Appendix B
Spherically symmetric solutions and
Birkhoff’s theorem

We wish to consider Einstein’s equations in the case of a spherically symmetric space–time. One might regard the essential feature of a spherically symmetric space–time as the existence of a world-line $\mathcal{L}$ such that the space–time is spherically symmetric about $\mathcal{L}$. Then all points on each spacelike two-sphere $\mathcal{S}^2$ centred on any point $p$ of $\mathcal{L}$, defined by going a constant distance $d$ along all geodesics through $p$ orthogonal to $\mathcal{L}$, are equivalent. If one permutes directions at $p$ by use of the orthogonal group $SO(3)$ leaving $\mathcal{L}$ invariant, the space–time is, by definition, unchanged, and the corresponding points of $\mathcal{S}^2$ are mapped into themselves; so the space–time admits the group $SO(3)$ as a group of isometries, with the orbits of the group the spheres $\mathcal{S}^2$. (There could be particular values of $d$ such that the surface $\mathcal{S}^2$ was just a point $p’$; then $p’$ would be another centre of symmetry. There can be at most two points ($p’$ and $p$ itself) related in this way.)

However, there might not exist a world-line like $\mathcal{L}$ in some of the space–times one would wish to regard as spherically symmetric. In the Schwarzschild and Reissner–Nordström solutions, for example, space–time is singular at the points for which $r = 0$, which might otherwise have been centres of symmetry. We shall therefore take the existence of the group $SO(3)$ of isometries acting on two-surfaces like $\mathcal{S}^2$ as the characteristic feature of a spherically symmetric space–time. Thus we shall say that space–time is spherically symmetric if it admits the group $SO(3)$ as a group of isometries, with the group orbits spacelike two-surfaces. These orbits are then necessarily two-surfaces of constant positive curvature.

For each point $q$ in any orbit $\mathcal{S}(q)$, there is a one-dimensional subgroup $I_q$ of isometries which leaves $q$ invariant (when there is a central axis $\mathcal{L}$, this is the group of rotations about $p$ which leaves the geodesic $pq$ invariant). The set $\mathcal{C}(q)$ of all geodesics orthogonal to $\mathcal{S}(q)$ at $q$ locally form a two-surface left invariant by $I_q$ (since $I_q$, which permutes directions in $\mathcal{S}(q)$ about $q$, leaves invariant directions perpendicular to $\mathcal{S}(q)$). At any other point $r$ of $\mathcal{C}(q)$, $I_q$ again permutes directions...
orthogonal to \( C(q) \), as it leaves \( C(q) \) invariant; since \( I_q \) must operate in the group orbit \( S_f(r) \) through \( r \), this orbit is orthogonal to \( C(q) \). Thus (Schmidt (1967)) the group orbits \( S \) are orthogonal to the surfaces \( C \). Further these surfaces define locally a one-one map between the group orbits, where the image \( f(q) \) of \( q \) in \( S_f(r) \) is the intersection of \( C(q) \) and \( S_f(r) \). Since this map is invariant under \( I_q \), vectors of equal magnitude in \( S_f(q) \) at \( q \) are mapped into vectors of equal magnitude in \( S_f(r) \) at \( f(q) \); and since all the points of \( S_f(q) \) are equivalent, the same magnitude multiplication factor occurs for the maps of vectors from any point in \( S_f(q) \) to its image in \( S_f(r) \). Thus (Schmidt (1967)) the orthogonal surfaces \( C \) map the trajectories \( S \) conformally onto each other.

If one chooses coordinates \( \{t, r, \theta, \phi\} \) so that the group orbits \( S \) are the surfaces \( \{t, r = \text{constant}\} \) and the orthogonal surfaces \( C \) are the surfaces \( \{\theta, \phi = \text{constant}\} \), it now follows that the metric takes the form \( ds^2 = dr^2(t, r) + Y^2(t, r) \, d\Omega^2(\theta, \phi) \), where \( dr^2 \) is an indefinite two-surface and \( d\Omega^2 \) is a surface of positive constant curvature. If one further chooses the functions \( t, r \) so that the curves \( \{t = \text{constant}\} \), \( \{r = \text{constant}\} \) are orthogonal in the two-surfaces \( C \) (cf. Bergmann, Cahen and Komar (1965)), one can write the metric in the form

\[
-8\pi q = \frac{2X}{Y} \left( \frac{Y'}{X} - \frac{X'}{YY} + \frac{Y'F'}{YF} \right),
\]

\[
8\pi \mu = 1 + 2F \left( \frac{Y'}{Y} \right)' - 3 \left( \frac{Y'}{XY} \right)^2 + 2F^2 \frac{X'Y'}{XY} + F^2 \left( \frac{Y'}{Y} \right)^2,
\]

\[
-8\pi p = 1 + 2F \left( \frac{Y'}{Y} \right)' + 3 \left( \frac{Y'}{Y} \right)^2 F^2 + 2 \frac{Y'F'}{YF} - \left( \frac{Y'}{Y} \right)^2 F^2,
\]

\[
4\pi(\mu + 3p) = 1 \left( \frac{F''}{FX} \right)' - F \left( \frac{F'}{X} \right)' - 2F \left( \frac{F'}{Y} \right)' - F^2 \left( \frac{Y'}{X} \right)^2
\]

\[
+ 2F^2 \frac{Y'}{YY} + \frac{1}{X^2} \left( \frac{F'}{F} \right)^2 - 2 \frac{Y'F'}{YF},
\]

where \( ' \) denotes \( \partial/\partial r \) and \( ' \) denotes \( \partial/\partial t \).
We first consider the empty space field equations \( R_{ab} = 0 \); this means that in (A 2)–(A 5) we must set \( \mu = p = q = 0 \). The local solution depends on the nature of the surfaces \( \{ Y = \text{constant} \} \); these surfaces may be timelike, spacelike or null, or they may not be defined (if \( Y \) is constant). In the exceptional case when \( Y;^aY;_a = 0 \) on some open set \( \mathcal{U} \) (this includes the case when \( Y \) is constant),

\[
\frac{Y'}{X} = FY'.
\]

(A 6)

holds in \( \mathcal{U} \). However when (A 6) holds, the value of \( Y'' \) determined by (A 2) is inconsistent with (A 3). Thus we may consider some point \( p \) where \( Y;^aY;_a < 0 \) or \( Y;^aY;_a > 0 \); the same inequality must hold in some open neighbourhood \( \mathcal{U}' \) of \( p \).

Consider first the situation when \( Y;^aY;_a < 0 \). Then the surfaces \( \{ Y = \text{constant} \} \) are timelike in \( \mathcal{U} \), and one can choose \( Y \) to be the coordinate \( r \). (Then \( r \) is an area coordinate, as the area of the two-surfaces \( \{ r, t = \text{constant} \} \) is \( 4\pi r^2 \).) Thus \( Y' = 0 \), \( Y' = 1 \) and (A 2) shows that \( X' = 0 \). Further (A 4) shows that \( (F'/F)' = 0 \), so one can choose a new time coordinate \( t'(t) \) in such a way as to set \( F = F(r) \). Then one has \( F = F(r) \), \( X = X(r) \), \( Y = r \); the solution is necessarily static. Equation (A 3) now shows \( d(r/X^2)/dr = 1 \), so solutions are of the form \( X^2 = (1 - 2m/r)^{-1} \) where \( 2m \) is a constant of integration. Equation (A 4) can be integrated, with a suitable choice of a constant of integration, to give \( F^2 = X^2 \), and then (A 5) is identically satisfied. With these forms of \( F \) and \( X \) the metric (A 1) becomes

\[
ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2);
\]

(A 7)

this is the Schwarzschild metric for \( r > 2m \).

Now suppose \( Y;^aY;_a > 0 \). Then the surfaces \( \{ Y = \text{constant} \} \) are spacelike in \( \mathcal{U} \), and one can choose \( Y \) to be the coordinate \( t \). Then \( Y' = 1 \), \( Y' = 0 \) and (A 2) shows \( F' = 0 \). One can choose the \( r \)-coordinate so that \( X = X(t) \); then \( F = F(t) \), \( X = X(t) \), \( Y = t \) and the solution is spatially homogeneous. Now (A 4) and (A 5) can be integrated to find the solution

\[
ds^2 = -\left(\frac{dt^2}{\left(\frac{2m}{t} - 1\right)} + \left(\frac{2m}{t} - 1\right)dr^2 + t^2(d\theta^2 + \sin^2\theta d\phi^2)\right).
\]

(A 8)

This is part of the Schwarzschild solution inside the Schwarzschild radius, for the transformation \( t \rightarrow r' \), \( r \rightarrow t' \) transforms this metric into
the form (A 7) with \( r' < 2m \). Finally, if the surfaces \( \{ Y = \text{constant} \} \) are spacelike in some part of an open set \( \mathcal{V} \) and timelike in another part, one can obtain solutions (A 8) and (A 7) in these parts, and then join them together across the surfaces where \( Y' a Y = 0 \) as in §5.5, obtaining a part of the maximal Schwarzschild solution which lies in \( \mathcal{V} \). Thus we have proved Birkhoff's theorem: any \( C^2 \) solution of Einstein's empty space equations which is spherically symmetric in an open set \( \mathcal{V} \), is locally equivalent to part of the maximally extended Schwarzschild solution in \( \mathcal{V} \). (This is true even if the space is \( C^0 \), piecewise \( C^1 \); see Bergmann, Cahen and Komar (1965).)

We now consider spherically symmetric static perfect fluid solutions. Then one can find coordinates \( \{ t, r, \theta, \phi \} \) such that the metric has the form (A 1), the fluid moves along the \( t \)-lines (so \( q = 0 \)), and \( F = F(r) \), \( X = X(r) \), \( Y = Y(r) \). The field equations (A 3), (A 4) now show that if \( Y' = 0 \), then \( \mu + p = 0 \); we exclude this as being unreasonable for a physical fluid, so we assume \( Y' \neq 0 \). One may therefore again choose \( Y \) as the coordinate \( r \); the metric then has the form

\[
\text{d}s^2 = -\frac{\text{d}t^2}{F^2(r)} + X^2(r) \text{d}r^2 + r^2(\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2).
\]

(A 9)

The contracted Bianchi identities \( T^{ab} ;_b = 0 \) now shows

\[
p' - (\mu + p) F'/F = 0;
\]

(A 10)

(A 5) is identically satisfied if (A 3), (A 4) and (A 10) are satisfied. Equation (A 3) can be directly integrated to show

\[
X^2 = \left( 1 - \frac{2\mathcal{M}}{r} \right)^{-1},
\]

(A 11)

where

\[
\mathcal{M}(r) = 4\pi \int_0^r \mu r^2 \text{d}r,
\]

and the boundary condition \( X(0) = 1 \) has been used (i.e. the fluid sphere has a regular centre). With (A 10), (A 11), equation (A 4) takes the form

\[
\frac{\text{d}p}{\text{d}r} = -\frac{(\mu + p)(\mathcal{M} + 4\pi pr^3)}{r(r - 2\mathcal{M})}
\]

(A 12)

which determines \( p \) as a function of \( r \), if the equation of state is known. Finally (A 10) shows that

\[
F(r) = C \exp \int_{\rho(0)}^{\rho(r)} \frac{\text{d}p}{\mu + p},
\]

(A 13)

where \( C \) is a constant. Equations (A 11)–(A 13) determine the metric inside the fluid sphere, i.e. up to the value \( r_0 \) of \( r \) representing the surface of the fluid.