An example of an amenable action from geometry

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Abstract. Let M be a compact manifold of not necessarily constant negative curvature. We observe that $\pi_1(M)$ acts amenably on the sphere at infinity of the universal cover of M with respect to a natural measure class. We also note that this action is of type III_1.

1. *Introduction*

In this work we describe a geometric version of the following basic example of an amenable action [**Z3**, example 5.4]:

Let Γ be a lattice in a connected real semisimple Lie group G of the non-compact type. Let P be a minimal parabolic subgroup of G. Then Γ acts amenably on G/P with respect to Haar measure.

If G has real rank 1 then G/P can be identified with the sphere at infinity [**E**], [**BGS**] of the globally symmetric space G/K, where K is a maximal compact subgroup of G. Then the action of Γ on G/P becomes the natural extension of the Γ-action on G/K to the sphere at infinity.

More generally, let M be a compact Riemannian manifold of negative sectional curvature K. Set $\Gamma = \pi_1(M)$. Then Γ acts on the sphere at infinity S of the universal cover $\tilde{M}$ of M. We describe a natural measure class on S (recall that in general S is only a topological manifold and does not have a canonical Lebesgue measure class): if $p \in \tilde{M}$, define a map ϕ_p from the unit sphere S_p in $T_p\tilde{M}$ to S by $\phi_p(v) = \gamma_v(\infty)$ where γ_v is the unique geodesic with $\dot{\gamma}_v(0) = v$. Then $\phi_p : S_p \to S$ is a homeomorphism [**BGS**, 3.2]. Moreover, if $q \in \tilde{M}$ then $\phi_q^{-1} \circ \phi_p : S_p \to S_q$ is absolutely continuous. Indeed if $v \in S_p$ and $w \in S_1$ then $\phi_p(v) = \phi_p(w)$ if and only if v and w lie on the same leaf of the weak stable foliation W^{ws} of the geodesic flow. Since S_p and S_q are nowhere tangent to W^{ws} the claim follows from the absolute continuity of W^{ws} [**A**, § 5]. Hence the image of Lebesgue measure on S_p under ϕ_p defines a measure class on S independent of $p \in M$. We denote it by $[\lambda]$.

We refer to [**M**] for the notions of amenability and type.

PROPOSITION 1. *The action of Γ on S is ergodic, amenable and of type* III_1 *with respect to* $[\lambda]$.

The third claim generalizes a result of D. Sullivan and R. J. Zimmer [**Su**, theorem 6], [**Z3**, example 5.14].

We reduce proposition 1 to a theorem of Bowen utilizing a certain duality between the Γ-action on S and the weak stable foliation W^{ws} of the geodesic flow on the unit tangent bundle SM of M. In the course of the proof we point out:

PROPOSITION 2. *The weak stable foliation of the geodesic flow on SM has type* III_1.

Finally we consider the Bowen-Margulis measure m_1 for the geodesic flow g_t on SM [Ma1], [Ma2], [B1], [S], [T], [PP]. It is the measure of maximal entropy for g_t. Recall that m_1 is absolutely continuous with respect to the stable and unstable foliations W^s and W^u of g_t [S, 2.10]. In fact, locally m_1 is a product $\mu^s \times \mu^u \times dt$ where μ^s and μ^u are measures on W^s and W^u respectively and dt is Lebesgue measure on $\mathbb{R}$ [Ma2], [PP], [T]. These data determine a lift m_2 of m_1 to $\widetilde{SM}$. Let $\pi_1 : \widetilde{SM} \to S$ be given by $\pi_1(v) = \gamma_v(\infty)$. Set $m = (\pi_1)_*(m_2)$. Clearly m is quasi-invariant under Γ. Note that m is rather awkward since typically the measure $m(A)$ of a set A in S is either 0 or ∞. However for m_2-a.e. $v \in \widetilde{SM}$ we can define a nice measure m_v on S in the same measure class as m by $m_v = (\pi_1)_* \mu_v^u$ where μ_v^u is the disintegration of m_2 to W_v^u.

Projecting μ_v^u to the horosphere of v we can think of the m_v as projections of measures on horospheres:
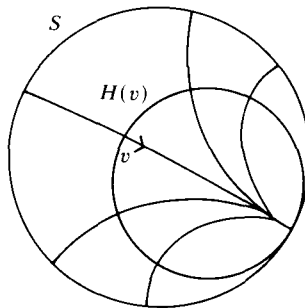
PROPOSITION 3. *The Γ-action on S is ergodic, amenable and has type* III_1 *with respect to the measure class of m.*
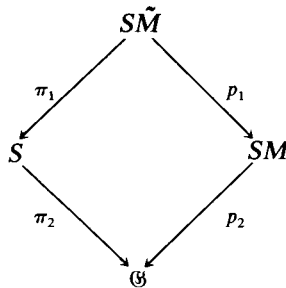
We expect that the propositions hold true for any compact non-positively curved manifold M of rank 1 [E]. Note that the measure class on the sphere at infinity is still well defined, essentially since the weak stable foliation for the geodesic flow is absolutely continuous.

Let us observe that if $M = \Gamma \backslash G / K$ is a compact locally symmetric space of non-compact type and (real) rank at least 2, then Γ acts amenable but not ergodically on the sphere at infinity S of G/K. Indeed, S splits measurably as $S = G/P \times C$ where P is the minimal parabolic subgroup expected in G and C is a Weyl chamber. The canonical measure class on S is the product of Haar measure on G/P and Lebesgue measure on C. The action of Γ on S is the extension of the Γ-action on G/P by the identity on C. Since an extension of an amenable action is amenable [Z2, theorem 2.4] Γ acts amenably on S. Clearly the Γ-action is not ergodic when rank $M \ge 2$.

We would like to thank J. Hawkins for several helpful discussions about type III_1.

2. *Proofs*

First we describe the duality between the Γ-action on S and the weak stable foliation on the unit tangent bundle SM of M: Let $S\tilde{M}$ be the unit tangent bundle of $\tilde{M}$ and let $p_1: S\tilde{M} \to SM$ be the derivative of the covering $\tilde{M} \to M$. Let $\pi_1: S\tilde{M} \to S$ be given by $\pi_1(v) = \gamma_v(\infty)$. Set $\mathfrak{G} = \Gamma \backslash S$ and let $\pi_2: S \to \mathfrak{G}$ be the projection map. Finally let p_2 be the projection from SM to the orbit space of the weak stable foliation on SM. Since π_1 is Γ-equivariant this orbit space can be identified with $\mathfrak{G}$ and the following diagram commutes:

$$
\begin{array}{ccc}
 & S\tilde{M} & \\
\pi_1 \swarrow & & \searrow p_1 \\
S & & SM \\
\pi_2 \searrow & & \swarrow p_2 \\
 & \mathfrak{G} &
\end{array}
$$

Set $\pi = \pi_2 \circ \pi_1$. Let μ and $\tilde{\mu}$ be the Liouville measures on SM and $S\tilde{M}$. Clearly, both p_1 and π_1 are non-singular with respect to $\tilde{\mu}$, μ and $\tilde{\mu}$, λ.

In [**B2**, theorem 7, corollary 11] Bowen established:

THEOREM. *The weak stable foliation W^{ws} of a transitive C^2 Anosov flow f_t on a compact manifold is hyperfinite and ergodic. If f_t does not have a global cross section then the ratio set of W^{ws} is $[0, \infty)$.*

Observe that Γ acts ergodically on S because π_1 and p_1 are non-singular and W^{ws} is ergodic on SM.

Given a map $f: X \to Y$ let $\sim_f$ be the equivalence relation on X defined by $x_1 \sim_f x_2$ if $f(x_1) = f(x_2)$.

By Bowen's theorem $\sim_{p_2}$ and therefore $\sim_\pi$ are amenable equivalence relations. Indeed, amenability and approximate finiteness are equivalent, and extensions of approximately finite relations are approximately finite [**M**, 5.1, 5.3].

LEMMA 1. *Let X and Y be standard measure spaces with measure classes $[\mu]$ and $[\lambda]$, and a non-singular projection map $\pi_1: X \to Y$. Suppose $\pi: X \to Z$ and $\pi_2: Y \to Z$ are maps such that $\pi = \pi_2 \circ \pi_1$. Then $\sim_{\pi_2}$ is amenable if $\sim_\pi$ is amenable.*

Proof. If $\sim_{\pi_2}$ is measurable then so is $\sim_\pi$. Let α be a homomorphism of $\sim_{\pi_2}$ into the group of isometries Iso (E) of a separable Banach space E. Let $y \mapsto K_y$ be a measurable field of weak*-compact sets in E^* invariant under α in the sense that $\alpha(y_1, y_2)^{*-1} K_{y_2} = K_{y_1}$ for a.e. $y_1 \sim_{\pi_2} y_2$. Composing α and K with π_1 we get a homomorphism of $\sim_\pi$ into Iso (E) with invariant field of sets $x \mapsto K_{\pi_1(x)}$. Since $\sim_\pi$ is amenable there is a fixed point ϕ in $K \circ \pi_1$, i.e. a Borel function $\phi: X \to E^*$ such that $\phi(x) \in K_{\pi_1(x)}$ and

$$
((\alpha \circ \pi_1)(x_1, x_2)^{*-1}) \phi(x_2) = \phi(x_1)
$$

for every x_1, x_2 outside a set of measure 0 (cf. [**R**] for the technical difference

between this and a.e. $x_1 \sim_\pi x_2$). In particular, we have that $\phi(x_1) = \phi(x_2)$ a.e. if $\pi_1(x_1) = \pi_1(x_2)$. By this and the non-singularity of π_1 we see that ϕ defines a fixed point for α in K.   □

LEMMA 2. *The action of* Γ *on* S *is free a.e.*

Proof. Since M is compact all $\gamma \neq 1$ in Γ are hyperbolic [**BGS**, 8.2]. By [**BGS**, 6.8] each $\gamma \neq 1$ has exactly two fixed points in S. Since Γ is countable there are only countably many points in S with non-trivial stabiliser.   □

Recall [**Z1**, proposition 3.2] that a free ergodic action of a countable discrete group is amenable if and only if the orbits define an amenable equivalence relation. Hence lemmas 1 and 2 show that Γ acts amenably on S with respect to $[\lambda]$.

   Next we prove proposition 2. By [**M**, 3.3] we need to show that the ratio set is $[0, \infty)$. By Bowen's theorem it suffices to prove that the geodesic flow g_t does not admit a global cross section. If it does then g_t is a continuous time change of a suspension of a homeomorphism. This is impossible by Arnold's theorem [**A**, § 23].

   Finally we show that the Γ-action is type III_1 or equivalently [**M**, 3.3] that the ratio set is $[0, \infty)$. Again we use duality and thus reduce the claim to proposition 2. First observe that $\sim_\pi$ is of type III_1 by proposition 2. Since the weak stable foliation W^{ws} of the geodesic flow on $S\tilde{M}$ is absolutely continuous with respect to the Liouville measure and each unstable leaf $W^u(x)$ is transverse to W^{ws} we can disintegrate $\tilde{\mu} = \int_{W^u(x)} \mu_y^{ws} \, d\mu_x^u(y)$, where $x \in S\tilde{M}$ and μ_y^{ws} and μ_x^u are measures on $W^{ws}(y)$ and $W^u(x)$ respectively. Clearly, $S\tilde{M} = W^{ws}(x) \times W^u(x)$ measurably and μ is equivalent to the product measure $\mu_x^{ws} \times \mu_x^u$. Also note that π_1 becomes the projection to the second factor. Let ν_1 be a Haar measure for $\sim_{\pi_2}$ [**M**, 3.1]. The product measure $\nu_2 \overset{\text{def}}{=} \mu_x^u \times \mu_x^u$ is a Haar measure for $\sim_{\pi_1}$. Then $\nu_1 \times \nu_2$ is a Haar measure for $\sim_\pi$. Let $\Delta, \Delta_1, \Delta_2$ be the modular functions for $\sim_\pi, \sim_{\pi_1}, \sim_{\pi_2}$ respectively [**M**, 3.1]. Then $\Delta = \Delta_1 \cdot \Delta_2 = \Delta_1$ since $\Delta_2 \equiv 1$ by our choice of ν_2. Hence the ratio sets [**M**, 3.3] of Δ_1 and Δ are the same. Therefore the ratio set of the Γ-action on S is $[0, \infty)$. This finishes the proof of proposition 1.

   Finally we consider the measure of maximal entropy. By [**S**, 2.10(ii)], W^{ws} is hyperfinite with respect to m_1. Proposition 3 follows from this and [**S**, 2.13] by the same argument as proposition 1.   □

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