Glasgow Math. J. **43** (2001) 425–456. © 2001 Glasgow Mathematical Journal Trust. Printed in the United Kingdom

HOROFUNCTIONS AND SYMBOLIC DYNAMICS ON GROMOV HYPERBOLIC GROUPS

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(Received 2 February, 2000; revised 9 November, 2000)

Abstract. Let X be a proper geodesic metric space which is δ-hyperbolic in the sense of Gromov. We study a class of functions on X, called horofunctions, which generalize Busemann functions. To each horofunction is associated a point in the boundary at infinity of X. Horofunctions are used to give a description of the boundary. In the case where X is the Cayley graph of a hyperbolic group Γ, we show, following ideas of Gromov sketched in his paper *Hyperbolic groups*, that the space of cocycles associated to horofunctions which take integral values on the vertices is a one-sided subshift of finite type.

2000 *Mathematics Subject Classification*. 20F67, 20F65, 20F69, 53C23, 53C21, 37D40, 37B10.

1. Introduction. Let X be a proper geodesic metric space which is δ-hyperbolic in the sense of Gromov [**10**]. By a *horofunction* on X, we mean a function $h : X \to \mathbb{R}$ which is quasi-convex and which satisfies the following "distance-like property": for all $x \in X$ and for every real number $\lambda \leq h(x)$, the point x is at distance $h(x) - \lambda$ from the level set $h^{-1}(\lambda) \subset X$.

Given a horofunction h on X, there are (descending) gradient rays starting at every point in X. All these gradient rays converge to a common point on the boundary at infinity ∂X of X. This point is called *the point at infinity of h*.

Two horofunctions which differ by a constant define the same point at infinity. Let Φ be the set of all horofunctions on X up to the equivalence relation which identifies two horofunctions when they differ by a constant. We equip Φ with the quotient of the topology of uniform convergence on compact sets. By taking the point at infinity associated to a horofunction, we obtain a map $\pi : \Phi \to \partial X$. The space Φ is compact and metrizable, and the map $\pi : \Phi \to \partial X$ is continuous and surjective. Furthermore, π is $\mathrm{Isom}(X)$-equivariant with respect to the natural actions of $\mathrm{Isom}(X)$ on the spaces Φ and ∂X.

Now assume that X is the Cayley graph of a word hyperbolic group Γ with respect to some finite symmetric generating set $A \subset \Gamma$. A horofunction $h : X \to \mathbb{R}$ is called an *integral horofunction* if $h(X^0) \subset \mathbb{Z}$, where $X^0 = \Gamma$ denotes the set of vertices of X. Let $\Phi_0 \subset \Phi$ denote the set of equivalence classes of integral horofunctions. The restriction map $\pi : \Phi_0 \to \partial X$ is surjective, Γ-equivariant and uniformly finite-to-one. We fix an arbitrary total order relation on A. Consider the map $\alpha : \Phi_0 \to \Phi_0$ defined by $\alpha(\varphi) = a^{-1}\varphi$ for $\varphi = [h] \in \Phi_0$ and where $a = a(\varphi)$ is the smallest element in A satisfying $h(Id) - h(a) = 1$. The main result in this paper is the following.

THEOREM. *The dynamical system* (Φ_0, α) *is topologically conjugate to a subshift of finite type.*

In [**8**], we use the ideas developed in this paper to obtain a symbolic coding for the geodesic flow associated to a word hyperbolic group.

The plan of this paper is the following.

In Section 2, we define horofunctions on a δ-hyperbolic space X and we study gradient arcs associated to horofunctions.

In Section 3, we use gradient rays to define the point at infinity associated to a horofunction. We establish the main properties of the map $\pi : \Phi \to \partial X$. We prove that any horofunction is entirely determined by its restriction to a 16δ-neighborhood of the image of any geodesic ray converging to its point at infinity. We prove also that every horofunction on X is 68-delta convex.

From Section 4 on, we take X to be the Cayley graph of a hyperbolic group Γ with respect to some finite symmetric generating system.

In Section 4, we prove the main properties of the map $\pi : \Phi_0 \to \partial\Gamma$.

In Section 5, we equip A with a total order relation. This allows us to define the map $\alpha : \Phi_0 \to \Phi_0$.

In Sections 6 and 7, we construct the subshift of finite type $\Sigma(\infty)$ and the homeomorphism $P : \Phi_0 \to \Sigma(\infty)$ conjugating α with the shift map on $\Sigma(\infty)$. Section 8 (the surjectivity of P) is the main difficult part of the proof of the theorem.

We are indebted to M. Gromov for the ideas that we found in [**10**]. This work started with an attempt to understand Section 8.5.Q of [**10**], where a result similar to the theorem above is stated.

Applications of symbolic dynamics to geometry started with the work of Hadamard and then of Morse (see [**11**] and [**12**]). There is an approach, developed by various authors, for the study the symbolic dynamics of hyperbolic groups using Cannon's cone types, and the theory of automatatic structures (see [**5**] and [**9**]). The work of Bourdon (see [**3**] and [**4**]) contains also a study of the symbolic dynamics of boundaries of hyperbolic groups. In the particular case of surface groups, much has been done by R. Bowen and C. Series, and later on by C. Series (see for instance [**1**] and [**13**]).

2. Horofunctions and cocycles. For basic facts about Gromov hyperbolic spaces, we refer the reader to [**10**] and [**6**]. In all this paper, X is a metric space which is proper, geodesic and δ-hyperbolic for some $\delta \geq 0$.

A function $f : X \to \mathbb{R}$ is said to be *quasi-convex* if there exists $\epsilon \geq 0$ such that for all geodesic segment $[x_0, x_1] \in X$ and for every $t \in [0, 1]$, we have

$$f(x_t) \leq (1 - t)f(x_0) + tf(x_1) + \epsilon,$$

where x_t is the point on $[x_0, x_1]$ satisfying $|x_0 - x_t| = t|x_0 - x_1|$. If we want to specify the constant ϵ occuring in this definition, then we say that f is ϵ-convex.

DEFINITION 2.1. Let $\epsilon \geq 0$. An *ϵ-horofunction* (or a *horofunction*) on X is a function $h : X \to \mathbb{R}$ satisfying the following two properties:

(i) "Quasi-convexity property": h is ϵ-convex.

(ii) "Distance-like property": For every $x \in X$ and for every λ satisfying $h(x) \geq \lambda$, we have

$$h(x) = \lambda + \operatorname{dist}\big(x, h^{-1}(\lambda)\big).$$

We note that this definition is slightly different from that given in [**10**], and which is the one we used in [**7**], in which the set $h^{-1}(\lambda)$ in Property (ii) is replaced by the set $h^{-1}(]-\infty, \lambda])$.

We call the level set $h^{-1}(\lambda)$ the *horosphere of radius* λ associated to the horofunction h.

The following is an easy consequence of the distance-like property.

PROPOSITION 2.2. *If $h : X \to \mathbb{R}$ is a horofunction, then h is 1-Lipschitz*

An important class of horofunctions is the class of Busemann functions. We recall the definition.

DEFINITION 2.4. Let $r : [0, \infty[\to X$ be a geodesic ray. The associated *Busemann function* $h_r : X \to \mathbb{R}$ is defined by

$$h_r(x) = \lim_{t \to \infty} (|x - r(t)| - t).$$

Using the triangle inequality, one can see easily that this limit exists and is finite. The proof of the following proposition is contained in [**7**], Chapter 3, with a slight modification to take into account the new definition of a horofunction.

PROPOSITION 2.5. *A Busemann function on X is a* 4δ*-horofunction.*

DEFINITION 2.6. A function $\varphi : X \times X \to \mathbb{R}$ is called an *ϵ-cocycle* if there exists an ϵ-horofunction $h : X \to \mathbb{R}$ such that $\varphi(x, y) = h(x) - h(y)$ for every x and y in X. We call such a function h a *primitive* for φ, and we say that φ is the *cocycle of h*.

As in the case of horofunctions, we shall use the term *cocycle* instead of ϵ-cocycle, unless it is necessary to specify the value of ϵ.

We note that if φ is a cocycle and if h is a primitive for φ, then the set of all primitives of φ consists exactly in the functions on X which are of the form $h + C$, with C being an arbitrary constant.

PROPOSITION 2.7. *Let φ be a cocycle. For every x, y, z and $t \in X$, we have*

(i) $\varphi(x, x) = 0$,

(ii) $\varphi(x, y) = -\varphi(y, x)$,

(iii) $\varphi(x, y) = \varphi(x, z) + \varphi(z, y)$ *(the "cocycle property")*,

(iv) $|\varphi(x, y)| \leq |x - y|$,

(v) $|\varphi(x, y) - \varphi(z, t)| \leq |x - z| + |y - t|$.

Proof. The proof follows immediately from Proposition 2.2.

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If φ is a cocycle, then the relation $\sim$ on X defined by $x \sim y \Longleftrightarrow \varphi(x, y) = 0$ is an equivalence relation. The equivalence classes of $\sim$ are called the *horospheres* of φ. If h is a primitive of φ, then the horospheres of φ coincide with the horospheres of h.

Let γ be an isometry of X and let $h : X \to \mathbb{R}$ is an ϵ-horofunction. The function $\gamma h : X \to \mathbb{R}$ defined by

$$\gamma h(x) = h(\gamma^{-1} x)$$

is clearly an ϵ-horofunction. In the same way, if $\varphi : X \times X \to \mathbb{R}$ is an ϵ-cocycle, then the function $\gamma\varphi : X \times X \to \mathbb{R}$ defined by

$$\gamma\varphi(x, y) = \varphi(\gamma^{-1}x, \gamma^{-1}y)$$

is an ϵ-cocycle. In fact, if φ is the cocycle of h, then $\gamma\varphi$ is the cocycle of γh.

Let Φ be the set of cocycles on X, that is, the set of all ϵ-cocycles, with all possible values of ϵ. We equip Φ with the quotient topology of the topology of uniform convergence on compact sets.

Let $\mathrm{Isom}(X)$ denote the group of isometries of X. We see easily from the definitions that the action of $\mathrm{Isom}(X)$ on Φ defined by $(\gamma, \varphi) \mapsto \gamma\varphi$ is continuous.

Definition 2.8. Let φ be a cocycle on X. A *gradient arc for* φ, or a φ-*gradient arc* is a path $g : I \to X$ parametrized by arclength and satisfying $\varphi(g(t), g(t')) = t' - t$ for every t and t' in I. In the case where $I = \mathbb{R}$, we say that g is a *gradient line*. In the case where $I = [0, \infty[$, we say that g is a *gradient ray*. If g is a gradient ray and if $x = g(0)$, then we say that g *starts at* x. If $h : X \to \mathbb{R}$ is a horofunction, then a *gradient arc for* h, or an h-*gradient arc* is a gradient arc for the cocycle of h.

The proof of the following lemma follows easily from the definitions.

Lemma 2.9 (*Concatenation of gradient arcs*). *Let φ be a cocycle on X and let $I \subset \mathbb{R}$ be an interval, with $a \in I$, $I_1 = I \cap]-\infty, a]$ and $I_2 = I \cap [a, \infty[$. If $g : I \to X$ is a path whose restrictions to I_1 and I_2 are φ-gradient arcs, then g is itself a φ-gradient arc.*

The next proposition establishes relations between gradient arcs and geodesics.

Proposition 2.10. *Let φ be a cocycle on X. Then*

(i) *Any φ-gradient arc $g : I \to X$ is a geodesic.*

(ii) *If x and y are points in X satisfying $\varphi(x, y) = |x - y|$, and if $g : [a, b] \to X$ is a geodesic joining x and y, then g is a φ-gradient arc.*

Proof. The proof follows from Properties (iii) and (iv) of Proposition 2.7.

If $Y \subset X$, and $x \in X$, then a *projection* of x on Y is a point $y_0 \in Y$ satisfying $\mathrm{dist}(x, Y) = |x - y_0|$, where $\mathrm{dist}(x, Y) = \inf_{y \in Y} |x - y|$.

Proposition 2.11. *Let h be a horofunction on X and let $g : I \to X$ be an h-gradient arc. Then, for every t and t' in I satisfying $t \le t'$, $g(t')$ is a projection of $g(t)$ on $h^{-1}\big(h(g(t'))\big)$.*

Proof. Since g is geodesic, we have $|g(t) - g(t')| = t' - t$. Since $t \leq t'$, we have $h\big(g(t)\big) \geq h\big(g(t')\big)$, which implies $h\big(g(t)\big) = h\big(g(t')\big) + \mathrm{dist}\big(g(t), h^{-1}(h(g(t')))\big)$. Therefore, we obtain $|g(t) - g(t')| = t' - t = h\big(g(t)\big) - h\big(g(t')\big) = \mathrm{dist}\big(g(t), h^{-1}(h(g(t')))\big)$.

PROPOSITION 2.12. *Let h be a horofunction on X, let $x \in X$ and let $\lambda \leq h(x)$. Then, there exists a projection of x on $h^{-1}(\lambda)$. Furthermore, if y is a projection of x on $h^{-1}(\lambda)$, then $h(y) = \lambda$ and every geodesic joining x and y is an h-gradient arc.*

Proof. By property (ii) of Definition 2.1, we have $\mathrm{dist}\big(x, h^{-1}(\lambda)\big) < \infty$. Therefore, $h^{-1}(\lambda) \neq \emptyset$. The set $h^{-1}(\lambda)$ is closed, and since X is proper, we can find a projection y of x on this set. We have $\mathrm{dist}\big(x, h^{-1}(\lambda)\big) = |x - y| = h(x) - \lambda$, and since $y \in h^{-1}(\lambda)$, we obtain $h(y) = \lambda$.

If $g : [a, b] \to X$ is now a geodesic arc joining x and y, we have $h\big(g(a)\big) - h\big(g(b)\big) = |x - y|$. By Proposition 2.10 (ii), g is an h-gradient arc.

PROPOSITION 2.13. *For every cocycle φ and for every x in X, there exists a φ-gradient ray $g : [0, \infty[\to X$ starting at x.*

Proof. Let h be a primitive for φ and let us fix an arbitrary real number $\lambda > 0$. We let $x_0 = x$ and for every integer $i \geq 0$, we take x_{i+1} to be a projection of x_i on $h^{-1}\big(h(x_i) - \lambda\big)$ (such a point x_{i+1} exists by Proposition 2.12). There is a unique path $g : [0, \infty[\to X$ starting at x_0, parametrized by arclength and whose image is obtained by concatenating the segments $[x_i, x_{i+1}]$. By Proposition 2.12, each subpath of g whose image is one of the geodesic segments $[x_i, x_{i+1}]$ is an h-gradient arc. Lemma 2.9 implies now that g is an h-gradient ray.

PROPOSITION 2.14. *Let φ be a cocycle on X, let $g : I \to X$ be a φ-gradient arc and let $\gamma \in Isom(X)$. Then $\gamma g : I \to X$ is a gradient arc for the cocycle $\gamma\varphi$.*

Proof. The proof follows easily from the definitions.

3. The point at infinity associated to a cocycle. Given a cocycle φ on X, any φ-gradient ray $g : [0, \infty[\to X$, being a geodesic, converges to a well-defined point $g(\infty) \in \partial X$.

PROPOSITION 3.1. *Let φ be a cocycle on X and let $g : [0, \infty[\to X$ and $g' : [0, \infty[\to X$ be two φ-gradient rays. Then $g(\infty) = g'(\infty)$.*

To prove Proposition 3.1, we shall use the following lemma, which is an extension of Lemma 3.2 in [**2**]:

LEMMA 3.2. *Let $x, y, p, q \in X$ and let m be the middle of some geodesic segment $[p, q]$. Assume that L and ϵ are real numbers satisfying the following three properties:*

(i) $L \leq |x - p|$ *and* $L \leq |y - q|$,
(ii) $|x - p| \leq |x - m| + \epsilon$ *and* $|y - q| \leq |y - m| + \epsilon$,
(iii) $|x - y| < 2L - 2\epsilon - 16\delta$.

Then, we have $|p - q| \leq 32\delta + 2\epsilon$.

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Proof. Let us draw three geodesic segments $[p, x]$, $[x, y]$ and $[y, q]$. By δ-hyperbolicity, there is a point u on $[p, x] \cup [x, y] \cup [y, q]$ such that $|m - u| \leq 8\delta$. We cannot have $u \in [x, y]$ since otherwise we can assume by symmetry that u lies between x and the middle m' of the geodesic segment $[x, y]$, and we would get then

$$\begin{aligned} |x - y| &= 2|x - m'| \\ &\geq 2|x - u| \\ &\geq 2(|x - m| - |m - u|) \quad \text{by the triangle inequality,} \\ &\geq 2(|x - m| - 8\delta) \\ &\geq 2(|x - p| - \epsilon - 8\delta) \quad \text{by (ii),} \\ &\geq 2L - 2\epsilon - 16\delta \ \text{ by (i),} \end{aligned}$$

which contradicts (iii). Therefore, we have $u \in [p, x] \cup [y, q]$. By symmetry, we can assume $u \in [p, x]$ (see Figure 1).

We have

$$|p - u| + |u - x| = |p - x| \leq |m - x| + \epsilon \leq |m - u| + |u - x| + \epsilon \leq 8\delta + |u - x| + \epsilon$$

which yields $|p - u| \leq 8\delta + \epsilon$. Therefore, we obtain

$$|p - q| = 2|p - m| \leq 2(|p - u| + |u - m|) \leq 32\delta + 2\epsilon.$$

Proof of Proposition 3.1. Suppose that φ is an ϵ-cocycle. We can assume, without loss of generality, that $a = \varphi\big(g(0), g'(0)\big) \geq 0$. Let $x = g(a)$ and $y = g'(0)$. The cocycle property implies then $\varphi(x, y) = 0$. Let h be the primitive of φ satisfying $h(x) = h(y) = 0$. Consider a real number $L \geq 0$ and let $p = g(a + L)$ and $q = g'(L)$. Since g and g' are gradient rays for h, we have $h(p) = h(q) = -L$.

Let m be the midpoint of some geodesic segment $[p, q]$ (see Figure 2).

By the ϵ-convexity of h, we have $h(m) \leq \frac{1}{2}h(p) + \frac{1}{2}h(q) + \epsilon = -L + \epsilon$. We obtain therefore $|x - p| = L \leq -h(m) + \epsilon = h(x) - h(m) + \epsilon \leq |x - m| + \epsilon$.

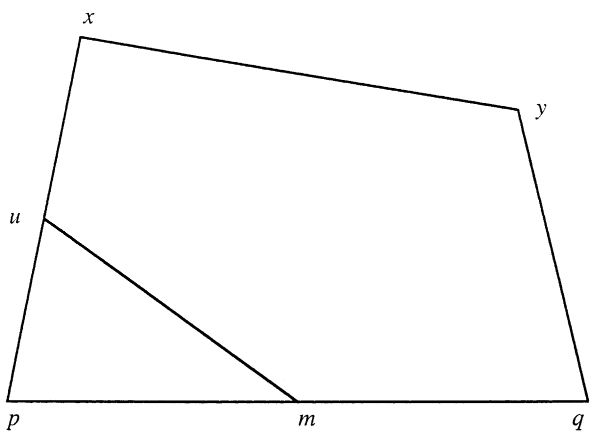

Figure 1.

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Similarly, we have $|y - q| \leq |y - m| + \epsilon$.

Using Lemma 3.2, we deduce that $|p - q| \leq 32\delta + 2\epsilon$, provided L satisfies $|x - y| < 2L - 2\epsilon - 16\delta$.

Therefore, we have $|g(a + L) - g'(L)| \leq 32\delta + 2\epsilon$ for all L large enough, which proves Proposition 3.1.

We can define now the a map $\pi : \Phi \to \partial X$ which associates to each cocycle $\varphi \in \Phi$ the endpoint of an arbitrary φ-gradient ray.

Proposition 3.3. *The map $\pi : \Phi \to \partial X$ is continuous, surjective and equivariant with respect to the actions of $Isom(X)$ on the spaces Φ and ∂X.*

Proof. For the surjectivity, let $\xi \in \partial X$ and let $r : [0, \infty[\to X$ be a geodesic ray converging to ξ. Let h_r be the associated Busemann function and let φ_r be the cocycle of h_r. For every $t \geq 0$, we have $h_r\big(r(t)\big) = -t$, from which it is easy to see that r is a φ_r-gradient ray. Therefore, we have $\pi(\varphi_r) = r(\infty) = \xi$.

To prove the continuity of π, let $(\varphi_n)_{n\geq 0}$ be a sequence of elements of Φ converging to $\varphi \in \Phi$, and for each $n \geq 0$, let h_n be a primitive of φ_n, normalized so as to take the same value on a fixed point $x \in X$ for all $n \geq 0$. In this way, the sequence (h_n) converges to a primitive h of φ. For each $n \geq 0$, we can construct, as in the proof of Proposition 2.13, a φ_n-gradient ray g_n starting at x by taking a sequence of successive projections on the horospheres of h_n. As the values of the horofunctions h_n are close to those of h uniformly on compact sets of X, we can manage so that the sequences of projections that we use to construct the rays g_n are uniformly close to the sequence of projections which are associated to φ. Thus, as $n \to \infty$, the geodesic rays g_n are uniformly close on every compact set from the φ-gradient ray. Therefore we have $\pi(\varphi_n) \to \pi(\varphi)$. This proves the continuity of π. The equivariance follows easily from the definitions.

For each cocycle φ, for each geodesic ray $r : [0, \infty[\to X$ satisfying $r(\infty) = \pi(\varphi)$ and for each $t \geq 0$, we set

$$R_{\varphi,t} = \{z \in X \ : \ \varphi\big(r(t), z\big) = 0\} \cap B\big(r(t), 16\delta\big).$$

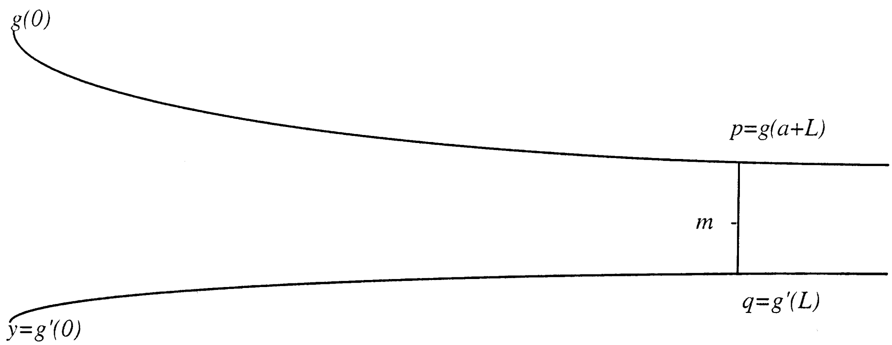

Figure 2.

PROPOSITION 3.4. *Let φ be a cocycle on X and let $r : [0, \infty[\to X$ be a geodesic ray such that $r(\infty) = \pi(\varphi)$. For all x in X and for all t satisfying $t > |x - r(0)| + 16\delta$, we have $\varphi\big(x, r(t)\big) = \mathrm{dist}(x, R_{\varphi,t})$*

Proof. Let $x \in X$ and let $t > |x - r(0)| + 16\delta$. Let $g : [0, \infty[\to X$ be a φ-gradient ray starting at x. By Proposition 3.1, the geodesic rays g and r converge to the same point at infinity. By δ-hyperbolicity, since we have $t > |x - r(0)| + 16\delta$, we can find $t' \geq 0$ such that $|r(t) - g(t')| \leq 8\delta$. We have

$$t' = |x - g(t')| \geq |r(0) - r(t)| - |x - r(0)| - |r(t) - g(t')| = t - |x - r(0)| - |r(t) - g(t')| \geq 8\delta.$$

Let us set $u = \varphi\big(x, r(t)\big)$.We have $u = \varphi\big(x, g(t')\big) + \varphi\big(g(t'), r(t)\big) = t' - \varphi\big(r(t), g(t')\big)$.

Using Proposition 2.7 (iv), we obtain $\varphi\big(r(t), g(t')\big) \leq |r(t) - g(t')| \leq 8\delta$. This shows that $u \geq 0$.

Using the cocycle property, we have $\varphi\big(r(t), g(u)\big) = \varphi\big(g(0), g(u)\big) - u = 0$. On the other hand, we have $|r(t) - g(u)| \leq |r(t) - g(t')| + |g(t') - g(u)| \leq 8\delta + |g(t') - g(u)|$. Since $|g(t') - g(u)| = |\varphi\big(g(t'), g(u)\big)| = |\varphi\big(g(t'), r(t)\big)| \leq |g(t') - r(t)| \leq 8\delta$, we deduce that $|r(t) - g(u)| \leq 16\delta$. Thus, $g(u) \in R_{\varphi,t}$. By Proposition 2.11, the point $g(u)$ is a projection of x on $R_{\varphi,t}$. Therefore, we have $\mathrm{dist}(x, R_{\varphi,t}) = |x - g(u)| = |g(0) - g(u)| = u$, that is, $\mathrm{dist}(x, R_{\varphi,t}) = \varphi\big(x, r(t)\big)$. This proves Proposition 3.4.

COROLLARY 3.5. *Let $r : [0, \infty[\to X$ be a geodesic ray, let $r(\infty) = \xi \in \partial X$ and let $(t_n)_{n\geq 0}$ be a sequence of nonnegative real numbers tending to infinity. For each $n \geq 0$, let B_n be the closed ball of radius 16δ centered at $r(t_n)$. Let φ and φ' be two elements of Φ satisfying $\pi(\varphi) = \pi(\varphi') = \xi$ and such that for every $n \geq 0$, φ and φ' have the same restriction on $B_n \times B_n$. Then $\varphi = \varphi'$.*

Proof. Let x and y be two arbitrary points in X. Since $t_n \to \infty$, we can find an integer $n \geq 0$ satisfying $t_n > \max(|x - r(0)|, |y - r(0)|) + 16\delta$. We fix such an integer n.

The hypotheses imply that $R_{\varphi,t_n} = R_{\varphi',t_n}$, and we have, by Proposition 3.4,

$$\varphi\big(x, r(t_n)\big) = \mathrm{dist}\big(x, R_{\varphi,t_n}\big) = \varphi'\big(x, r(t_n)\big)$$

and

$$\varphi\big(y, r(t_n)\big) = \mathrm{dist}\big(y, R_{\varphi,t_n}\big) = \varphi'\big(y, r(t_n)\big).$$

Using the cocycle property, we obtain $\varphi(x, y) = \varphi'(x, y)$. This proves Corollary 3.5.

COROLLARY 3.6. *Let $h : X \to \mathbb{R}$ and $h' : X \to \mathbb{R}$ be horofunctions on X having the same point at infinity $\xi \in \partial X$. Let $r : [0, \infty[\to X$ be a geodesic ray with $r(\infty) = \xi$. Assume that h and h' have the same restrictions to the closed 16δ-neighborhood of $r([0, \infty[)$. Then $h = h'$.*

Proof. Consider a sequence $(t_n)_{n\geq 0}$ of nonnegative real numbers tending to infinity and for every $n \geq 0$, let $B_n = B\big(r(t_n), 16\delta\big)$. By hypothesis, the cocycles associated to h and h' coincide on $B_n \times B_n$, for every $n \geq 0$. By Corollary 3.5, these cocycles are equal. Since $h\big(r(0)\big) = h'\big(r(0)\big)$, this shows that $h = h'$.

COROLLARY 3.7. *Every horofunction on X is 68δ-convex.*

Proof. Let $h : X \to \mathbb{R}$ be a horofunction and let φ denote the cocycle of h. We use the notations of Proposition 3.4. Let x and $y \in X$ and for $t \geq 0$, let p and q be respectively projections of x and y on $R_{\varphi,t}$. Consider a geodesic segment $[x, y]$ and let z be a point on this segment satisfying $|x - z| = u|x - y|$ for some $u \in [0, 1]$.

By the quasi-convexity of the distance-function (see [7], Chapter 3, Lemma 3.2.), we have $|z - p| \leq (1 - u)|x - p| + u|y - p| + 4\delta$. Since the diameter of $R_{\varphi,t}$ is bounded by 32δ, we have $|y - p| \leq |y - q| + 32\delta$.

Letting s be a projection of z on $R_{\varphi,t}$, we can write now

$$\begin{aligned} |z - s| &\leq |z - p| + 32\delta \leq (1 - u)|x - p| + u|y - p| + 4\delta + 32\delta \\ &= (1 - u)|x - p| + u|y - q| + 68\delta. \end{aligned}$$

Let $t > \max(|x - r(0)|, |y - r(0)|) + |x - y| + 16\delta$. By the triangle inequality, this implies that for every $z \in [x, y]$, we have $t > |z - r(0)| + 16\delta$. Proposition 3.4 implies now that $h(x) = h\big(r(t)\big) + |x - p|$, $h(y) = h\big(r(t)\big) + |y - q|$ and $h(z) = h\big(r(t)\big) + |z - s|$. Therefore, we obtain

$$h(z) \leq (1 - u)h(x) + uh(y) + 68\delta.$$

This proves Corollary 3.7.

For $F : X \times X \to \mathbb{R}$, we define $\|F\|_\infty = \sup_{(x,y)\in X\times X} |F(x, y)|$.

COROLLARY 3.8. *Let φ and $\varphi' \in \Phi$. Then, the following three perperties are equivalent :*

(i) $\pi(\varphi) = \pi(\varphi')$
(ii) $\|\varphi - \varphi'\|_\infty \leq 64\delta$
(iii) $\|\varphi - \varphi'\|_\infty < \infty$.

Proof. Suppose first that $\pi(\varphi) = \pi(\varphi') = \xi$ and let $r : [0, \infty[\to X$ be a geodesic ray satisfying $r(0) = Id$ and $r(\infty) = \xi$. Let x and y be arbitrary points in X and let $t \geq \max\{|x - r(0)|, |y - r(0)|\} + 16\delta$. Proposition 3.4 implies then $\varphi\big(x, r(t)\big) = \mathrm{dist}(x, R_{\varphi,t})$ and $\varphi\big(y, r(t)\big) = \mathrm{dist}(y, R_{\varphi,t})$.

By the cocycle property, we obtain therefore

$$\varphi(x, y) = \varphi\big(x, r(t)\big) - \varphi\big(y, r(t)\big) = \mathrm{dist}(x, R_{\varphi,t}) - \mathrm{dist}(y, R_{\varphi,t}).$$

In the same way, we have $\varphi'(x, y) = \mathrm{dist}(x, R_{\varphi',t}) - \mathrm{dist}(y, R_{\varphi',t})$.

Therefore, we have

$$\begin{aligned} |\varphi(x, y) - \varphi'(x, y)| &= |\mathrm{dist}(x, R_{\varphi,t}) - \mathrm{dist}(y, R_{\varphi,t}) - \big(\mathrm{dist}(x, R_{\varphi',t}) - \mathrm{dist}(y, R_{\varphi',t})\big)| \\ &\leq |\mathrm{dist}(x, R_{\varphi,t}) - \mathrm{dist}(x, R_{\varphi',t})| + |\mathrm{dist}(y, R_{\varphi,t}) - \mathrm{dist}(y, R_{\varphi',t})|. \end{aligned}$$

Since the sets $R_{\varphi,t}$ and $R_{\varphi',t}$ are both contained in the closed ball $B(r(t), 16\delta)$ whose diameter is bounded above by 32δ, we have $|\mathrm{dist}(x, R_{\varphi,t}) - \mathrm{dist}(x, R_{\varphi',t})| \leq 32\delta$

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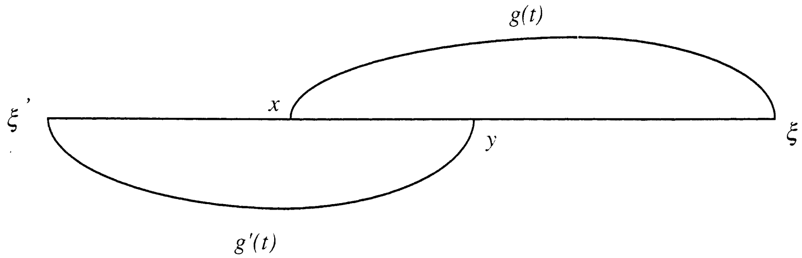

Figure 3.

and $|\mathrm{dist}(y, R_{\varphi,t}) - \mathrm{dist}(y, R_{\varphi',t})| \leq 32\delta$. Therefore, we obtain $|\varphi(x, y) - \varphi'(x, y)| \leq 32\delta + 32\delta = 64\delta$, which proves that (i)⇒(ii).

The implication (ii)⇒(iii) is clear. We prove now (iii)⇒(i).

Let φ and $\varphi' \in \Phi$ satisfy $\pi(\varphi) = \xi$ and $\pi(\varphi') = \xi'$, with $\xi \neq \xi'$, and let $\ell : \mathbb{R} \to X$ be a geodesic line such that $\ell(-\infty) = \xi'$ and $\ell(\infty) = \xi$. We set $x = \ell(0)$ and for $t > 0$, we set $y = \ell(t)$. Let $g : [0, \infty[\to X$ be a φ-gradient ray starting at x. We have therefore $g(\infty) = \xi$. By δ-hyperbolicity, we have $|y - g(t)| \leq 4\delta$. Proposition 2.7 (iv) gives $|\varphi\big(y, g(t)\big)| \leq 4\delta$. By the cocycle property, we obtain $|\varphi(x, y) - t| \leq 4\delta$.

Let $g' : [0, \infty[\to X$ be a φ'-gradient ray starting at y (see Figure 3). In the same way, we obtain $|\varphi'(y, x) - t| \leq 4\delta$.

We conclude that $|\varphi(x, y) - \varphi'(x, y) - 2t| \leq 8\delta$. Since $t \geq 0$ is arbitrary, we obtain $\|\varphi - \varphi'\|_\infty = \infty$. This proves that (iii)⇒(i), which concludes the proof of Corollary 3.8.

Proposition 3.9. *The space* Φ *is compact.*

Proof. Let us fix a point $x_0 \in X$ and let H denote the space of horofunctions on X which vanish at x_0. The map which associates to each element of H its cocycle is clearly a homeomorphism between H and Φ. Let us prove that H is compact.

Every function $h \in H$ is 1-Lipschitz (Proposition 2.2) and satisfies $|h(x)| \leq |x - x_0|$ for all $x \in X$. Therefore, to prove that H is compact, it is sufficient by Ascoli's theorem to show that H is a closed subset of the space of continuous functions on X (for the topology of uniform convergence on compact sets).

Consider a function f on X which is the limit of a sequence (h_n) of elements in H and let us show that $f \in H$. The function f is 68δ-convex since every h_n is 68δ-convex (Corollary 3.7). Let us show that f satisfies the distance-like property. Let $x \in X$ and let $\lambda < f(x)$. The function f is 1-Lipschitz since every h_n is 1-Lipschitz. This implies that

$$f(x) \leq \lambda + \mathrm{dist}(x, f^{-1}(\lambda)). \tag{3.9.1}$$

Let (λ_n) be a sequence of real numbers which converges to λ and such that $\lambda < \lambda_n < f(x)$ for all n. Since the distance-like property is satisfied by h_n, we can find, for n large enough, a point $p_n \in X$ such that

$$h_n(x) = \lambda_n + |x - p_n| \tag{3.9.2}$$

and

$$h_n(p_n) = \lambda_n. \tag{3.9.3}$$

After possibly replacing the sequence (p_n) by a subsequence, we can assume that (p_n) converges to a point $p \in X$. By taking limits, we deduce from (3.9.2) that

$$f(x) = \lambda + |x - p|. \tag{3.9.4}$$

Since $|h_n(p_n) - h_n(p)| \leq |p_n - p|$, we have, using (3.9.3),

$$f(p) = \lim_{n\to\infty} h_n(p) = \lim_{n\to\infty} h_n(p_n) = \lambda. \tag{3.9.5}$$

It follows from (3.9.4) and (3.9.5) that

$$f(x) \geq \lambda + \mathrm{dist}(x, f^{-1}(\lambda)). \tag{3.9.6}$$

We deduce from (3.9.1) and (3.9.6) that $f(x) = \lambda + \mathrm{dist}(x, f^{-1}(\lambda))$. Thus f satisfies the distance-like property. We have shown that f is a horofonction. Since $h_n(x_0) = 0$ for all n, we have $f(x_0) = 0$. Therefore $f \in H$. This completes the proof of Proposition 3.9.

Theorem 3.10. *Let $\sim$ be the equivalence relation on Φ defined by*

$$\varphi \sim \varphi' \iff \|\varphi - \varphi'\|_\infty < \infty.$$

Then the map $\pi : \Phi \to \partial X$ induces an $\mathrm{Isom}(X)$*-equivariant homeomorphism from the quotient space $\Phi/\sim$ onto ∂X.*

Proof. This quotient map is $\mathrm{Isom}(X)$-equivariant, continous and bijective, by Proposition 3.3 and Corollary 3.8. Since Φ is compact and since ∂X is Hausdorff, the quotient map is a homeomorphism.

4. Integral cocycles on hyperbolic groups. In all what follows, Γ is a group which is δ-hyperbolic with respect to some fixed finite set of generators A, and X is the Cayley graph associated to the pair (Γ, A). We denote by $X^0 = \Gamma$ the set of vertices and by X^1 the set of edges of X.

Definition 4.1. An *integral horofunction* on X is a horofunction $h : X \to \mathbb{R}$ satisfying $h(X^0) \subset \mathbb{Z}$. An *integral cocycle* is a cocycle having an integral horofunction as a primitive. Equivalently, an integral cocycle is a cocycle taking integral values on $X^0 \times X^0$.

Proposition 4.2. *Let $r : [0, \infty[\to X$ be a geodesic ray starting at a point in $X^0 = \Gamma$. Then, the associated Busemann function h_r is an integral 4δ-horofunction.*

Proof. The proof is easy, using Proposition 2.5.

Proposition 4.3. *Let $h : X \to \mathbb{R}$ be an integral horofunction and let x and y be adjacent vertices of X. Then $h(y)$ is equal either to $h(x)$, to $h(x) - 1$ or to $h(x) + 1$.*

Furthermore, the restriction of h to the segment $[x, y]$ is entirely determined by the values $h(x)$ and $h(y)$.

Proof. The proof is easy, using Proposition 2.2, Proposition 2.10 (ii) and Proposition 2.13.

COROLLARY 4.4. *An integral cocycle φ is completely determined by its values on the set $X^0 \times X^0 = \Gamma \times \Gamma$.*

Thus, we can regard an integral cocycle φ on X as a function from $\Gamma \times \Gamma$ to $\mathbb{Z}$. Let $\Phi_0 \subset \Psi$ be the space of integral cocycles on X. The topology induced by Φ on Φ_0 is the topology of pointwise convergence on $\Gamma \times \Gamma$.

For simplicity, we still denote by $\pi : \Phi_0 \to \partial\Gamma$ the restriction of the map $\pi : \Phi \to \partial\Gamma$ defined in Section 3.

PROPOSITION 4.5. *The map $\pi : \Phi_0 \to \partial\Gamma$ is continuous, Γ-equivariant, onto and uniformly finite to one. In fact, we have, for every $\xi \in \partial\Gamma$,*

$$\mathrm{card}\{\varphi \in \Phi \ : \ \pi(\varphi) = \xi\} \leq (2N_0 + 1)^{N_1},$$

where N_0 is the integral part of $16\delta + 1$ and where N_1 is the number of elements in Γ contained in the closed ball of radius N_0 centered at the identity.

Proof. The continuity and the Γ-equivariance of the map π follow from Proposition 3.3, and the surjectivity follows from Proposition 4.2. To prove the last statement in the proposition, we need the following lemma, which will also be useful in Section 7 below.

LEMMA 4.6. *Let $B = B(x_0, N_0)$ be a closed ball in X centered at $x_0 \in \Gamma$ and whose radius is an integer $N_0 \geq 0$. Then the number of distinct restrictions to $B \times B$ of elements $\varphi \in \Phi_0$ is bounded above by $(2N_0 + 1)^{N_1}$, where N_1 is the number of elements of Γ at distance $\leq N_0$ from the identity.*

Proof. By Proposition 4.3 and the cocycle property, the restriction of φ to $B \times B$ is determined by the function $f : B \cap \Gamma \to \mathbb{Z}$ defined by $f(x) = \varphi(x_0, x)$. We have, for every x in B, using Proposition 2.7(iv), $|f(x)| = |\varphi(x_0, x)| \leq |x - x_0| \leq N_0$. Since the cardinality of $B \cap \Gamma$ is N_1, the assertion follows easily.

Consider now an arbitrary finite subset $F \subset \pi^{-1}(\xi)$ and let $N' = \mathrm{card}(F)$. Let $r : [0, \infty[\to X$ be a geodesic ray starting at a vertex of X, with $r(\infty) = \xi$, let $(t_n)_{n \geq 0}$ be a sequence of nonnegative real numbers and for every $n \geq 0$, let B_n be the closed ball in X of radius N_0 centered at $r(t_n)$. We choose the sequence (t_n) in such a way that for every $n \geq 0$, $t_{n+1} - t_n = |r(t_n) - r(t_{n+1})| > N_0 + 16\delta$.

Let φ and φ' be two distinct elements of F. By Corollary 3.5, we can find an integer $m \geq 0$ such that the restrictions of φ and φ' to $B_m \times B_m$ are distinct.

We claim now that if $n \geq 1$ is an integer such that φ and φ' have the same restrictions on $B_n \times B_n$, then φ and φ' have also the same restrictions on $B_{n-1} \times B_{n-1}$. Indeed, for $x \in B_{n-1}$, we have

$$\begin{aligned} t_n &= |r(0) - r(t_n)| \\ &= |r(0) - r(t_{n-1})| + |r(t_{n-1}) - r(t_n)| \\ &\geq (|r(0) - x| - |x - r(t_{n-1})|) + |r(t_{n-1}) - r(t_n)| \\ &> (|r(0) - x| - N_0) + (N_0 + 16\delta) \\ &= |r(0) - x| + 16\delta. \end{aligned}$$

Therefore, Proposition 3.4 gives $\varphi(x, r(t_n)) = \mathrm{dist}(x, R_{\varphi,t_n})$.

For x and y arbitrary in B_{n-1}, we obtain $\varphi(x, y) = \varphi(x, r(t_n)) - \varphi(y, r(t_n)) = \mathrm{dist}(x, R_{\varphi,t_n}) - \mathrm{dist}(y, R_{\varphi,t_n})$.

Thus, the value of $\varphi(x, y)$ depends only on the restriction of the cocycle φ on B_n. This proves the claim.

Therefore, there exists an integer $n_0 \geq 0$ such that for every $n \geq n_0$, we have $\varphi_{|B_n \times B_n} \neq \varphi_{|B_n \times B_n}$. Since F is finite, we can find, by taking n large enough, a ball B_n with the property that the restriction to $B_n \times B_n$ of all of the N' cocycles in F are distinct. By Lemma 4.6, we obtain therefore $N' \leq (2N_0 + 1)^{N_1}$. This proves Proposition 4.5.

5. The map $\alpha : \Phi_0 \to \Phi_0$. We start with the following

LEMMA 5.1. *For every element $\varphi \in \Phi_0$ and for every $x \in \Gamma = X^0$, there exists an element $a \in A$ satisfying $\varphi(x, xa) = 1$.*

Proof. By Proposition 2.13, we can find a φ-gradient ray $g : [0, \infty[\to X$ starting at x. In particular, we have $\varphi(x, g(1)) = 1$. Furthermore, since g is a geodesic, we have $|x - g(1)| = 1$, which implies that $g(1) = xa$ for some a in A.

We fix now, and for the rest of this paper, a total order relation on the generating set A. Let $x \in X$. The lexicographic order on $A^{\mathbb{N}}$ induces a total order on the set of φ-gradient rays starting at x.

PROPOSITION 5.2. *Let $x \in \Gamma$. The set of φ-gradient rays starting at x has a smallest element.*

Proof. We define first by induction a sequence $(x_n)_{n\geq 0}$ of vertices of X. We start by letting $x_0 = x$. Assuming that x_n has been defined, we let $a \in A$ be the smallest element of A such that $\varphi(x_n, x_n a) = 1$ and we take then $x_{n+1} = x_n a$. Let $g : [0, \infty[\to X$ be the uniquely defined ray, parametrized by arclength and satisfying $g(n) = x_n$ for every $n \in \mathbb{N}$. By Lemma 2.9, g is a φ-gradient ray. It is clear that this ray g is smallest among all the gradient rays staring at x.

We call the gradient ray provided by Proposition 5.2 *the smallest φ-gradient ray starting at x*.

PROPOSITION 5.3. *Let $\varphi \in \Phi_0$, $x \in X$ and $\gamma \in \Gamma$. If $g : [0, \infty[\to X$ is the smallest φ-gradient ray starting at x, then for every $n \geq 0$, the smallest φ-gradient ray starting at $g(n)$ is the ray $g_n : [0, \infty[\to X$ defined by $g_n(t) = g(t + n)$, for every $t \geq 0$.*

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Proof. The proof follows from the construction of the smallest φ-gradient ray given in the proof of Proposition 5.2.

PROPOSITION 5.4. *Let $\varphi \in \Phi_0$, let $x \in \Gamma$ and let $g : [0, \infty[\to X$ be the smallest φ-gradient ray starting at x. Then for all $\gamma \in \Gamma$, the map $\gamma g : [0, \infty[\to X$ is the smallest $\gamma\varphi$-gradient ray starting at γx.*

Proof. The proof is easy, using the construction of the smallest gradient ray described in the proof of Proposition 5.2.

DEFINITION 5.5. We define the map $\alpha : \Phi_0 \to \Phi_0$ by letting, for every $\varphi \in \Phi_0$, $\alpha(\varphi) = a^{-1}\varphi$, where a is the smallest element in A satisfying $\varphi(Id, a) = 1$.

PROPOSITION 5.6. *The map $\alpha : \Phi_0 \to \Phi_0$ is continuous.*

Proof. Let φ_n be a sequence of elements in Φ_0 converging to $\varphi \in \Phi_0$. There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the cocycles φ and φ_n coincide on $B(Id, 1) \times B(Id, 1)$. Therefore, there exists $a \in A$ such that $\forall n \geq n_0$, $\alpha(\varphi) = a^{-1}\varphi$ and $\alpha(\varphi_n) = a^{-1}\varphi_n$. Since Γ acts continuously on Φ_0, this shows that $\alpha(\varphi_n) \to \alpha(\varphi)$ as $n \to \infty$.

PROPOSITION 5.7. *Let $\varphi \in \Phi_0$ and let $g : [0, \infty[\to X$ be the smallest φ-gradient ray starting at the identity. For every $n \in \mathbb{N}$, let $a_n \in A$ be the label of the oriented edge going from $g(n)$ to $g(n+1)$ and let $g_n : [0, \infty[\to X$ be the smallest $\alpha^n(\varphi)$-gradient ray starting at the identity. Then, we have*

(i) $\alpha^n(\varphi) = g(n)^{-1}\varphi$.

(ii) *For every $t \geq 0$, $g_n(t) = g(n)^{-1}g(t+n)$.*

(iii) *For every $k \in \mathbb{N}$, the label of the oriented edge going from $g_n(k)$ to $g_n(k+1)$ is a_{k+n}.*

Proof. Let us first prove (i) by induction on n. For $n = 0$, the statement is obviously true. Suppose now that $\alpha^n(\varphi) = g(n)^{-1}\varphi$ for some $n \in \mathbb{N}$. Then, $\alpha^{n+1}(\varphi) = \alpha(\alpha^n(\varphi)) = \alpha(g(n)^{-1}\varphi)$. By definition, we have $\alpha(g(n)^{-1}\varphi) = a^{-1}g(n)^{-1}\varphi$ where a is the smallest element in A satisfying $g(n)^{-1}\varphi(Id, a) = 1$, or equivalently $\varphi(g(n), g(n)a) = 1$. By the construction of the φ-gradient ray given in Proposition 5.2, we see that $a = a_n$. Therefore we have $\alpha(g(n)^{-1}\varphi) = a_n^{-1}(g(n)^{-1}\varphi) = (g(n)a_n)^{-1}\varphi = g(n+1)^{-1}\varphi$, which completes the induction and the proof of (i).

Proposition 5.4 implies that for every $n \geq 0$, $g(n)g_n$ is the smallest φ-gradient ray starting at $g(n)$. By Proposition 5.3, wo obtain $g(n)g_n(t) = g(t+n)$ for every $t \geq 0$, or equivalently $g_n(t) = g(n)^{-1}g(n+t)$, which proves (ii). Now (ii) clearly implies (iii). This completes the proof of Proposition 5.7.

6. Consistent sequences. Let S be a finite set. We denote by Σ the set $S^{\mathbb{N}}$ of sequences $(\sigma_n)_{n\geq 0}$ with $\sigma_n \in S$ for every $n \geq 0$. The sets $\mathbb{N}$ and S are equipped with the discrete topology, and Σ with the product topology, *i.e.* the topology of pointwise convergence. The *shift map* is the continuous map $T : \Sigma \to \Sigma$ defined, for $\sigma = (\sigma_n)_{n\geq 0} \in \Sigma$, by $T(\sigma) = \sigma'$ where σ' is the sequence $(\sigma'_n)_{n\geq 0}$ such that $\sigma'_n = \sigma_{n+1}$ for every $n \geq 0$.

The dynamical system (Σ, T) is usually called the *one-sided Bernoulli shift on S*.

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DEFINITION 6.1. Let $k \geq 1$ be an integer and let W be a subset of S^{k+1}. The set

$$\Sigma_W = \{\sigma = (\sigma_n) \in \Sigma \ : \ \forall n \in \mathbb{N}, \big(\sigma_n, \ldots, \sigma_{n+k}\big) \in W\}$$

is a closed T-invariant subset of of Σ. The dynamical system (Σ_W, T) is called the *subshift of finite type* associated to the pair (S, W).

In this section, we define a subshift of finite type $(\Sigma(\infty), T)$ and a map $P : \Phi_0 \to \Sigma(\infty)$. We show that P is continuous, injective, and that it satisfies the relation $P \circ \alpha = T \circ P$. In Section 8, we shall prove that P is surjective. Thus, P is a topological conjugacy between the dynamical systems (Φ_0, α) and $(\Sigma(\infty), T)$.

We continue using the notations of the preceding section. We take a real number $R_0 \geq 100\delta + 1$ and an integer $L_0 \geq 2R_0 + 32\delta + 1$.

For every subset $Y \subset X$ and for every $\epsilon \geq 0$, we set

$$N(Y, \epsilon) = \{x \in X : \operatorname{dist}(x, Y) \leq \epsilon\}.$$

Let $\varphi \in \Phi_0$ and let $g : [0, \infty[\to X$ be the smallest φ-gradient ray starting at Id. We set

$$V(\varphi) = N\big(g([0, L_0]), R_0\big).$$

Note that we have $V(\varphi) \subset B(Id, L_0 + R_0)$.

For each $\varphi \in \Phi_0$, let $\rho(\varphi) : V(\varphi) \to \mathbb{R}$ be the function defined by

$$\rho(\varphi)(x) = \varphi(x, Id),$$

for each $x \in V(\varphi)$. We note that $\rho(\varphi)$ is the restriction to $V(\varphi)$ of the primitive h of φ satisfying $h(Id) = 0$.

Let S be the set of functions $\rho(\varphi)$, with φ ranging over Φ_0. Thus, we have a surjective map $\rho : \Phi_0 \to S$ which associates to each $\varphi \in \Phi_0$ the function $\rho(\varphi)$. Given an element $s \in S$, we denote by $V(s)$ the domain of definition of this function s.

LEMMA 6.2. *The set S is finite.*

Proof. Consider an integer $N_0 \geq L_0 + R_0$ and let $B = B(Id, N_0)$. The cardinality of S is bounded by the number of distinct restrictions of elements $\varphi \in \Phi_0$ to $B \times B$. By Lemma 4.6, this number is bounded above by $(2N_0 + 1)^{N_1}$, where N_1 is the number of elements of Γ at distance $\leq N_0$ from the identity.

Let $P : \Phi_0 \to \Sigma$ be the map which associates to each $\varphi \in \Phi_0$ the element $\sigma = (\sigma_n)_{n\geq 0} \in \Sigma$ defined by $\sigma_n = \rho\big(\alpha^n(\varphi)\big)$ for all $n \geq 0$.

LEMMA 6.3. *We have $P \circ \alpha = T \circ P$.*

Proof. Let $P(\varphi) = (\sigma_n)_{n\geq 0}$ and let $P \circ \alpha(\varphi) = (\sigma'_n)_{n\geq 0}$. Then, we have, for every $n \geq 0$,

$$\sigma'_n = \rho\big(\alpha^n(\alpha(\varphi))\big) = \rho\big(\alpha^{n+1}(\varphi)\big) = \sigma_{n+1}.$$

LEMMA 6.4. *The map $P : \Phi_0 \to \Sigma$ is continuous.*

Proof. Let $(\varphi_n)_{n\geq 0}$ be a sequence in Φ_0 converging to $\varphi \in \Phi_0$ and consider an integer $k \geq 0$. By Proposition 5.7, the map $\alpha^k : \Phi_0 \to \Phi_0$ is continuous. Therefore, we have $\alpha^k(\varphi_n) \to \alpha^k(\varphi)$ as $n \to \infty$. Hence, for all n large enough, the cocycles $\alpha^k(\varphi_n)$ and $\alpha^k(\varphi)$ have the same restriction on the product $B(Id, L_0 + R_0) \times B(Id, L_0 + R_0)$. This implies that $\rho\big(\alpha^k(\varphi_n)\big) = \rho\big(\alpha^k(\varphi)\big)$. Therefore, for all n large enough, the sequences $P(\varphi_n)$ and $P(\varphi)$ have the same $(k+1)$-th coordinate. This shows that $P(\varphi_n)$ converges to $P(\varphi)$.

We note now that since $R_0 \geq 1$, then for every $s \in S$, its domain $V(s)$ contains the closed ball $B(Id, 1)$. Therefore the value $s(a)$ is well-defined for all $a \in A$. Since the set A is equipped with a total order relation, we can define $w(s) \in A$ to be the *smallest* element $a \in A$ satisfying $s(a) = -1$.

Let $\sigma = (\sigma_n)_{n\geq 0} \in \Sigma$. We associate to σ the sequence $\big(\gamma_n(\sigma)\big)_{n\geq 0}$ of elements in $X^0 = \Gamma$, defined by setting $\gamma_0(\sigma) = Id$ and for every $n \geq 1$,

$$\gamma_n(\sigma) = w(\sigma_0)\ldots w(\sigma_{n-1}).$$

For every $n \geq 0$, let

$$V_n(\sigma) = \gamma_n(\sigma)V(\sigma_n) = \{\gamma_n(\sigma)x \;:\; x \in V(\sigma_n)\}.$$

We note that the set $V_n(\sigma)$ depends only on the first $n+1$ coordinates of σ.

For every $n \geq 0$, we define the function $f_n(\sigma) : V_n(\sigma) \to \mathbb{R}$ by

$$f_n(\sigma)(x) = \sigma_n(\gamma_n(\sigma)^{-1}x) - n.$$

We note that $f_n(\sigma)$ is the restriction to $V_n(\sigma)$ of a horofunction. In fact, if $h : X \to \mathbb{R}$ is a horofunction whose restriction to $V_n(\sigma)$ is σ_n, then $\gamma_n(\sigma)h - n$ is a horofunction whose restriction to $V(\sigma_n)$ is $f_n(\sigma)$.

LEMMA 6.5. *Let $\varphi \in \Phi_0$, let $\sigma = P(\varphi)$ and let $g : [0, \infty[\to X$ be the smallest φ-gradient ray starting at Id. Then, for every $n \geq 0$, we have the following:*

(i) $\gamma_n(\sigma) = g(n)$

(ii) $V_n(\sigma) = N\big(g([n, n + L_0]), R_0\big)$

(iii) *$f_n(\sigma)$ is the restriction to $V_n(\sigma)$ of the primitive h of φ satisfying $h(Id) = 0$.*

Proof. By Proposition 5.7 (iii), $w(\sigma_n)$ is the label of the oriented edge going from $g(n)$ to $g(n+1)$. Therefore, we have $\gamma_n(\sigma) = g(n)$. This proves (i).

To prove (ii), we note that Proposition 5.7 (ii) implies that the $\alpha^n(\varphi)$-gradient say starting at the identity is defined, for $t \geq 0$, by $t \mapsto g(n)^{-1}g(t+n)$. Therefore, we have

$$V(\sigma_n) = N\big(g(n)^{-1}g([n, n + L_0]), R_0\big).$$

Thus, we obtain

$$V_n(\sigma) = \gamma_n(\sigma)V(\sigma_n) = g(n)V(\sigma_n) = N\big(g([n, n + L_0]), R_0\big).$$

To prove (iii), let $x \in V_n(\sigma)$. Then, we have

$$\begin{aligned} f_n(\sigma)(x) &= \sigma_n\big(\gamma_n(\sigma)^{-1}x\big) - n \\ &= \rho\big(\alpha^n(\varphi)\big)\big(\gamma_n(\sigma)^{-1}x\big) - n \\ &= \alpha^n(\varphi)\big(\gamma_n(\sigma)^{-1}x, Id\big) - n \\ &= \alpha^n(\varphi)\big(g(n)^{-1}x, Id\big) - n \\ &= \varphi\big(x, g(n)\big) - n \text{ (Proposition 5.7(i))} \\ &= h(x) - h\big(g(n)\big) - n \\ &= h(x). \end{aligned}$$

LEMMA 6.6. *The map $P : \Phi_0 \to \Sigma$ is injective.*

Proof. Let φ and φ' be two elements of Φ_0 such that $P(\varphi) = P(\varphi')$ and let $g : [0, \infty[\to X$ and $g' : [0, \infty[\to X$ be the smallest gradient rays starting at the identity and associated respectively to φ and φ'. By Lemma 6.5 (i), we have $g(n) = g'(n)$ for every $n \geq 0$. This implies that $g = g'$. Using proposition 5.7 (i), we have, for every $n \geq 0$, $\alpha^n(\varphi) = g(n)^{-1}\varphi$. Since $g(n)^{-1}\varphi$ and $g'(n)^{-1}\varphi'$ coincide on $B\big(Id, R_0\big) \times B\big(Id, R_0\big)$ then φ and φ' coincide on $B\big(g(n), R_0\big) \times B\big(g(n), R_0\big)$. As $R_0 > 16\delta$, Proposition 3.5 implies that $\varphi = \varphi'$. This proves Lemma 6.6.

DEFINITION 6.7. Let $\sigma \in \Sigma$ and consider an integer $k \geq 1$. We say that σ is *consistent up to order* k if for all i and $j \in \mathbb{N}$ satisfying $i \leq j \leq i + k$, we have

$$f_i(\sigma)(x) = f_j(\sigma)(x) \quad \forall x \in V_i(\sigma) \cap V_j(\sigma).$$

We say that σ is *consistent* if it is consistent up to order k for all $k \geq 1$, *i.e*. if we have $f_i(\sigma)(x) = f_j(\sigma)(x)$ for all $i, j \in \mathbb{N}$ and for all $x \in V_i(\sigma) \cap V_j(\sigma)$.

For every integer $k \geq 1$, we let $\Sigma(k) \subset \Sigma$ denote the set of sequences which are consistent up to order k, and we let $\Sigma(\infty) \subset \Sigma$ denote the set of consistent sequences.

LEMMA 6.8. *We have $P(\Phi_0) \subset \Sigma(\infty)$.*

Proof. Let $\varphi \in \Phi_0$ and let $\sigma = P(\varphi)$. By Lemma 6.5 (iii), we have, for every $x \in V_n(\sigma)$, $f_n(\sigma)(x) = h(x)$, where h is the primitive of φ which vanishes at the identity. The right hand side does not depend on n, which proves the assertion.

PROPOSITION 6.9. *For every integer $k \geq 1$, the set $\Sigma(k) \subset \Sigma$ is a subshift of finite type.*

Proof. An element $\sigma \in \Sigma$ is consistent up to order k if and only if we have

$$\sigma_i(\gamma_i(\sigma)^{-1}x) - i = \sigma_j(\gamma_j(\sigma)^{-1}x) - j. \tag{6.9.1}$$

for every i and $j \in \mathbb{N}$ such that $i \leq j \leq i + k$ and for every $x \in V_i(\sigma) \cap V_j(\sigma)$.

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Let $y = \gamma_i(\sigma)^{-1}x$ and let $\gamma_{i,j}(\sigma) = \gamma_i(\sigma)^{-1}\gamma_j(\sigma)$. Then, equation (6.9.1) is equivalent to $\sigma_i(y) = \gamma_{i,j}(\sigma)\sigma_j(y) - (j-i)$, for all $y \in V(\sigma_i) \cap \gamma_{i,j}(\sigma)V(\sigma_j)$.

We have $\gamma_{i,j}(\sigma) = w(\sigma_{i-1})^{-1} \dots w(\sigma_0)^{-1} w(\sigma_0) \dots w(\sigma_{j-1}) = w(\sigma_i) \dots w(\sigma_{j-1})$. Let $W \subset S^{k+1}$ be the set of words $(s_0, \dots, s_k)$ satisfying $s_0(y) = \big(w(s_0) \dots w(s_{m-1})\big)s_m(y) - m$ for every m such that $1 \leq m \leq k$ and for every $y \in V(s_0) \cap w(s_0) \dots w(s_{m-1})V(s_m)$. Then $\Sigma(k) = \Sigma_W$. Thus, $\Sigma(k)$ is a subshift of finite type. This proves Lemma 6.9.

For $A \subset X$ and $R > 0$, we define

$$S(A, R) = \{x \in X \ : \ \mathrm{dist}(x, A) = R\}.$$

It is easy to see that if if $A \subset X$ is a bounded set, then, for every $R > 0$, the set $S(A, R)$ is finite.

LEMMA 6.10. *Let $A_1, \dots, A_n$ be a collection of bounded subsets of X, let $R_1, \dots, R_n$ be a collection of positive real numbers and let $\mathcal{I} \subset \cup_{i=1}^n N(A_i, R_i)$ be a topological segment (that is, $\mathcal{I}$ is a subset of X which is homeomorphic to a compact interval). Then, we can find a sequence of consecutive points $z_0, \dots, z_k \in \mathcal{I}$ such that for every $i = 1, \dots, k-1$, there exists an integer $j = j(i) \in \{1, \dots, n\}$ such that the topological segment $[z_i, z_{i+1}] \subset \mathcal{I}$ is contained in $N(A_j, R_j)$.*

Proof. Let $\Omega = \cup_{i=1}^n S(A_i, R_i)$. Thus, Ω is a finite set. Let $z_0, \dots, z_n$ be the sequence of points in $\mathcal{I} \cap \Omega$, appearing in this order. For each $i = 1, \dots, n-1$, let z_i' be a point in the interior of the topological segment $[z_i, z_{i+1}] \subset \mathcal{I}$. The point z_i' belongs therefore to a set $N(A_j, R_j)$ for some $j = j(i)$. The intermediate value theorem implies then that the segment $[z_i, z_{i+1}]$ is contained in $N(A_j, R_j)$. This proves Lemma 6.10.

Let $\sigma = (\sigma_n)_{n \geq 0} \in \Sigma$. Since $\gamma_{n+1}(\sigma) = \gamma_n(\sigma)\, w(\sigma_n)$ for every $n \geq 0$, we have $|\gamma_n(\sigma) - \gamma_{n+1}(\sigma)| = 1$. Therefore, there is a unique path $r(\sigma) : [0, \infty[\to X$ parametrized by arclength and satisfying $r(\sigma)(n) = \gamma_n(\sigma)$ for every $n \geq 0$.

LEMMA 6.11. *Let $k \geq 2(L_O + R_0)$ be an integer and let $\sigma = (\sigma_n)_{n \geq 0} \in \Sigma$ be consistent up to order k. Then, $r(\sigma)$ is a geodesic ray and σ is a consistent sequence.*

Proof. To fix our notations, let us record the fact that the sequence $(\sigma_n)_{n \geq 0}$ is consistent up to order k implies the following:

(6.11.1) For every i and $j \in \mathbb{N}$ satisfying $i \leq j \leq i + 2(L_0 + R_0)$ and for every x in $V_i(\sigma) \cap V_j(\sigma)$, we have $f_i(\sigma)(x) = f_j(\sigma)(x)$.

We use induction on n to prove simultaneously the following two properties for every $n \geq 0$:

($\mathcal{P}_n$) The restriction of $r(\sigma)$ to $[0, n]$ is geodesic.

($\mathcal{Q}_n$) For every $0 \leq i \leq j \leq n$ and for every $x \in V_i(\sigma) \cap V_j(\sigma)$, we have $f_i(\sigma)(x) = f_j(\sigma)(x)$.

This will clearly prove the lemma.

$\mathcal{P}_0$ and $\mathcal{Q}_0$ are trivially satisfied. Let us suppose that $\mathcal{P}_n$ and $\mathcal{Q}_n$ are satisfied for some $n \in \mathbb{N}$ and let us prove $\mathcal{P}_{n+1}$ and $\mathcal{Q}_{n+1}$.

By property $\mathcal{Q}_n$, there exists a function $H_n(\sigma) : V_0(\sigma) \cup \dots \cup V_n(\sigma) \to \mathbb{R}$ whose restriction to each $V_i(\sigma)$ is equal to $f_i(\sigma)$ for every $i \leq n$.

From the definitions, we have

$$f_{n+1}(\sigma)\big(r(\sigma)(n+1)\big) = f_{n+1}(\sigma)\big(\gamma_{n+1}(\sigma)\big) = \sigma_{n+1}(Id) - (n+1) = -(n+1).$$

Condition (6.11.1) implies in particular that $f_{n+1}(\sigma)$ coincides with $f_n(\sigma)$ on $V_n(\sigma) \cap V_{n+1}(\sigma)$. We have $r(\sigma)(n+1) \in V_n(\sigma) \cap V_{n+1}(\sigma)$. Therefore

$$H_n(\sigma)\big(r(\sigma)(n+1)\big) = f_n(\sigma)\big(r(\sigma)(n+1)\big) = -(n+1).$$

Consider a geodesic segment $[r(\sigma)(0), r(\sigma)(n+1)]$ and let T be the geodesic triangle in X whose vertices are $r(\sigma)(0)$, $r(\sigma)(n)$ and $r(\sigma)(n+1)$, and whose sides are $[r(\sigma)(0), r(\sigma)(n+1)]$, the edge joining $r(\sigma)(n)$ to $r(\sigma)(n+1)$ and the segment $r(\sigma)([0,n])$.

By δ-hyperbolicity, if z is an arbitrary point on $[r(\sigma)(0), r(\sigma)(n+1)]$, there exists an integer i satisfying $0 \leq i \leq n$ such that $|z - r(\sigma)(i)| \leq 4\delta + 1$. Since the set $V_i(\sigma)$ contains the closed ball of radius $R_0 \geq 100\delta + 1$ centered at $r(\sigma)(i)$, we have $z \in V_i(\sigma)$. Therefore, $[r(\sigma)(0), r(\sigma)(n+1)] \subset V_0(\sigma) \cup \ldots \cup V_n(\sigma)$.

By Lemma 6.10, there exists a sequence $Id = z_0, z_1, \ldots, z_k = r(\sigma)(n+1)$ of consecutive points on $[r(\sigma)(0), r(\sigma)(n+1)]$ such that for every $i = 0, 1, \ldots, k-1$, $[z_i, z_{i+1}] \subset V_j(\sigma)$, for some $j = j(i) \in \{0, \ldots, n\}$ (see Figure 4).

We can write, for every $i = 0, 1, \ldots, k-1$,

$$|H_n(\sigma)(z_i) - H_n(\sigma)(z_{i+1})| = |f_j(\sigma)(z_i) - f_j(\sigma)(z_{i+1})|.$$

Since $f_j(\sigma)$ is the restriction to $V_j(\sigma)$ of a horofunction, and since horofunctions are 1-Lipschitz (Proposition 2.2), we obtain $|H_n(\sigma)(z_i) - H_n(\sigma)(z_{i+1})| \leq |z_i - z_{i+1}|$.

We have therefore

$$\begin{aligned} |H_n(\sigma)\big(r(\sigma)(0)\big) - H_n(\sigma)\big(r(\sigma)(n+1)\big)| &= \Big|\sum_{i=0}^{k-1} H_n(\sigma)(z_i) - H_n(\sigma)(z_{i+1})\Big| \\ &\leq \sum_{i=0}^{k-1} |z_i - z_{i+1}| \\ &= |r(\sigma)(0) - r(\sigma)(n+1)|. \end{aligned}$$

Thus, we obtain

$$\text{length}(r(\sigma)_{|[0,n+1]}) = n+1 = |H_n(\sigma)\big(Id\big) - H_n(\sigma)\big(r(n+1)\big)| \leq |r(\sigma)(0) - r(\sigma)(n+1)|,$$

which proves that $r(\sigma)_{|[0,n+1]}$ is a geodesic. This proves property $\mathcal{P}_{n+1}$.

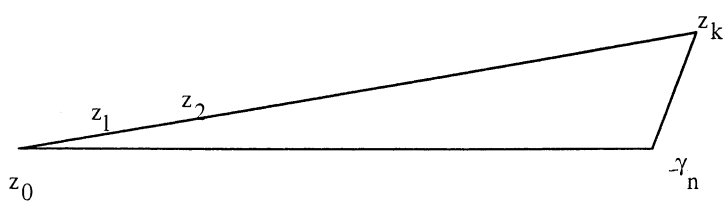

Figure 4.

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We prove now property $\mathcal{Q}_{n+1}$. Let $0 \leq i \leq j \leq n+1$. If $|i-j| \leq 2(L_0 + R_0)$, then $f_i(\sigma)(x) = f_j(\sigma)(x)$ for all $x \in V_i(\sigma) \cap V_j(\sigma)$, since σ is consistent up to order k. Assume now that $|i-j| > 2(L_0 + R_0)$. We have $V_i(\sigma) \subset B\big(r(\sigma)(i), L_0 + R_0\big)$ and $V_j(\sigma) \subset B\big(r(\sigma)(j), L_0 + R_0\big)$. By property $\mathcal{P}_{n+1}$, $r(\sigma)_{|[0,n+1]}$ is a geodesic, which implies $|r(\sigma)(i) - r(\sigma)(j)| = |i-j| > 2(L_0+R_0)$. Therefore, $B\big(r(\sigma)(i), L_0+R_0\big) \cap B\big(r(\sigma)(j), L_0+R_0\big) = \emptyset$. Thus, we have $V_i(\sigma) \cap V_j(\sigma) = \emptyset$. This proves property $\mathcal{Q}_{n+1}$. This proves Lemma 6.11.

Lemma 6.12. *Let $k \geq 2(L_0 + R_0)$ be an integer. Then, $\Sigma(k) = \Sigma(\infty)$.*

Proof. This is a consequence of Lemma 6.11.

We obtain finally the following

Theorem 6.13. *The set of consistent sequences $\Sigma(\infty) \subset \Sigma$ is a subshift of finite type.*

Proof. This follows from Lemmas 6.9 and 6.12.

7. The horofunction associated to a consistent sequence. Let $\sigma = (\sigma_n)_{n \geq 0} \in \Sigma(\infty)$ be a consistent sequence. Our aim is to construct an element $\varphi \in \Phi_0$ such that $P(\varphi) = \sigma$, in order to prove the surjectivity of the map $P : \Phi_0 \to \Sigma(\infty)$. This will require several lemmas.

We fix an element $\sigma \in \Sigma(\infty)$ and we set, in order to simplify the notations, $r(\sigma) = r$ and for every $n \geq 0$, $V_n(\sigma) = V_n$, $f_n(\sigma) = f_n$, $\gamma_n(\sigma) = \gamma_n$ and $w(\sigma_n) = w_n$.

We let $V = \cup_{n \geq 0} V_n$, and we define $f : V \to \mathbb{R}$ to be the function whose restriction to each V_n is equal to f_n.

Lemma 7.1. *We have*

(i) *For all $n \geq 0$, $V(\sigma_n) = N\big(\gamma_n^{-1} r([n, n + L_0]), R_0\big)$.*
(ii) *For all $n \geq 0$, $V_n = N\big(r([n, n + L_0]), R_0\big)$.*
(iii) *$V = N(r([0, \infty[), R_0)$.*

Proof. For every integer $n \geq 0$, we let $h_n : X \to \mathbb{R}$ be a horofunction with associated cocycle $\varphi_n : X \times X \to \mathbb{R}$, such that $\sigma_n = \rho(\varphi_n)$. We recall that we have a function $f_n : V_n \to \mathbb{R}$ defined by

$$f_n(\gamma_n x) = \sigma_n(x) - n, \ \ \forall x \in V(\sigma_n).$$

We recall also that for every integer $k \in [0, L_0]$, we have $\gamma_{n+k+1} = \gamma_{n+k} w_{n+k}$, where w_{n+k} is the smallest $a \in A$ satisfying $\sigma_{n+k}(a) = -1$. Thus, w_{n+k} is the smallest $a \in A$ satisfying $f_{n+k}(\gamma_{n+k} a) - (n+k) = -1$.

Let us fix an integer $n \geq 0$ and let $g_n : [0, \infty[\to X$ be the smallest φ_n-gradient ray starting at Id. We prove by induction on k, $0 \leq k \leq L_0$, the following property :

$(\mathcal{P}_k) : g_n([0, k]) = \gamma_n^{-1} r([n, n+k])$.

Assertion (i) of the Lemma follows then by taking $L = L_0$.

Property $(\mathcal{P}_0)$ is true since $r(n) = \gamma_n$ and $g_n(0) = Id$.

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Suppose that $(\mathcal{P}_k)$ is true for some $k < L_0$, and let us prove $(\mathcal{P}_{k+1})$. For that, it suffices to prove that $g_n(k+1) = \gamma_n^{-1} r(n+k+1)$, that is, $g_n(k+1) = \gamma_n^{-1}\gamma_{n+k+1}$. Using property $(\mathcal{P}_k)$ and the fact that $\gamma_{n+k+1} = \gamma_{n+k} w_{n+k}$, it suffices to prove now that $g_n(k+1) = g_n(k) w_{n+k}$.

From the definition of the φ_n-gradient ray g_n, we have $g_n(k+1) = g_n(k)a$, where a is the smallest element in A satisfying $h_n\big(g_n(k)a\big) = h_n\big(g_n(k)\big) - 1$.

Since $k \leq L_0$ and since $V(\sigma_n) = N\big(g_n([0, L_0]), R_0\big)$, $g_n(k)$ and $g_n(k)a$ belong to $V(\sigma_n)$. Thus, a is the smallest element in A such that $\sigma_n\big(g_n(k)a\big) = \sigma_n\big(g_n(k)\big) - 1$. Equivalently, a is the smallest element in A such that $f_n\big(\gamma_n g_n(k)a\big) = f_n\big(\gamma_n g_n(k)\big) - 1$, or equivalently (using property $(\mathcal{P}_k)$), $f_n\big(r(n+k)a\big) = f_n\big(r(n+k)\big) - 1$.

The elements $r(n+k) = \gamma_{n+k}$ and $r(n+k)a = \gamma_{n+k}a$ belong to $\gamma_{n+k}B(Id, 1) \subset \gamma_{n+k}V(\sigma_{n+k}) = V_{n+k}$, and the functions f_n and f_{n+k} have the same restriction on $V_n \cap V_{n+k}$. Thus, a is the smallest element in A satisfying $f_{n+k}\big(r(n+k)a\big) = f_{n+k}\big(r(n+k)\big) - 1$, or equivalently $f_{n+k}\big(\gamma_{n+k}a\big) = -(n+k) - 1$.

That is, a is the smallest element in A satisfying $\sigma_{n+k}(a) = -1$, which implies that $a = w_{n+k}$. This proves (i).

Property (ii) follows from (i) since $V_n = \gamma_n V(\sigma_n)$.

We prove now Property (iii). For all $n \geq 0$, we have, from Property (ii), $V_n = N\big(r([n, n+L_0]), R_0\big) \subset N(r([0, \infty[), R_0)$. Therefore, we have $V \subset N(r([0, \infty[), R_0)$.

Conversely, let $x \in N(r([0, \infty[), R_0)$. If $x \notin r([0, \infty[)$, then since X is a simplicial complex of dimension 1, the projection of x on this ray is necessarily a vertex $r(n)$. Then, we have $x \in B\big(r(n), R_0\big)$, which implies that $x \in V_n$. This shows that $N(r([0, \infty[), R_0) \subset V$, which completes the proof of Lemma 7.1.

We have the following:

Lemma 7.2. *Let x and $y \in V$, and suppose that there exists a geodesic segment $[x, y]$ contained in V. Then, we have $|f(x) - f(y)| \leq |x - y|$.*

Proof. There exists an integer n such that $[x, y] \subset V_0 \cup \ldots \cup V_n$ and by Lemma 7.1 (ii), each V_n is a set of the form $V_n = N\big(r([n, n+L_0]), R_0\big)$. Therefore, by Lemma 6.10, there exists a sequence $x_0 = z_0, z_1, \ldots, z_k = y$ of consecutive points on $[x, y]$ such that for every $i = 0, 1, \ldots, k-1$, the geodesic sub-segment $[z_i, z_{i+1}] \subset [x, y]$ is contained in a set V_j for some $j = j(i) \in \mathbb{N}$. We have, using Proposition 2.2,

$$|f(z_i) - f(z_{i+1})| = |f_j(z_i) - f_j(z_{i+1})| \leq |z_i - z_{i+1}|.$$

By taking the sum, we obtain $|f(x) - f(y)| \leq |x - y|$.

Lemma 7.3. *For every $t \geq 0$, we have $f\big(r(t)\big) = -t$.*

Proof. For every integer $n \geq 0$, we have $f\big(r(n)\big) = f_n(\gamma_n) = -n$. For $t \geq 0$, we let n be an integer satisfying $n \leq t < n+1$. We recall that since $R_0 \geq 1$, the domain V_n of f_n contains the closed ball of radius one centered at $\gamma_n = r(n)$. Therefore, V_n contains the segment $[r(n), r(n+1)]$ and we have $f_{|[r(n), r(n+1)]} = f_{n|[r(n), r(n+1)]}$.

The function f_n is the restriction of a horofunction $h_n : X \to \mathbb{R}$ in such a way that $r_{|[n, n+1]}$ is an h_n-gradient arc. Since h_n takes the values $-n$ and $-(n+1)$ respectively at $r(n)$ and $r(n+1)$, we have $h_n\big(r(t)\big) = -t$.

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For each $t \geq 0$, we define the set

$$S_t = f^{-1}(-t) \cap N\big(r([0, \infty[), 8\delta\big).$$

LEMMA 7.4. *For every $t \geq 0$, S_t is a nonempty closed subset of X and we have $S_t \subset B\big(r(t), 16\delta\big)$.*

Proof. Let $t \geq 0$. Lemma 7.3 shows that S_t is nonempty. Since the function f, restricted to any of the closed sets V_n, is equal to f_n, f is continuous. This implies that $f^{-1}(-t)$ is closed in V, which in turn implies that $S_t = f^{-1}(-t) \cap N\big(r([0, \infty[), 8\delta\big)$ is closed in V. Now V is a closed subset of X (see for instance Lemma 7.1(iii)). This implies that S_t is a nonempty closed subset of X.

It remains to prove the inclusion $S_t \subset B\big(r(t), 16\delta\big)$. Let $y \in S_t$. Then, we have $f(y) = -t$, and $|y - r(t')| \leq 8\delta$ for some $t' \geq 0$. Furthermore, any geodesic segment $[y, r(t')]$ is contained in V. Using Lemma 7.2, we have

$$|r(t') - r(t)| = |t' - t| = |f(y) - f\big(r(t')\big)| \leq |y - r(t')| \leq 8\delta,$$

which implies $|y - r(t)| \leq |y - r(t')| + |r(t') - r(t)| \leq 16\delta$. This proves Lemma 7.4.

We introduce now the following definition:

Let $I \subset \mathbb{R}$ be an interval. A rectifiable path $g : I \to V$ is called an *f-gradient arc* if g is parametrized by arclength and if for all t and $t' \in I$, we have $f\big(g(t)\big) - f\big(g(t')\big) = t' - t$.

If $I = [0, \infty[$, then g is also called an *f-gradient ray*.

LEMMA 7.5. *The geodesic ray $r : [0, \infty[\to X$ is an f-gradient ray.*

Proof. This is an easy consequence of Lemma 7.3.

LEMMA 7.6. *An f-gradient arc whose image is contained in $N(r([0, \infty[), 16\delta)$ is a geodesic arc.*

Proof. Let $g : I \to N(r([0, \infty[), 16\delta)$ be an f-gradient arc. Consider two real numbers t and t' satisfying $0 \leq t \leq t'$. Consider a geodesic segment $[g(t), g(t')]$. Since $g(t)$ and $g(t') \in N(r[0, \infty[), 16\delta)$, we can find points u and $u' \in [0, \infty[$ such that $|g(t) - r(u)| \leq 16\delta$ and $|g(t') - r(u')| \leq 16\delta$. Consider geodesic segments $[g(t), r(u)]$ and $[g(t'), r(u')]$ and the geodesic quadrilateral whose sides are $[g(t), r(u)]$, $[r(u), r(u')]$, $[g(t'), r(u')]$ and $[g(t), g(t')]$. Since X is δ-hyperbolic, every point on $[g(t), g(t')]$ is at distance $\leq 8\delta$ from $[g(t), r(u)] \cup [r(u), r(u')] \cup [g(t'), r(u')]$. Thus, $[g(t), g(t')] \subset N(r([0, \infty[), 24\delta) \subset N(r([0, \infty[), R_0) = V$, since $R_0 \geq 24\delta$. Using Lemma 7.2, we have

$$\text{length}(g_{|[t,t']}) = |t' - t| = |f\big(g(t)\big) - f\big(g(t')\big)| \leq |g(t) - g(t')|.$$

This proves Proposition 7.6.

LEMMA 7.7. *Let $x \in V$. Then, there exists an integer $n \geq 0$, a real number $t \in [n, n + L_0]$ and a point $y \in V_n$ such that the following three properties hold:*

(i) $|x - r(n)| \leq R_0$

(ii) $|y - r(t)| \leq 8\delta$

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(iii) $f(x) - f(y) = |x - y| = R_0 + 8\delta + 1$.

Furthermore, if $x \in N(r([0, \infty[), 8\delta) \subset V$, *then we can find* n, t *and* y *satisfying the three properties above and such that every geodesic segment* $[x, y]$ *is contained in* V_n.

Proof. Let $x \in V$. Then, $\mathrm{dist}\big(x, r([0, \infty[)\big) \leq R_0$. Either the point x is on $r([0, \infty[)$, and in this case x is at distance $\leq 1/2$ from a vertex on $r([0, \infty[)$, or a projection of x on $r([0, \infty[)$ is a vertex of X, that is, an element of the form $r(n)$ for some $n \in \mathbb{N}$. In any case, there exists an integer n such that $|x - r(n)| \leq R_0$. Let $h_n : X \to \mathbb{R}$ be a horofunction whose restriction to V_n is equal to f_n. The map $g_0 : [0, L_0] \to X$ defined by setting $g_0(t) = r(n + t)$ for every $t \in [0, L_0]$ is an h_n-gradient arc. Using Proposition 2.13 and Proposition 2.9, we extend g_0 to a map $g : [0, \infty[\to X$ which is an h_n-gradient ray starting at $r(n)$.

Let $g' : [0, \infty[\to X$ be an h_n-gradient ray starting at x and let $y = g'(R_0 + 8\delta + 1)$. Then, we have $h_n(x) - h_n(y) = |x - y| = R_0 + 8\delta + 1$. By Proposition 3.1, the two rays g and g' converge to the point at infinity of h_n. Consider a geodesic segment $[x, r(n)]$, and the geodesic triangle with one point at infinity whose sides are $[x, r(n)]$, $g([0, \infty[)$ and $g'([0, \infty[)$.

By δ-hyperbolicity, there exists a point $z \in [x, r(n)] \cup g([0, \infty[)$ such that $|y - z| \leq 8\delta$. Such a point z cannot be on $[x, r(n)]$. Indeed, if $z \in [x, r(n)]$, then we would have

$$|x - y| \leq |x - r(n)| + |z - y| \leq R_0 + 8\delta < R_0 + 8\delta + 1,$$

a contradiction. Therefore, we have $z \in g([0, \infty[)$. Thus,

$$|r(n) - z| \leq |r(n) - x| + |x - y| + |y - z| \leq 2R_0 + 8\delta + 1 \leq L_0.$$

Therefore, $z \in g([0, L_0]) = r([n, n + L_0])$. Let $z = r(t)$ (see Figure 5). Since $|y - r(t)| \leq 8\delta$, we conclude that $y \in V_n$.

It remains to prove now the last part of the lemma.

Suppose first that $x \notin r([0, \infty[)$. In this case, there is an integer n such that $|x - r(n)| \leq 8\delta$. We consider this integer n, and we use the same t and y provided by the above construction.

Consider a geodesic quadrilateral whose sides are $[x, y]$, $[x, r(n)]$, $[r(n), r(t)]$ and $[r(t), y]$. Let $v \in [x, y]$ and let us prove that $v \in V_n$. By the δ-hyperbolicity of X, there exists a point $w \in [x, r(n)] \cup r([n, t]) \cup [r(t), y]$ such that $|v - w| \leq 8\delta$.

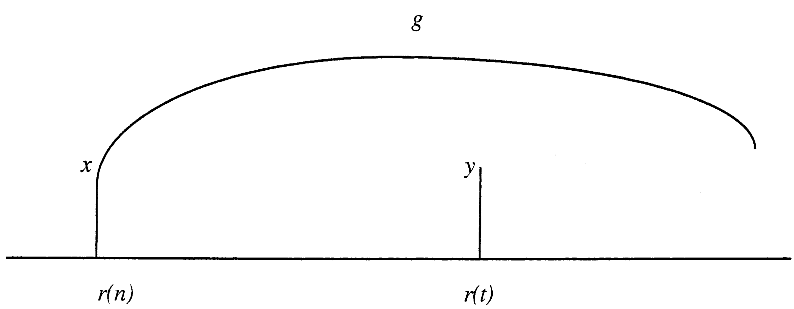

Figure 5.

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Since $[x, r(n)] \cup r([n, t]) \cup [r(t), y] \subset N\big(r([n, n + L_0])8\delta\big)$, we have $v \in N\big(r([n, n+ L_0])16\delta\big) \subset V_n$.

In the case where $x \in r([0, \infty[)$, $x = r(t_0)$ for some $t_0 \geq 0$. Then, the required properties are satisfied with n equal to the integral part of t_0, $y = r(t_0 + R_0 + 8\delta + 1)$ and $t = t_0 + R_0 + 8\delta + 1$.

LEMMA 7.8. *Let* $x \in N(r([0, \infty[), 8\delta)$. *Then, there exists an f-gradient ray* $g : [0, \infty[\to X$ *starting at* x *and satisfying* $g([0, \infty[) \subset N(r([0, \infty[), 16\delta)$. *Furthermore, if* $g : [0, \infty[\to X$ *is such an f-gradient ray and if* $t > 16\delta - f(x)$, *then we have* $g\big(t + f(x)\big) \in S_t$.

Proof. Using Lemma 7.7, we construct, by induction, a sequence $(x_i)_{i\geq 0}$ of points in X with $x_0 = x$ and such that for each $i \geq 0$, the following three properties hold:

(i) $x_i \in N(r([0, \infty[), 8\delta)$

(ii) there exists an integer $n = n(i)$ such that every geodesic segment $[x_i, x_{i+1}]$ is contained in V_n

(iii) $f(x_i) - f(x_{i+1}) = |x_i - x_{i+1}| = R_0 + 8\delta + 1$.

For each $i \geq 0$, consider a geodesic segment $[x_i, x_{i+1}]$.

We have $[x_i, x_{i+1}] \subset N(r([0, \infty[), 16\delta)$. Indeed, let $p_i \in r([0, \infty[)$ satisfy $|x_i - p_i| \leq 8\delta$ and consider a geodesic segment $[x_i, p_i]$. By δ-hyperbolicity, every point on $[x_i, x_{i+1}]$ is at distance $\leq 8\delta$ from the union $[x_i, p_i] \cup [p_i, p_{i+1}] \cup [p_{i+1}, x_{i+1}]$. The triangle inequality implies now $[x_i, x_{i+1}] \subset N(r([0, \infty[, 16\delta)$.

Let $g : [0, \infty[\to X$ be the ray starting at x obtained by concatenating the geodesic segments $[x_i, x_{i+1}]$. We have $g([0, \infty[) \subset N(r([0, \infty[, 16\delta)$ and, for all $i \geq 0$, $x_i = g(t_i)$, with $t_i = (R_0 + 8\delta + 1)i$.

By Property (ii), we have $g([t_i, t_{i+1}]) \subset V_n$. By Property (iii), we have

$$f_n\big(g(t_i)\big) - f_n\big(g(t_{i+1})\big) = t_{i+1} - t_i.$$

The function $f_n : V_n \to \mathbb{R}$ is the restriction to V_n of a horofunction $h_n : X \to \mathbb{R}$. By Proposition 2.10(ii), g is an h_n-gradient arc. Since f coincides with h_n on V_n, this shows that for every $i \geq 0$, $g_{|[t_i, t_{i+1}]}$ is an f-gradient arc. Thus, g is an f-gradient ray.

By Lemma 7.6, g is geodesic. The two geodesic rays g and r have the same point at infinity, since $g([0, \infty[) \subset N\big(r([0, \infty[, 16\delta)\big)$. Let $t' = -f(x)$ and let $t > 16\delta + t'$. We take a geodesic segment $[x, r(t')]$, and we consider the geodesic triangle with one vertex at infinity, whose sides are $[x, r(t')]$, $r([t', \infty[)$ and $g([0, \infty[)$. By δ-hyperbolicity, each point on $g([0, \infty[)$ is situated at distance $\leq 8\delta$ from $[x, r(t')] \cup r([t', \infty[)$.

Let $q = g\big(t + f(x)\big) = g(t - t')$ and let us prove that $q \in S_t$. Since $t > t' + 16\delta$, we have

$$\mathrm{dist}(q, [x, r(t')]) \geq |q - x| - |x - r(t')| = t - t' - |x - r(t')| > 16\delta - 8\delta = 8\delta.$$

Therefore, $q \in N(r([0, \infty[), 8\delta)$. To prove that $q \in S_t$, it remains to prove that $f(q) = -t$. Now since g is an f-gradient ray, we have $f\big(g(0)\big) - f(q) = t - t' - 0 = t - t'$. On the other hand, we have $f\big(g(0)\big) = f(x) = -t'$, which implies $f\big(q\big) = -t$. Thus, $q \in S_t$. This completes the proof of Lemma 7.8.

For each $x \in X$, we define the function $\Delta_x : [0, \infty[\to \mathbb{R}$ by

$$\Delta_x(t) = \mathrm{dist}\big(x, S_t\big) - t.$$

Lemma 7.9. *Let $x \in X$. Then, there exists $t_0 \geq 0$ such that for every $t \geq t_0$, we have* $\Delta_x(t) = \Delta_x(t_0)$.

Proof. The proof is given in two steps.

Step 1. Let $t_1 = 2|x - r(0)| + 24\delta$. We prove that for all $t > t_1$ and for all $t' \geq t + |x - r(0)| + 40\delta$, we have $\Delta_x(t') \geq \Delta_x(t)$.

Consider t and t' satisfying these conditions and let q' be a projection of x on $S_{t'}$. This projection exists since $S_{t'}$ is a nonempty closed subset of X (Lemma 7.4). Consider now a geodesic segment $[x, q']$.

We claim that $[x, q'] \cap S_t \neq \emptyset$.

To prove the claim, we take geodesic segments $[x, r(0)]$ and $[q', r(t')]$. Since the space X is δ-hyperbolic, every point on $[x, q']$ is at distance $\leq 8\delta$ from some point on $[x, r(0)] \cup r([0, t']) \cup [q', r(t')]$. Since $S_{t'} \subset B\big(r(t'), 16\delta\big)$, we have

$$|x - q'| \geq |r(0) - r(t')| - |x - r(0)| - |q' - r(t')| \geq t' - |x - r(0)| - 16\delta.$$

Thus,

$$|x - q'| \geq t + |x - r(0)| + 40\delta - |x - r(0)| - 16\delta = t + 24\delta > 2|x - r(0)| + 48\delta.$$

We can take therefore two points y_1 and y_2 on $[x, q']$ satisfying $|x - y_1| = |x - r(0)|+ 24\delta$ and $|y_2 - q'| = 24\delta$.

Since $|x - q'| > 2|x - r(0)| + 48\delta$, the points y_1 and y_2 are distinct (see Figure 6).

Let $[y_1, y_2]$ be the geodesic segment joining y_1 and y_2 and contained in $[x, q']$.

If y is a point in the interior of $[y_1, y_2]$, then

$$\mathrm{dist}(y, [x, r(0)]) > |y_1 - x| - |x - r(0)| = |x - r(0)| + 24\delta - |x - r(0)| = 24\delta.$$

Likewise, we have

$$\mathrm{dist}(y, [q', r(t')]) > |y_2 - q'| - |q' - r(t')| \geq 24\delta - 16\delta = 8\delta.$$

We conclude that for every $y \in [y_1, y_2]$, we have $\mathrm{dist}(y, r([0, t'])) \leq 8\delta$.

Thus, to prove that $[x, q'] \cap S_t \neq \emptyset$, it suffices to prove that there exists a point $y \in [y_1, y_2]$ such that $f(y) = -t$. For that purpose, we begin by proving the two inequalities $f(y_2) \leq -t$ and $f(y_1) \geq -t$.

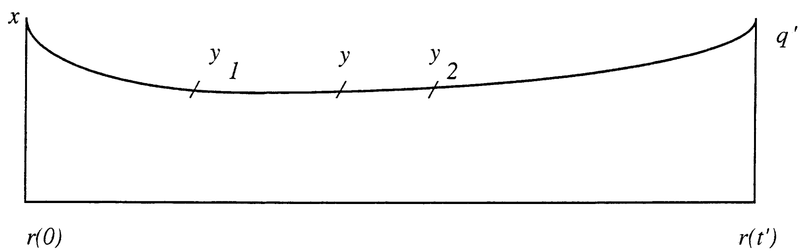

Figure 6.

We note first that since $\operatorname{dist}(y_2, r([0,\infty[)) \leq 8\delta$ and $\operatorname{dist}(q', r([0,\infty[)) \leq 8\delta$, we have $[y_2, q'] \subset V$. Since $|y_2 - q'| = 24\delta$, Lemma 7.2 implies $f(y_2) \leq 24\delta + f(q') = 24\delta - t'$. Therefore, $f(y_2) \leq 24\delta - t - |x - r(0)| - 40\delta = -t - (|x - r(0)| + 16\delta) \leq -t$, which is the first inequality we wanted.

To prove now that $f(y_1) \geq -t$, we write

$$|y_1 - r(0)| \leq |y_1 - x| + |x - r(0)| = |x - r(0)| + 24\delta + |x - r(0)| = 2|x - r(0)| + 24\delta.$$

Let us choose a segment $[y_1, r(0)]$. Since $\operatorname{dist}(y_1, r([0,\infty[)) \leq 8\delta$, we have $[y_1, r(0)] \subset V$.

By Lemma 7.2, we have

$$|f(y_1) - f(r(0))| \leq |y_1 - r(0)| \leq 2|x - r(0)| + 24\delta,$$

which implies

$$f(y_1) - f(r(0)) \geq -2|x - r(0)| - 24\delta,$$

that is,

$$f(y_1) \geq f(r(0)) - 2|x - r(0)| - 24\delta = -2|x - r(0)| - 24\delta.$$

Since $t > 2|x - r(0)| + 24\delta$, we obtain $f(y_1) \geq -t$.

Consider now the restriction of f to the geodesic segment $[y_1, y_2]$. Since $f(y_2) \leq -t \leq f(y_1)$ and since the function f is continuous on V, the intermediate value theorem implies that there exists a point $y \in [y_1, y_2]$ such that $f(y) = -t$. This proves that $[x, q'] \cap S_t \neq \emptyset$.

We continue now the proof of Step 1.

Let $q_1 \in [x, q'] \cap S_t$. Consider the geodesic segment $[q_1, q'] \subset [x, q']$ and let q be a projection of x on S_t. We have therefore $|x - q_1| \geq |x - q|$.

We have $[q_1, q'] \subset V$. Therefore, Lemma 7.2 implies $|q_1 - q'| \geq |f(q_1) - f(q')| = t' - t$.

We obtain therefore $|x - q'| = |x - q_1| + |q_1 - q'| \geq |x - q| + t' - t$, which gives $|x - q'| - t' \geq |x - q| - t$. Therefore, we have, for all $t > t_1$ and for all $t' \geq t$, $\operatorname{dist}(x, S_{t'}) - t' \geq \operatorname{dist}(x, S_t) - t$, that is, $\Delta_x(t') \geq \Delta_x(t)$. This finishes the proof of Step 1.

Step 2. We prove that for all $t \geq 0$ and for all $t' > t + 16\delta$, we have $\Delta_x(t) \geq \Delta_x(t')$.

To prove Step 2, let $t \geq 0$ and let q be a projection of x on S_t. Then, $q \in N(r([0,\infty[, 8\delta[)$ and Lemma 7.8 implies that there exists an f-gradient ray $g : [0,\infty[\to X$ starting at q and satisfying $g([0,\infty[) \subset N(r([0,\infty[), 16\delta)$ and such that for all $t' > t + 16\delta$, we have $g(t' - t) \in S_{t'}$.

Since g is geodesic, we have $|q - q'| = |g(t' - t) - g(0)| = t' - t$. Since $q' \in S_{t'}$, we have $|x - q'| \geq \operatorname{dist}(x, S_{t'})$. Using the triangle inequality, we obtain $|x - q| \geq \operatorname{dist}(x, S_{t'}) - t' + t$.

Therefore, for all $t \geq 0$ and for all $t' > t + 16\delta$, we have $\operatorname{dist}(x, S_t) - t \geq \operatorname{dist}(x, S_{t'}) - t'$, that is, $\Delta_x(t) \geq \Delta_x(t')$.

This finishes the proof of Step 2.

Let $t_1 = 2|x - r(0)| + 24\delta$ be the constant provided by Step 1. Step 1 implies in particular that for all $t' > t_1 + |x - r(0)| + 40\delta$, we have $\Delta_x(t') \geq \Delta_x(t_1)$. Step 2

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implies that for all $t' \geq t_1 + 16\delta$, we have $\Delta_x(t') \leq \Delta_x(t_1)$. Therefore, $\Delta_x(t)$ is constant for all $t > t_1 + |x - r(0)| + 40\delta$. This proves Lemma 7.9.

Consider now the function $F : X \to \mathbb{R}$ defined by $F(x) = \lim_{t\to\infty} \Delta_x(t)$. (The limit exists and is finite, by Lemma 7.9.) We have the following

LEMMA 7.10. *For every $x \in V$, we have $F(x) = f(x)$.*

Proof. Let $x \in V$ and let us prove first that $F(x) \leq f(x)$. By Lemma 7.7, there exists $y \in N(r([0, \infty[), 8\delta)$ such that $|x - y| = f(x) - f(y)$. By Lemma 7.8, there exists an f-gradient ray $g : [0, \infty[\to X$ starting at y and satisfying $g([0, \infty[) \subset N(r([0, \infty[), 16\delta)$. By Lemma 7.6, the gradient ray g is geodesic. Lemma 7.8 implies also that for all $t > 16\delta - f(y)$, we have $S_t \cap g([0, \infty[) \neq \emptyset$

Let $t > 16\delta - f(y)$ and let $z \in S_t \cap g([0, \infty[)$. We have

$$|x - z| \leq |x - y| + |y - z| = f(x) - f(y) + f(y) - f(z) = f(x) + t.$$

Therefore, we have, for all $t > 16\delta - f(y)$, $\mathrm{dist}(x, S_t) \leq f(x) + t$, which implies $\Delta_x(t) \leq f(x)$. This proves $F(x) \leq f(x)$.

We now prove that $F(x) \geq f(x)$ by showing that for all t large enough, we have $\Delta_x(t) \geq f(x)$.

Since $x \in V$, there exists an integer $n \geq 0$ such that $|x - r(n)| \leq R_0$. Let us take $t \geq n$ and t large enough so that $\mathrm{dist}(x, S_t) > R_0 + 8\delta + 1$ and let p be a projection of x on S_t. Choose a geodesic segment $[x, p]$ and let x' be a point on this segment satisfying $|x - x'| = R_0 + 8\delta + 1$. Choose geodesic segments $[x, r(n)]$ and $[p, r(t)]$ and consider the geodesic quadrilateral $[x, r(n)] \cup r([n, t]) \cup [p, r(t)] \cup [x, p]$ (Figure 7). By δ-hyperbolicity, the point x' is at distance $\leq 8\delta$ from $[x, r(n)] \cup r([n, t]) \cup [p, r(t)]$.

We have $\mathrm{dist}(x', [x, r(n)]) \geq |x' - x| - |x - r(n)| \geq R_0 + 8\delta + 1 > 8\delta$, which implies that x' is at distance $\leq 8\delta$ from the union $r([n, t]) \cup [p, r(t)]$.

Since $p \in S_t \subset B\big(r(t), 16\delta\big)$ (Proposition 7.4), x' is at distance $\leq 14\delta$ from some point $x'' \in r([n, t])$.

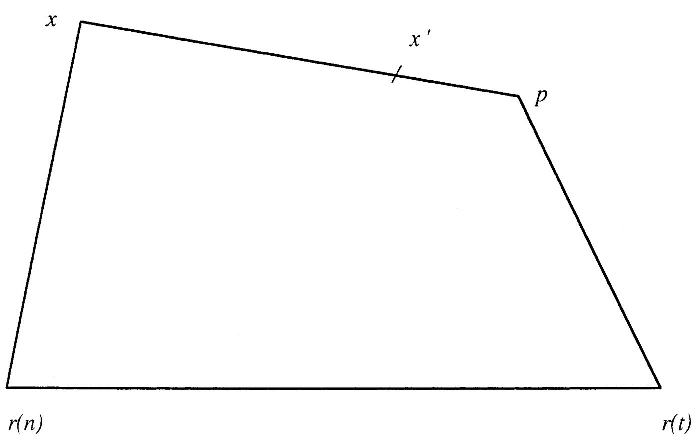

Figure 7.

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We have

$$|r(n) - x''| \leq |r(n) - x| + |x - x'| + |x' - x''| \leq R_0 + R_0 + 8\delta + 1 + 24\delta$$
$$= 2R_0 + 32\delta + 1 \leq L_0.$$

We conclude that $x' \in N(r([n, n + L_0]), 24\delta) \subset N(r([n, n + L_0]), R_0) = V_n$.

Let $h_n : X \to \mathbb{R}$ be a horofunction whose restriction to V_n is equal to f_n. Then, since x and $x' \in V_n$, we have

$$f(x) - f(x') = h_n(x) - h_n(x') \leq |x - x'|.$$

Consider now the geodesic segment $[x', p]$. Since x' and p are in $N(r([0, \infty[), 24\delta)$, the fact that X is δ-hyperbolic implies that $[x', p] \subset N(r([0, \infty]), 32\delta) \subset N(r([0, \infty]), R_0) \subset V$, and Lemma 7.2 implies that

$$f(x') - f(p) \leq |x' - p|.$$

We conclude that

$$f(x) - f(p) = f(x) - f(x') + f(x') - f(p) \leq |x - x'| + |x' - p| = |x - p|.$$

Since p is a projection of x on S_t, we have $|x - p| = \mathrm{dist}(x, S_t)$ and $f(p) = -t$. Thus, we obtain

$$f(x) + t \leq \mathrm{dist}(x, S_t),$$

that is, $\Delta_x(t) \geq f(x)$ for t large enough. This proves Lemma 7.10.

LEMMA 7.11. *The function F is 1-Lipschitz.*

Proof. Let x and $y \in X$. For every $t \geq 0$, we have $\Delta_x(t) - \Delta_y(t) = \mathrm{dist}(x, S_t) - \mathrm{dist}(y, S_t)$.

Since the function "distance to a nonempty set" is 1-Lipschitz, we obtain, for all $t \geq 0$, $|\Delta_x(t) - \Delta_y(t)| \leq |x - y|$.

By making $t \to \infty$, we obtain $|F(x) - F(y)| \leq |x - y|$, which proves Lemma 7.11.

LEMMA 7.12. *Let $x \in X$ and let $\lambda \in \mathbb{R}$ satisfy $F(x) \geq \lambda$. Then there exists a point $p \in X$ such that $|x - p| = F(x) - \lambda$ and $F(p) = \lambda$.*

Proof. For $t \geq 0$, let q_t be a projection of x on S_t and consider a geodesic segment $[x, q_t]$. As $\mathrm{dist}(x, S_t) \to \infty$ as $t \to \infty$, there exists $t \geq t_0$ such that for all $t \geq t_0$, we have $|x - q_t| \geq F(x) - \lambda$.

In what follows, we take $t \geq t_0$. There exists a (unique) point $p_t \in [x, q_t]$ satisfying $|x - p_t| = F(x) - \lambda$.

We have $|x - q_t| = |x - p_t| + |p_t - q_t|$, which implies

$$|x - q_t| - t = F(x) - \lambda + \big(|p_t - q_t| - t\big). \tag{7.12.1}$$

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We note now that q_t is a projection of p_t on S_t. Indeed, for every $z \in S_t$, we have

$$|x - q_t| \leq |x - z| \leq |x - p_t| + |p_t - z|,$$

which implies

$$|p_t - q_t| = |x - q_t| - |x - p_t| \leq |p_t - z|,$$

which proves that q_t is a projection of p_t on S_t.

Thus, we have $\mathrm{dist}(p_t, S_t) = |p_t - q_t|$ and we obtain from (7.12.1):

$$\mathrm{dist}(x, S_t) - t = F(x) - \lambda + \big(\mathrm{dist}(p_t, S_t) - t\big). \tag{7.12.2}$$

As t varies, the points p_t are all contained in the closed ball of radius $F(x) - \lambda$ centered at x. Since the closed balls in X are compact, there exists a sequence $(t_i)_{i \geq 0}$, with $t_i \to \infty$ as $i \to \infty$, such that p_{t_i} converges to a point $p \in X$. Since $|x - p_{t_i}| = F(x) - \lambda$ for every i, we have $|x - p| = F(x) - \lambda$. It remains to prove that $F(p) = \lambda$.

We have, using (7.12.2), $\mathrm{dist}(p_t, S_t) - t = \big(\mathrm{dist}(x, S_t) - t\big) - (F(x) - \lambda)$. Hence, we obtain

$$\lim_{t \to \infty} \big(\mathrm{dist}(p_t, S_t) - t\big) = F(x) - (F(x) - \lambda) = \lambda. \tag{7.12.3}$$

On the other hand, since the function "distance to a nonempty set" is 1-Lipschitz, we have

$$|(\mathrm{dist}(p, S_t) - t) - (\mathrm{dist}(p_t, S_t) - t)| = |\mathrm{dist}(p, S_t) - \mathrm{dist}(p_t, S_t)| \leq |p - p_t|.$$

Since $|p - p_{t_i}| \to 0$ as $i \to \infty$, we conclude, using (7.12.3), that $\lim_{i \to \infty} \big(\mathrm{dist}(p, S_{t_i}) - t_i\big) = \lambda$. Hence, $F(p) = \lambda$. This completes the proof of Lemma 7.12.

LEMMA 7.13. *Let $x \in X$ and let $\lambda \in \mathbb{R}$ satisfy $F(x) \geq \lambda$. Then, we have*

$$F(x) = \lambda + \mathrm{dist}\big(x, F^{-1}(\lambda)\big).$$

Proof. Let $y \in F^{-1}(\lambda)$. By Lemma 7.11, we have $F(x) - F(y) = |F(x) - F(y)| \leq |x - y|$. Therefore, we have $F(x) \leq F(y) + |x - y| \leq \lambda + |x - y|$. Taking the infimum over $y \in F^{-1}(\lambda)$, we obtain $F(x) \leq \lambda + \mathrm{dist}\big(x, F^{-1}(\lambda)\big)$. On the other hand, Lemma 7.12 gives $F(x) \geq \lambda + \mathrm{dist}\big(x, F^{-1}(\lambda)\big)$.

LEMMA 7.14. *The function F is 68δ-convex.*

Proof. Let x and $y \in X$, let $t \geq 0$ and let p (respectively q) be a projection of x (respectively y) on S_t. Consider a geodesic segment $[x, y]$, let $u \in [0, 1]$ and let z be the point on $[x, y]$ satisfying $|x - z| = u|x - y|$.

By [**7**], Chapter 3, Lemma 3.2, we have $|z - p| \leq (1 - u)|x - p| + u|y - p| + 4\delta$.

By Lemma 7.4, the diameter of the set S_t is bounded by 32δ. Therefore, we have $|y - p| \leq |y - q| + 32\delta$.

Letting m be a projection of z on S_t, we can write now

$$\begin{aligned}\operatorname{dist}(z, S_t) &= |z-m| \\ &\leq |z-p| + 32\delta \\ &\leq (1-u)|x-p| + u|y-p| + 4\delta + 32\delta \\ &\leq (1-u)\operatorname{dist}\big(x, S_t\big) + u\big(\operatorname{dist}\big(y, S_t\big) + 32\delta\big) + 4\delta + 32\delta \\ &\leq (1-u)\operatorname{dist}\big(x, S_t\big) + u\operatorname{dist}\big(y, S_t\big) + 68\delta.\end{aligned}$$

Thus, we have

$$\operatorname{dist}(z, S_t) - t \leq (1-u)(\operatorname{dist}(x, S_t) - t) + u(\operatorname{dist}(y, S_t) - t) + 68\delta.$$

Letting t tend to infinity, we obtain

$$F(z) \leq (1-u)F(x) + uF(y) + 68\delta,$$

which proves Lemma 7.14.

LEMMA 7.15. *The function $F : X \to \mathbb{R}$ is a 68δ-horofunction.*

Proof. This follows from Lemmas 7.13 and 7.14.

Let φ be the cocycle associated to the horofunction F.

LEMMA 7.16. *The geodesic ray r is the smallest φ-gradient ray starting at Id.*

Proof. We recall that for all $n \geq 0$, we have $r(n) = \gamma_n$ and that $r(n+1) = r(n)w(\sigma_n)$, with $w(\sigma_n)$ being the smallest $a \in A$ satisfying $\sigma_n(a) = -1$.
The functions $\sigma_n : V(\sigma_n) \to \mathbb{R}$ and $f_n : V_n \to \mathbb{R}$ are related by

$$\sigma_n(x) = f_n(\gamma_n x) + n,$$

for all $x \in V(\sigma_n)$.
Therefore, $w(\sigma_n)$ is the smallest $a \in A$ satisfying $f_n(\gamma_n a) + n = -1$. Since $n = -f_n(\gamma_n)$, we see now that $w(\sigma_n)$ is the smallest $a \in A$ satisfying $f_n(\gamma_n a) - f_n(\gamma_n) = -1$, that is, $\varphi(\gamma_n a, \gamma_n) = -1$. This proves Lemma 7.16.

LEMMA 7.17. *We have $P(\varphi) = \sigma$.*

Proof. Let $P(\varphi) = (\sigma'_n)_{n\geq 0}$ and let us show that $\sigma'_n = \sigma_n$ for all $n \geq 0$.
Let $n \geq 0$. From the definitions, we have $\sigma'_n = \rho\big(\alpha^n(\varphi)\big)$. Using Proposition 5.7 (i) and Lemma 7.16, we obtain $\sigma'_n = \rho(\gamma_n^{-1}\varphi)$. Thus, σ'_n is the function with domain $V(\gamma_n^{-1}\varphi)$, defined by $\sigma'_n(x) = \gamma_n^{-1}\varphi(x, Id)$ for all $x \in V(\gamma_n^{-1}\varphi)$, that is,

$$\sigma'_n(x) = F(\gamma_n x) + n, \quad \forall x \in V(\gamma_n^{-1}\varphi). \tag{7.17.1}$$

Let us study now the domain $V(\gamma_n^{-1}\varphi)$. We have $V(\gamma_n^{-1}\varphi) = N(g'([0, L_0]), R_0)$, where $g' : [0, \infty[\to X$ is the smallest $\gamma_n^{-1}\varphi$-gradient ray starting at Id. By Proposition 5.4, we have $g' = \gamma_n^{-1} g_n$, where $g_n : [0, \infty[\to X$ is the smallest φ-gradient ray starting at γ_n. Proposition 5.3 and Lemma 7.16 imply now $g_n(t) = r(t+n)$ for all $n \geq 0$. This

https://doi.org/10.1017/S0017089501030063 Published online by Cambridge University Press

shows that $V(\gamma_n^{-1}\varphi) = \gamma_n^{-1}N(r([n, n+L_0]), R_0)$, that is, $V(\gamma_n^{-1}\varphi) = \gamma_n^{-1}V_n = V(\sigma_n)$. Thus, for all $x \in V(\gamma_n^{-1}\varphi)$, we have $\gamma_n x \in V_n$ and formula (7.17.1) becomes $\sigma_n'(x)f_n(\gamma_n x) + n = \sigma_n(x), \quad \forall x \in V_n$. This proves Lemma 7.17.

Now we obtain the following

THEOREM 7.18. *The set $\Sigma(\infty) \subset \Sigma$ of consistent sequences is a subshift of finite type and the map $P : \Phi_0 \to \Sigma(\infty)$ is a homeomorphism satisfying $P \circ \alpha = T \circ P$.*

Proof. By Theorem 6.13, $\Sigma(\infty)$ is a subshift of finite type. By Lemma 6.4, the map $P : \Phi_0 \to \Sigma$ is continuous. By Lemma 6.6, $P : \Phi_0 \to \Sigma$ is injective. By Lemma 7.17, P is surjective. Since Φ_0 is compact, this shows that P is a homeomorphism. By Lemma 6.3, we have $P \circ \alpha = T \circ P$. This proves the theorem.

Finally, let us note that the map α is related to the action of Γ on ∂X in the following manner:

For $s \in S$, let $S(s) \subset \Sigma$ be the cylinder set defined as

$$S(s) = \{\sigma \in \Sigma \ : \ \sigma(0) = s\}.$$

For every $\varphi \in P^{-1}\big(C(s)\big) \cap \Sigma(\infty)$, we have

$$\alpha(\varphi) = w(s)^{-1}\varphi,$$

and therefore

$$\pi \circ \alpha(\varphi) = w(s)^{-1}\pi(\varphi).$$

REFERENCES

1. R. Bowen, C. Series, Markov maps associated with Fuchsian groups. *Inst. Hautes Etudes Sci. Publ. Math.* **50** (1979), 153–170.

2. M. Bonk, Quasi-geodesic segments and Gromov hyperbolic spaces, *Geometriae Dedicata* **62** (1996), 281–289.

3. M. Bourdon, *Actions quasi-convexes d'un groupe hyperbolique, flot geodesique*, Thèse de doctorat, Université de Paris-Sud (Orsay) (1993).

4. M. Bourdon, Structure conforme au bord et flot géodésique d'un CAT(-1)-espace, *L'enseignement Mathématique* **41** (1995), 63–102.

5. J. Cannon, The combinatorial structure of cocompact discrete hyperbolic groups, *Geometriae Dedicata* **16** (1984), 123–148.

6. M. Coornaert, T. Delzant and A. Papadopoulos, *Géométrie et théorie des groupes*, Lecture Notes in Mathematics, No. 1441 (Springer Verlag, 1990).

7. M. Coornaert and A. Papadopoulos, *Symbolic dynamics and hyperbolic groups*, Lecture Notes in Mathematics, No. 1539 (Springer Verlag, 1993).

8. M. Coornaert and A. Papadopoulos, Symbolic coding for the geodesic flow associated to a word hyperbolic group, preprint (U. of Strasbourg, 1999).

9. D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. Levy, M. S. Paterson, W. P. Thurston, *Word processing in groups* (Jones and Bartlett, Boston, 1992).

10. M. Gromov, *Hyperbolic groups*, in *Essays in group theory* (S. M. Gersten, Ed.) (Springer Verlag, 1987).

11. J. Hadamard, Les surfaces à courbures opposées et leurs lignes géodésiques, *J. Math. Pures Appl.* **5** (1898), 27–74.

12. M. Morse, A one-to-one representation of geodesics on a surface of negative curvature, *Am. J. Math*. **43** (1921), 33–51.

13. C. Series , Geometrical Markov coding of geodesics on surfaces of constant negative curvature, *Ergodic Theory Dynamical Systems* **6** (1986), 601–625.