we have a global 1-form ω such that

$$\iota(T)\omega = 1, \quad \mathcal{D} := \ker \omega \simeq Q,$$

where Q denotes the normal bundle for $\mathcal{F}$. We call ω the characteristic form of T for $(\mathcal{F}, \Phi)$. Consider the following multiplicative filtration of $(\Omega^*(M), d)$ for $\mathcal{F}$ defined by

$$F^r\Omega^k := \{\alpha \in \Omega^k(M) \mid \iota(X_1)\cdots\iota(X_{k-r+1})\alpha = 0\},$$

for $X_i \in \Gamma(F)$, where F denotes the tangent bundle to $\mathcal{F}$.

The mean curvature form for such a flow $\mathcal{F}$ is defined by

$$\kappa := \mathcal{L}_T\omega. \tag{2.1}$$

It should be noted that $\kappa \in F^1\Omega^1$. If $\kappa \in \Omega_B^1(\mathcal{F})$ (respectively $\kappa = 0$ on M) then $\mathcal{F}$ is said to be tense (respectively geodesible). From Rummler's formula [24]

$$d\omega + \kappa \wedge \omega =: \varphi_0 \in F^2\Omega^2, \tag{2.2}$$

we find that $\varphi_0 = 0$ if and only if ω is integrable, that is, the distribution $\mathcal{D}$ is integrable.

EXAMPLES.

(1) If $\mathcal{F}$ is the contact flow on a contact manifold (M, ω) generated by the Reeb vector field T given by (1.1), then $(\mathcal{F}, \Phi = d\omega)$ is geodesible and transversally symplectic. In this case, $\varphi_0 \neq 0$ on M.

(2) Consider an almost cosymplectic manifold (M, η, Φ), that is, $d\eta = 0$, $d\Phi = 0$ and $\eta \wedge \Phi^n \neq 0$ on M [3]. Then we have the corresponding Reeb vector field ξ characterized by

$$\iota(\xi)\eta = 1, \quad \iota(\xi)\Phi = 0.$$

If $\mathcal{F}$ is the flow generated by ξ (we also call it the contact flow), then $(\mathcal{F}, \Phi)$ is geodesible and transversally symplectic. In this case, $\varphi_0 = 0$ on M. Thus, the distribution $\mathcal{D} = \ker \eta$ defines a codimension-one foliation $\mathcal{F}^\perp$ transversal to $\mathcal{F}$.

(3) More generally, a locally conformal almost cosymplectic manifold (M, η, Φ) is defined as an open covering $\{U_s\}$ of M endowed with smooth functions $\sigma_s : U_s \longrightarrow \mathbb{R}$ such that over each U_s the local conformal change given by

$$\eta_s := e^{-\sigma_s}\eta, \quad \Phi_s := e^{-2\sigma_s}\Phi$$

of (η, Φ) is almost cosymplectic. It is easy to see that this manifold is characterized by the existence of a closed 1-form ψ satisfying

$$d\eta = \psi \wedge \eta, \quad d\Phi = 2\psi \wedge \Phi. \tag{2.3}$$

Then by (2.2) the flow $\mathcal{F}$ generated by the Reeb vector field ξ is tense with mean curvature $-\psi$ and $\mathcal{D} = \ker \eta$ defines a codimension-one foliation $\mathcal{F}^\perp$ transversal to $\mathcal{F}$. From $\psi \in F^1\Omega^1$ and (2.3) we have on $\mathcal{D}$

$$d\Phi = 2\psi \wedge \Phi.$$

It follows that $\mathcal{D}$ admits a locally conformal symplectic structure Φ with Lee form ψ (see [25]). Namely, $\mathcal{F}$ is a transversally locally conformal symplectic flow. However, $\mathcal{F}$ is not necessarily transversally symplectic. Observe that $\mathcal{F}$ is transversally symplectic if and only if the Lee form ψ vanishes on M. Therefore, we deduce the following proposition.

PROPOSITION 2.1. *Let (M, η, Φ) be a locally conformal almost cosymplectic manifold. Then the flow $(\mathcal{F}, \Phi)$ generated by the Reeb vector field ξ is transversally locally conformal symplectic. Furthermore, it is transversally symplectic if and only if M is almost cosymplectic.*

In viewing the above examples, it is natural to consider a geodesible, transversally symplectic flow $(\mathcal{F}, \Phi)$. In this case, (2.2) becomes $\varphi_0 = d\omega$, so that it defines $[\varphi_0] \in H^2_B(\mathcal{F})$. Then we have the following theorem.

THEOREM 2.2. *Let $(\mathcal{F}, \Phi)$ be a geodesible, transversally symplectic flow generated by a nonsingular vector field T on a manifold M and ω be the characteristic form of T:*

(1) *if* $[\varphi_0] = 0$ *in* $H^2_B(\mathcal{F})$ *then* M *can be equipped with an almost cosymplectic structure;*
(2) *if* $[\Phi] = [\varphi_0](\neq 0)$ *in* $H^2_B(\mathcal{F})$*, then* M *can be equipped with a contact structure.*

PROOF.
(1) Since $[\varphi_0] = 0$ in $H^2_B(\mathcal{F})$ there exists $\beta \in \Omega^1_B(\mathcal{F})$ such that $\varphi_0 = d\beta$. Take $\tilde{\omega} := \omega - \beta \in \Omega^1(M)$. Then

$$\tilde{\omega} \wedge \Phi^n = \omega \wedge \Phi^n.$$

It follows that $(\tilde{\omega}, \Phi)$ is an almost cosymplectic structure on M.
(2) From the hypothesis we can write

$$\Phi = \varphi_0 + d\gamma$$

for some $\gamma \in \Omega^1_B(\mathcal{F})$. By taking $\tilde{\omega} := \omega + \gamma \in \Omega^1(M)$, we obtain a contact structure $\tilde{\omega}$ on M. □

REMARKS.
(1) Observe that under the situation as in Theorem 2.2, φ_0 and Φ define de Rham cohomology classes $[\varphi_0], [\Phi] \in H^2(M)$. Thus, if the second Betti number of M vanishes, then (M, ω) is a contact manifold. There are several results on the vanishing of the second Betti number on a closed Sasakian manifold [4].
(2) In the presence of the metric, Molino [14] discussed some classifications of transversally symplectic Riemannian foliations on a closed Riemannian manifold. In [16, 17], the authors studied the problem of when a Riemannian flow on a closed Einstein(–Weyl) manifold admits transversally almost complex structure. The vanishing of $[\varphi_0]$ was discussed in [21] when $\mathcal{F}$ is an isometric flow (which is generated by a Killing vector field).

Let $(\mathcal{F}, \Phi)$ be a transversally symplectic flow generated by a nonsingular vector field T on a manifold M of dimension $2n + 1$ and ω be its characteristic form of T. Define a map $\flat : \Gamma(TM) \longrightarrow \Omega^1(M)$ of $C^\infty(M)$-modules by

$$\flat(X) := \iota(X)\Phi + \omega(X)\omega,$$

where $\Gamma(\cdot)$ is the $C^\infty(M)$-module of all smooth sections of a vector bundle $(\cdot)$. Since Φ plays a role as a symplectic structure on the distribution $\mathcal{D}$, $\flat$ is an isomorphism. The map $\flat$ can be extended to an isomorphism of the space $\mathcal{X}^k(M)$ of all skew-symmetric k-vector fields onto $\Omega^k(M)$ by setting

$$\flat(X_1 \wedge \cdots \wedge X_k) := \flat(X_1) \wedge \cdots \wedge \flat(X_k), \quad \forall X_1, \ldots, X_k \in \Gamma(TM). \tag{2.4}$$

Let

$$\mathcal{X}_B(\mathcal{F}) := \{X \in \Gamma(\mathcal{D}) \mid [X, T] \in \Gamma(F)\} \subset \Gamma(TM),$$

where F denotes the subbundle of TM tangent to $\mathcal{F}$. An element in $\mathcal{X}_B(\mathcal{F})$ is called a basic vector field.

https://doi.org/10.1017/S1446788708000190 Published online by Cambridge University Press

LEMMA 2.3. *Let $(\mathcal{F}, \Phi)$ be a transversally symplectic flow generated by a nonsingular vector field T on a manifold M of dimension $2n+1$ and ω be its characteristic form of T. Then $\flat$ induces an isomorphism $\mathcal{X}_B(\mathcal{F}) \longrightarrow \Omega^1_B(\mathcal{F})$.*

PROOF. Let $X \in \mathcal{X}_B(\mathcal{F})$. It is obvious that $\iota(T)\flat(X) = 0$. Moreover, using the identity

$$[\mathcal{L}_T, \iota(K)] = \iota(\mathcal{L}_T K), \tag{2.5}$$

for $K \in \mathcal{X}^k(M)$ yields

$$\mathcal{L}_T \flat(X) = \iota(X)\mathcal{L}_T \Phi + \iota([T, X])\Phi = 0.$$

It follows that $\flat(X) \in \Omega^1_B(\mathcal{F})$.

Conversely, for $\alpha \in \Omega^1_B(\mathcal{F})$ there exists $X \in \Gamma(TM)$ such that $\alpha = \flat(X)$. Then

$$0 = \mathcal{L}_T \alpha = \iota([T, X])\Phi,$$

which implies that $[T, X] \in \Gamma(F)$ because of the nondegeneracy of Φ on $\mathcal{D}$. Hence, $X \in \mathcal{X}_B(\mathcal{F})$. □

By Lemma 2.3, together with (2.4), $\flat$ can be naturally extended to an isomorphism

$$\flat : \mathcal{X}^k_B(\mathcal{F}) \longrightarrow \Omega^k_B(\mathcal{F}), \tag{2.6}$$

where $\mathcal{X}^k_B(\mathcal{F})$ denotes the space of all basic skew-symmetric k-vector fields.

In terms of the transversal volume form $\nu := (\Phi^n/n!)$ for $\mathcal{F}$ we define the star operator $*_\mathcal{D}$ by the formula

$$*_\mathcal{D}\alpha := \iota(\flat^{-1}(\alpha))\nu, \tag{2.7}$$

for $\alpha \in F^k\Omega^k$.

COROLLARY 2.4. *Let $(M, \mathcal{F}, \Phi, T, \omega)$ be as in Lemma 2.3. Then $*_\mathcal{D} : \Omega^k_B(\mathcal{F}) \longrightarrow \Omega^{2n-k}_B(\mathcal{F})$ is well defined.*

PROOF. Let $\alpha \in \Omega^k_B$. Then by definition and (2.6)

$$\mathcal{L}_T *_\mathcal{D}\alpha = \mathcal{L}_T \iota(\flat^{-1}(\alpha))\nu = \iota(\mathcal{L}_T \flat^{-1}(\alpha))\nu = 0,$$

from which it follows that $*_\mathcal{D}\alpha \in \Omega^{2n-k}_B(\mathcal{F})$. □

As an application of Corollary 2.4, we have further properties.

https://doi.org/10.1017/S1446788708000190 Published online by Cambridge University Press

PROPOSITION 2.5. *Let $(M, \mathcal{F}, \Phi, T, \omega)$ be as in Lemma 2.3. Moreover, suppose that M is closed and $\mathcal{F}$ is tense. Then:*

(1) $d\kappa = 0$, *so* $[\kappa] \in H^1_B(\mathcal{F})$;
(2) $\varphi_0 \in \Omega^2_B(\mathcal{F})$.

PROOF.

(1) The proof essentially follows [24]. Since $\kappa \in \Omega^1_B(\mathcal{F})$, there exists $\alpha \in \Omega^{2n-2}_B(\mathcal{F})$ such that $*_{\mathcal{D}}\alpha = d\kappa$ by virtue of Corollary 2.4. Then (2.2) implies

$$\begin{aligned}\|\alpha\|^2 := \int_M \alpha \wedge *_{\mathcal{D}}\alpha \wedge \omega &= \int_M \alpha \wedge d\kappa \wedge \omega \\ &= \int_M \alpha \wedge (\kappa \wedge d\omega + d\varphi_0) \\ &= \int_M d(\alpha \wedge \varphi_0).\end{aligned}$$

Therefore, $\alpha = 0$, so that $d\kappa = 0$.

(2) From (2.1), (2.2) and (1), a direct computation yields

$$\mathcal{L}_T\varphi_0 = \mathcal{L}_T(d\omega + \kappa \wedge \omega) = d\kappa = 0.$$

This means that $\varphi_0 \in \Omega^2_B(\mathcal{F})$. □

3. Transversally symplectic harmonic forms

Let $(\mathcal{F}, \Phi)$ be a transversally symplectic flow generated by a nonsingular vector field T on a manifold M of dimension $2n + 1$ and ω be its characteristic form of T.

We define the star operator $* : \Omega^k(M) \longrightarrow \Omega^{2n+1-k}(M)$ by

$$*\alpha := \iota(\flat^{-1}(\alpha))(\omega \wedge \nu), \tag{3.1}$$

in terms of the canonical volume form $\omega \wedge \nu = \omega \wedge (\Phi^n/n!)$ on M. A k-form α is said to be harmonic if $d\alpha = 0$ and $\delta\alpha := (-1)^k * d * \alpha = 0$. Denote the space of all harmonic forms on M by $\mathcal{H}^*(M)$.

Now we need an operator $e(\omega)$ on $\Omega^*(M)$ defined by

$$e(\omega)\alpha := \omega \wedge \alpha, \qquad \forall \alpha \in \Omega^k(M).$$

Then we have the following lemma.

LEMMA 3.1. *Let $(\mathcal{F}, \Phi)$ be a transversally symplectic flow generated by a nonsingular vector field T on a manifold M of dimension $2n + 1$ and ω be its characteristic form of T. Then for each k the map $e(\omega) : F^k\Omega^k \longrightarrow \Omega^{k+1}(M)$ is an injective isomorphism.*

PROOF. It suffices to note that if $\alpha \in F^k\Omega^k$ satisfies $e(\omega)\alpha = 0$, then

$$\alpha = \iota(T)e(\omega)\alpha + e(\omega)\iota(T)\alpha = 0,$$

which means that $e(\omega)$ is injective. □

https://doi.org/10.1017/S1446788708000190 Published online by Cambridge University Press

LEMMA 3.2. *Under the same situation as in Lemma 3.1, the operator* $\Lambda := *L*$ *defined on* $\Omega^*(M)$ *preserves* $\Omega^*_B(\mathcal{F})$.

PROOF. A direct computation for $\alpha \in \Omega^k_B(\mathcal{F})$ gives rise to

$$\begin{aligned}\Lambda\alpha &= *L*\alpha = *L[\iota(\flat^{-1}(\alpha))(\omega \wedge \nu)] \\ &= (-1)^k *(\omega \wedge L*_{\mathcal{D}}\alpha) \\ &= \iota(\flat^{-1}(L*_{\mathcal{D}}\alpha))\nu \\ &= *_{\mathcal{D}}L*_{\mathcal{D}}\alpha.\end{aligned}$$

Thus, Corollary 2.4 implies that $\Lambda\alpha \in \Omega^{k-2}_B(\mathcal{F})$. □

For convenience, we set

$$\Omega^{0,k} := F^k\Omega^k, \quad \Omega^{1,k} := e(\omega)(F^k\Omega^k),$$

in the sense of Lemma 3.1. For $\ell = 0, 1$ let $\Pi_\ell : \Omega^k(M) \longrightarrow \Omega^{\ell,k-\ell}$ be the natural projection. Introduce a differential operator $d_{0,1} := \Pi_\ell \circ d$ on $\Omega^*(M)$. It is observed that $d_{0,1} = d_B$ on $\Omega^*_B(\mathcal{F})$ and

$$d_{0,1}\Omega^{\ell,k} \subset \Omega^{\ell,k+1}, \tag{3.2}$$

which implies from (2.2) that

$$d_{0,1}\omega = -\kappa \wedge \omega. \tag{3.3}$$

We define a codifferential on $\Omega^k(M)$ by

$$\delta_{0,1} := (-1)^k *d_{0,1}*. \tag{3.4}$$

Then it is obvious from (3.1) and (3.2) that

$$\delta_{0,1}\Omega^{\ell,k} \subset \Omega^{\ell,k-1}. \tag{3.5}$$

Furthermore, we have the following.

LEMMA 3.3. *Let* $(M, \mathcal{F}, \Phi, T, \omega)$ *be as in Lemma 3.1. Suppose that* $\mathcal{F}$ *is tense. Then the operator*

$$\delta_{0,1} : \Omega^k_B(\mathcal{F}) \longrightarrow \Omega^{k-1}_B(\mathcal{F}),$$

is well defined.

PROOF. By applying (3.3) we find for $\alpha \in \Omega^k_B(\mathcal{F})$

$$\begin{aligned}*d_{0,1}*\alpha &= *d_{0,1}[\iota(\flat^{-1}(\alpha))(\omega \wedge \nu)] \\ &= (-1)^k *[d_{0,1}\omega \wedge *_{\mathcal{D}}\alpha - \omega \wedge d_{0,1}*_{\mathcal{D}}\alpha] \\ &= (-1)^{k+1} *[\omega \wedge (d_B*_{\mathcal{D}}\alpha - \kappa \wedge *_{\mathcal{D}}\alpha)] \\ &= *_{\mathcal{D}}(d_B - \kappa\wedge)*_{\mathcal{D}}\alpha.\end{aligned}$$

Therefore, Corollary 2.4 and the tenseness of $\mathcal{F}$ imply $\delta_{0,1}\alpha \in \Omega^{k-1}_B(\mathcal{F})$. □

https://doi.org/10.1017/S1446788708000190 Published online by Cambridge University Press

In what follows, $(\mathcal{F}, \Phi)$ considered is a tense, transversally symplectic flow. By virtue of Lemma 3.3 we denote the restriction of $\delta_{0,1}$ to $\Omega_B^*(\mathcal{F})$ by δ_B. Then we define the space

$$\mathcal{H}_B^k(\mathcal{F}) := \{\alpha \in \Omega_B^*(\mathcal{F}) \mid d_B\alpha = \delta_B = 0\}, \tag{3.6}$$

which is called the transversally symplectic harmonic space for $\mathcal{F}$.

REMARK. In the presence of the Riemannian metric g on a manifold M, we can defined an ordinary basic harmonic space $\mathcal{H}_g^*(\mathcal{F})$ for a flow with transversal mean curvature and transversal volume forms [18]. In general, $\mathcal{H}_g^*(\mathcal{F})$ does not coincide with $\mathcal{H}_B^*(\mathcal{F})$.

From now on we prove Theorem B by a similar argument as in [27] for the point foliation. We need an operator $A := \sum_0^{2n}(n-k)\pi_k$, where $\pi_k : \Omega_B^*(\mathcal{F}) \longrightarrow \Omega_B^k(\mathcal{F})$ is the natural projection. From (1.3), Lemmas 3.2 and 3.3, we know that the operators d_B, δ_B, L, Λ and A preserve $\Omega_B^*(\mathcal{F})$. Furthermore, we have the following lemma.

LEMMA 3.4. *On $\Omega_B^*(\mathcal{F})$, it holds that:*

(1) $[\Lambda, L] = A$, $[A, L] = -2L$, $[A, \Lambda] = 2\Lambda$;
(2) $[L, d_B] = 0$, $[\Lambda, d_B] = \delta_B$.

PROOF. (1) By applying induction on p we can show a more general formula on $\Omega^k(M)$

$$[\Lambda, L^p] = p[(n+1-p-k)L^{p-1} + e(\omega)\iota(T)L^{p-1}],$$

where p is any nonnegative integer and $L^{-1} := 0$ (see [19] for the case of a contact manifold). The proof of the rest of (1) is trivial.

The first part of (2) is obvious since Φ is closed. The second part of (2) is due to [5, 27]. □

Lemma 3.4 means that $\{A, L, \Lambda\}$ spans the Lie algebra $sl(2)$. Thus, the space $\Omega_B^*(\mathcal{F})$ is a $sl(2)$-module on which A acts diagonally with only finitely many different eigenvalues. The next result follows.

COROLLARY 3.5 (Duality on transversally symplectic harmonic forms).

$$L^k : \mathcal{H}_B^{n-k}(\mathcal{F}) \longrightarrow \mathcal{H}_B^{n+k}(\mathcal{F})$$

is an isomorphism.

Now we are in a position to complete the proof of Theorem B. Assume that $\mathcal{H}_B^*(\mathcal{F}) = H_B^*(\mathcal{F})$. Consider the following the commutative diagram.

$$\begin{array}{ccc} \mathcal{H}_B^{n-k}(\mathcal{F}) & \xrightarrow{L^k} & \mathcal{H}_B^{n+k}(\mathcal{F}) \\ \downarrow & & \downarrow \\ H_B^{n-k}(\mathcal{F}) & \xrightarrow{L^k} & H_B^{n+k}(\mathcal{F}) \end{array}$$

Since the two vertical arrows are surjective by means of (1) in Theorem B, Corollary 3.5 implies that the second horizontal arrow is also surjective.

Conversely, assume that for any $k \le n$, $L^k : H_B^{n-k}(\mathcal{F}) \longrightarrow H_B^{n+k}(\mathcal{F})$ is surjective. We apply an induction on the degree of the basic cohomology classes for $\mathcal{F}$. It is obvious that any 0-cocycle and 1-cocycle are transversally symplectic harmonic forms. Suppose that the assertion (1) is true for r-cocycle with $r < n - k$. We must show that any class in $H_B^{n-k}(\mathcal{F})$ also contains a transversally symplectic harmonic representative.

To begin with, we observe that

$$H_B^{n-k}(\mathcal{F}) = \operatorname{im} L + P_{n-k},$$

where $P_{n-k} := \{[\alpha] \in H_B^{n-k}(\mathcal{F}) \mid L^{k+1}([\alpha]) = 0\}$. Indeed, by virtue of (2) in Theorem B there exists $[\beta] \in H_B^{n-k-2}(\mathcal{F})$ with $L^{k+1}([\alpha]) = L^{k+2}([\beta])$. Then $[\alpha] - L([\beta]) \in P_{n-k}$.

Next, it can be shown from Lemma 3.4 that any class in $\operatorname{im} L$ contains a transversally symplectic harmonic representative. Therefore, it remains to verify that any class in P_{n-k} contains a transversally symplectic harmonic representative.

Let $[z] \in P_{n-k}$. Then $L^{k+1}([z]) = 0$ in $H_B^{n+k+2}(\mathcal{F})$. Thus, there exists $\gamma \in \Omega_B^{n+k+1}(\mathcal{F})$ such that $L^{k+1}z = d_B\gamma$. By virtue of (2), we can take $\theta \in \Omega_B^{n-k-1}(\mathcal{F})$ such that $\gamma = L^{k+1}\theta$. Then $\beta := z - d_B\theta$ is as desired, that is, a transversally symplectic harmonic form satisfying $[\beta] = [z]$.

4. Transversally Kähler flows

In this section we consider the case of special tense, transversally symplectic flows. By a tense, transversally Kähler flow $(\mathcal{F}, \Phi)$ on a Riemannian manifold (M, g) we mean:

(1) $(\mathcal{F}, \Phi)$ is tense, transversally symplectic on M;

(2) g is a bundle-like metric for $\mathcal{F}$ which induces a transversally Kähler structure $(g_{\mathcal{D}}, J, \Phi)$ on the distribution $\mathcal{D} := \ker \omega$, where ω denotes its characteristic form.

THEOREM 4.1. *Let $(\mathcal{F}, \Phi)$ be a tense, transversally Kähler flow on a closed Riemannian manifold (M, g) of dimension $2n + 1$. Then any basic cohomology class for $\mathcal{F}$ has a transversally symplectic harmonic representative.*

PROOF. Since $(\mathcal{F}, \Phi)$ is transversal Kähler with respect to g, g induces a Kähler structure $(g_{\mathcal{D}}, J, \Phi)$ on the distribution $\mathcal{D}$.

https://doi.org/10.1017/S1446788708000190 Published online by Cambridge University Press

Now we compare two star operators $*_{\mathcal{D}}$ given in (2.7) and $*_{g_{\mathcal{D}}}$ associated to $g_{\mathcal{D}}$. Since the complex structure J is integrable, J naturally yields an orthogonal decomposition of complexified forms on $\mathcal{D}$, so an orthogonal decomposition of complexified basic forms

$$\Omega_B^k(\mathcal{F}) \otimes \mathbb{C} = \bigoplus_{p+q=k} \Omega_B^{p,q}(\mathcal{F}).$$

Then a similar computation as in [5] gives rise to

$$*_{\mathcal{D}} = (\sqrt{-1})^{p-q} *_{g_{\mathcal{D}}} \quad \text{on } \Omega_B^{p,q}(\mathcal{F}). \tag{4.1}$$

It follows that the codifferential $\delta_{g_{\mathcal{D}}} := (-1)^{k+1} *_{g_{\mathcal{D}}} d *_{g_{\mathcal{D}}}$ on $\Omega_B^k(\mathcal{F})$ associated to $g_{\mathcal{D}}$ is equal to a multiple of δ_B given in (3.4). Hence,

$$\mathcal{H}_B^*(\mathcal{F}) = \ker \Delta_{g_{\mathcal{D}}} \quad \text{on } \Omega_B^*(\mathcal{F}), \tag{4.2}$$

where $\Delta_{g_{\mathcal{D}}} := d_B \delta_{g_{\mathcal{D}}} + \delta_{g_{\mathcal{D}}} d_B$ denotes the ordinary transversal Laplacian associated to $g_{\mathcal{D}}$.

On the other hand, it was obtained in [10] that the tense Riemannian foliation $\mathcal{F}$ on a closed Riemannian manifold holds the basic Hodge decomposition

$$\Omega_B^*(\mathcal{F}) = \ker \Delta_{g_{\mathcal{D}}} \oplus \operatorname{im} d_B \oplus \operatorname{im} \delta_{g_{\mathcal{D}}}. \tag{4.3}$$

This implies that

$$H_B^*(\mathcal{F}) = \ker \Delta_{g_{\mathcal{D}}}. \tag{4.4}$$

Therefore, we conclude from (4.2) and (4.4) that any basic cohomology class for $\mathcal{F}$ has a transversally symplectic harmonic representative. □

REMARKS.

(1) Theorem 4.1 is found in [6] for the case where $\mathcal{F}$ is the contact flow on a closed cosymplectic manifold. When $\mathcal{F}$ is the contact flow on a Sasakian manifold, (4.3) was established in [7].

(2) The assumption of tenseness of $\mathcal{F}$ in Theorem 4.1 is redundant. Indeed, all of the arguments in Theorem 4.1 go through if, instead of the assumption $\kappa \in \Omega_B^1(\mathcal{F})$, we use the basic component κ_B of the mean curvature form κ arising from the orthogonal decomposition

$$\Omega^*(M) = \Omega_B^*(\mathcal{F}) \oplus \Omega_B^*(\mathcal{F})^{\perp}, \quad \kappa = \kappa_B + \kappa_B^{\perp},$$

for a Riemannian foliation on a closed Riemannian manifold [1].

(3) We can easily find counter-examples of contact manifolds which do not satisfy the transversal hard Lefschetz theorem (Theorem B(2)) via constructing principal circle bundles over symplectic manifolds (see section 1 for such manifolds). For almost cosymplectic manifolds, we take the products of symplectic manifolds with circles (refer to [6, 9]).

https://doi.org/10.1017/S1446788708000190 Published online by Cambridge University Press

References

[1] J. Alvarez-Lopez, 'The basic component of the mean curvature of Riemannian foliations', *Ann. Global Anal. Geom.* **10** (1992), 179–194.
[2] C. Benson and C. Gordon, 'Kähler and symplectic structures on nilmanifolds', *Topology* **27** (1988), 513–518.
[3] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics, 509 (Springer, Berlin, 1976).
[4] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Mathematics, 203 (Birkhäuser, Basel, 2001).
[5] J. L. Brylinski, 'A differential complex for Poisson manifold', *J. Differential Geom.* **28** (1988), 93–114.
[6] D. Chinea, M. de León and J. C. Marrero, 'Topology of cosymplectic manifolds', *J. Math. Pures Appl.* **72** (1993), 567–591.
[7] D. Chinea, M. de León and J. C. Marrero, 'Spectral sequences on Sasakian and cosymplectic manifolds', *Houston J. Math.* **23** (1997), 631–649.
[8] S. Haller, 'Harmonic cohomology of symplectic manifolds', *Adv. Math.* **180** (2003), 87–103.
[9] R. Ibanez, 'Harmonic cohomology classes of almost cosymplectic manifolds', *Michigan Math. J.* **44** (1997), 183–199.
[10] F. Kamber and Ph. Tondeur, 'De Rham–Hodge theory for Riemannian foliations', *Math. Ann.* **277** (1987), 415–431.
[11] H. Kitahara and H. K. Pak, 'A note on harmonic forms on a compact manifold', *Kyungpook Math. J.* **43** (2003), 1–10.
[12] O. Mathieu, 'Harmonic cohomology classes of symplectic manifolds', *Comment. Math. Helv.* **70** (1995), 1–9.
[13] D. McDuff, 'Examples of simply-connected symplectic non-Kähler manifolds', *J. Differential Geom.* **20** (1984), 267–277.
[14] P. Molino, 'Orbit-like foliations', in: *Geometric Study of Foliations* (World Scientific, Singapore, 1994), pp. 97–119.
[15] Y. Ogawa, 'On C-harmonic forms on a compact Sasakian space', *Tohoku Math. J.* **19** (1967), 267–296.
[16] H. K. Pak, 'On one-dimensional metric foliations in Einstein spaces', *Illinois J. Math.* **36** (1992), 594–599.
[17] H. K. Pak, 'Canonical flows of Einstein–Weyl manifolds', *The 3rd Pacific Rim Geometry Conference* (International Press, Cambridge, MA, 1998), pp. 307–318.
[18] H. K. Pak, 'On harmonic theory in flows', *Canad. Math. Bull.* **46** (2003), 617–631.
[19] H. K. Pak and T. Takahashi, 'Harmonic forms in a compact contact manifold', *Tohoku Math. Publ.* **20** (2001), 125–138.
[20] M. Rumin, 'Formes différéntielles sur les variétés de contact', *J. Differential Geom.* **39** (1994), 281–330.
[21] M. Saralegui, *The Euler Class for Flows of Isometries*, Research Notes in Mathematics, 131 (Pitman, Boston, MA, 1985), pp. 220–227.
[22] S. Tachibana, 'On a decomposition of C-harmonic forms on a compact Sasakian space', *Tohoku Math. J.* **19** (1967), 198–212.
[23] S. Tachibana, 'On harmonic tensors in a compact Sasakian spaces', *Tohoku Math. J.* **17** (1965), 271–284.
[24] Ph. Tondeur, *Geometry of Foliations*, Monographs in Mathematics, 90 (Birkhäuser, Basel, 1997).
[25] I. Vaisman, 'Locally conformal symplectic manifolds', *Int. J. Math. Sci.* **8** (1985), 521–536.
[26] S. M. Webster, 'Pseudo-hermitian structure on a real hypersurface', *J. Differential Geom.* **13** (1978), 25–41.
[27] D. Yan, 'Hodge structure on symplectic manifolds', *Adv. Math.* **120** (1996), 143–154.

HONG KYUNG PAK, Faculty of Information and Science, Daegu Haany University,
Kyungsan 712-715, Korea
e-mail: hkpak@dhu.ac.kr