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# ON CIRCULANT MATRICES FOR CERTAIN PERIODIC SPLINE AND HISTOSPLINE PROJECTIONS

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We present a unified treatment of the band circulant matrices which occur in the periodic spline and histospline projection theory with equispaced knots. Explicit bounds for the norm of these matrices are given.

## 1. Introduction

For the $n \geq 1$ degree periodic spline and histospline projections on a uniform partition of a periodic function we have to consider linear systems of the form $p_n^0(v, P)\, s_n^{(k)} = p_n^k(u, P)\, b$ $(k = 0, \ldots, n)$. The column vectors $s_n^{(k)}$ and $b$ of $\mathbb{R}^N$ correspond respectively to the $k^{th}$ derivative of the spline and to the data, and the circulant matrices $p_n^\ell(t,P)(\ell = 0, \ldots, n,\ t \in [0, 1])$ of order $N \geq n + 1$ are generated by the polynomials $p_n^\ell(t,x)$ and a permutation matrix $P$. These systems come from the linear dependence relationships that exist between a spline

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and its $k^{th}$ derivative (see [7]).

The regularity properties of the matrices $p_n^0(v,P)$ are useful in establishing existence results and, together with bounds for the uniform matrix norm of the inverses, in obtaining convergence results (see [5]).

The object of this paper is to review the properties of the polynomials $p_n^\ell(t,x)$ and, using elementary facts about circulant matrices, to present a unified treatment of the matrices $p_n^\ell(t,P)$ .

Throughout this paper $||A||_\infty$ is the uniform matrix norm of the matrix $A$ and $A = \text{circ}\,(a_1, a_2, \ldots a_N)$ means that $A$ is a circulant matrix of order $N$ with $a_1, a_2, \ldots, a_N$ on its first row [1, p.66].

## 2. Definition and Examples

Let $t \in \mathbb{R}$ . The polynomials $p_n^k(t,x)$ are defined as follows

(1) $$p_n^k(t,x) = \sum_{j=0}^{n} c_n^k(t,j)\, x^j$$

where $c_n^k(t,j) = (-1)^k\, \nabla^{n+1}[(j+1-t)^{n-k} \chi_{[0,\infty)}(j)]$ and $\nabla$ is the backward difference operator.

From (1) we have $p_n^k(t,x) = \frac{1}{(n)_k} \frac{\partial^k}{\partial t^k} p_n^0(t,x)$ and it is known (see [3] or [7]) that

$$\frac{x}{(1-x)^{n+1}}\, p_n^0(t,x) = (x\,\frac{\partial}{\partial x} - t)^n\, (\frac{x}{1-x}) = \sum_{\ell=1}^{\infty} (\ell-t)^n\, x^\ell, \quad |x| < 1 .$$

It follows that $p_0^0(t,\cdot) \equiv 1$ , and for $n \geq 1$ we have recurrence relations for the polynomials $p_n^0(t,\cdot)$

(2) $$p_n^0(t,x) = [(1-t) + (n-1+t)x]\, p_{n-1}^0(t,x) + x(1-x)\, \frac{\partial}{\partial x}\, p_{n-1}^0(t,x)$$

and for the coefficients $c_n^0(t,j)$

(3) $$c_n^0(t,j) = (n-j+t)\, c_{n-1}^0(t,j-1) + (j+1-t)\, c_{n-1}^0(t,j) .$$

The circulant matrix $p_n^\ell(t,P)$ is generated by the permutation matrix $P = \text{circ}\ (0,\ 1,\ 0,\ \ldots,\ 0)$ of order $N \geq n+1$ and the polynomial $p_n^\ell(t,x)$ .

For example, $p_0^0(t,P) = I$ and for $n = 1,\ \ldots,\ 8$ and $t = 0$ or $t = \frac{1}{2}$ we have

(i) $\quad t = 0$: $p_1^0(0,P) = I,$

$$p_2^0(0,P) = \text{circ}(1,1,0,\ldots,0),$$

$$p_3^0(0,P) = \text{circ}(1,4,1,0,\ldots,0),$$

$$p_4^0(0,P) = \text{circ}(1,11,11,1,0,\ldots,0),$$

$$p_5^0(0,P) = \text{circ}(1,26,66,26,1,0,\ldots,0),$$

$$p_6^0(0,P) = \text{circ}(1,57,302,302,57,1,0,\ldots,0),$$

$$p_7^0(0,P) = \text{circ}(1,120,1191,2416,1191,120,1,0,\ldots,0),$$

$$p_8^0(0,P) = \text{circ}(1,247,4293,15619,15619,4293,247,1,0,\ldots,0);$$

(ii) $\quad t = \frac{1}{2}$: $p_1^0(\frac{1}{2},P) = \frac{1}{2}\ \text{circ}(1,1,0,\ldots,0),$

$$p_2^0(\tfrac{1}{2},P) = (\tfrac{1}{2})^2\ \text{circ}(1,6,1,0,\ldots,0),$$

$$p_3^0(\tfrac{1}{2},P) = (\tfrac{1}{2})^3\ \text{circ}(1,23,23,1,0,\ldots,0),$$

$$p_4^0(\tfrac{1}{2},P) = (\tfrac{1}{2})^4\ \text{circ}(1,76,230,76,1,0,\ldots,0),$$

$$p_5^0(\tfrac{1}{2},P) = (\tfrac{1}{2})^5\ \text{circ}(1,237,1682,1682,237,1,0,\ldots,0),$$

$$p_6^0(\tfrac{1}{2},P) = (\tfrac{1}{2})^6\ \text{circ}(1,722,10543,23548,10543,722,1,0,\ldots,0),$$

$$p_7^0(\tfrac{1}{2},P) = (\tfrac{1}{2})^7\ \text{circ}(1,2179,60657,259723,259723,60657,2179,1,0,\ldots,0),$$

$$p_8^0(\tfrac{1}{2},P) = (\tfrac{1}{2})^8\ \text{circ}(1,6552,331612,2485288,4675014,2485288,331612,6552,1,0,\ldots,0).$$

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**Remark 1.** If $c_n^k(t,0) = 0$ or $c_n^k(t,n) = 0$ we can consider $P$ of order $N \geq n$ instead of order $N \geq n+1$.

## 3. Properties of the polynomials $p_n^k(t,\cdot)$

The polynomials $p_n^0(t,\cdot)$ have been analyzed by several authors (see [3], [6], [7], [8] and [9]) and are closely related to the exponential Euler polynomials. In this section we recall their properties without proof.

**THEOREM 2.** *$p_0^0(t,x) = 1$ and for $n \geq 1$*

(i) $p_n^0(t,\cdot)$ *is a polynomial of degree* $\begin{cases} n & \text{if } t \in (0,1]. \\ n-1 & \text{if } t = 0; \end{cases}$

(ii) $p_n^0(1,x) = x\, p_n^0(0,x)$;

(iii) $p_n^k(t,x) = (x-1)^k\, p_{n-k}^0(t,x) \qquad$ *for* $k = 0, \ldots, n$;

(iv) $p_n^0(t,x) = x^n\, p_n^0(1-t,1/x)$. □

**THEOREM 3.** *For all $n \geq 0$ we have*

(i) $p_n^0(t,1) = n!$ ,

(ii) $p_n^0(t,-1) = (-2)^n E_n(t)$

*where $E_n(\cdot)$ is the Euler polynomial degree $n$.* □

The next theorem has been obtained by several authors for $t \in [0,1]$ (see [6], [7], [9]) and the extension to $t \in (-\varepsilon,\ 1+\varepsilon)$ is given in [3].

**THEOREM 4.** *For all $n \geq 1$ there exist a strictly positive real number $\varepsilon$ and $n$ functions, denoted $x_{n,i}(\cdot)$ for $i = 1, \ldots, n$, such that $x_{n,i}(t)$ is the ith root of $p_n^0(t,x)$ for all $t \in (-\varepsilon,\ 1+\varepsilon)$ (except for $i = n$ when $t = 0$ ). These functions are such that* *(i)* $x_{n,1}(\cdot) \in C^{\infty}((-\varepsilon, 1+\varepsilon);\mathbb{R})$, $x_{n,1}(\cdot)$ *is strictly increasing over* $(-\varepsilon, 1]$ *and strictly increasing (respectively decreasing), over* $[1, 1+\varepsilon)$ *when* $n$ *is odd (respctively even);*

*(ii) for* $i = 2, \ldots, n-1$, $x_{n,i}(\cdot) \in C^{\infty}((-\varepsilon, 1+\varepsilon);\mathbb{R})$ *and* $x_{n,i}(\cdot)$ *is strictly increasing over* $(-\varepsilon, 1+\varepsilon)$*;*

*(iii)* $x_{n,n}(\cdot) \in C^{\infty}((-\varepsilon, 0) \cup (0, 1+\varepsilon);\mathbb{R})$, $x_{n,n}(\cdot)$ *is strictly increasing (respectively decreasing) over* $(-\varepsilon, 0)$ *if* $n$ *is odd (respectively even), and*

$$\lim_{t\to 0^+} x_{n,n}(t) = -\infty \quad \text{and} \quad \lim_{t\to 0^-} x_{n,n}(t) = \begin{cases} -\infty & \text{if } n \text{ is odd,} \\ +\infty & \text{if } n \text{ is even.} \end{cases}$$

Moreover $x_{n,n+1-i}(t) = 1/x_{n,i}(1-t)$ when the two roots exist, $x_{n,1}(1) = 0$ and for $i = 2, \ldots, n$ we have

$$x_{n,i-1}(0) = x_{n,i}(1)$$

and

$$x_{n,i}(t) < x_{n-1,i-1}(t) < x_{n,i-1}(t)$$

for all $t \in (0,1)$ . □

It is important to observe that for $t \in [0,1]$ the roots of $p_n^0(t,x)$ are real, distinct and nonpositive. This result can be obtained from the fact that the coefficients of the polynomials form a Polya frequency sequence (see [9]). But this is not the case when $t \notin [0,1]$ .

From these theorems we can prove the following consequences.

COROLLARY 5. *Let* $n \geq 1$ *and* $k \in \{0, \ldots, n\}$. *There exist a strictly positive real number* $\varepsilon$ *such that*

$$t \in (-\varepsilon, 1+\varepsilon) \quad \text{and} \quad p_n^k(t,-1) = 0$$

*if and only if*

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$$n-k \geq 1 \quad \text{and} \quad \begin{cases} n-k & \text{is odd and} \quad t=\tfrac{1}{2} \\ \text{or} & \\ n-k & \text{is even and} \quad t=0 \text{ or } 1. \end{cases}$$

*In this case we have* $p_n^k(t,x) = (x+1)\; q_n^k(t,x)$ *where*

$$q_n^k(t,x) = \sum_{j=0}^{n-1} d_n^k(t,j)\; x^j \quad \text{and} \quad d_n^k(t,j) = \sum_{\ell=0}^{j} (-1)^{j-\ell} c_n^k(t,\ell)\;.$$

*Moreover for* $n$ *odd* $x_{n,\frac{n+1}{2}}(\tfrac{1}{2}) = -1$ *and for* $n$ *even*

$x_{n,\frac{n}{2}}(0) = -1 = x_{n,\frac{n}{2}+1}(1)$ □

COROLLARY 6. *For each* $n \geq 2$ *there exist a strictly positive real number* $\varepsilon$ *such that*

$$\max_{t\in(-\varepsilon,1+\varepsilon)} |p_n^0(t,-1)| = \begin{cases} |p_n^0(0,-1)| = |p_n^0(1,-1)| & \text{if } n \text{ is odd,} \\ |p_n^0(\tfrac{1}{2},-1)| & \text{if } n \text{ is even.} \end{cases}$$ □

## 4. Regularity properties and explicit bounds

From the factorization of the polynomial $p_n^0(t,x)$ we obtain the following decomposition for the matrix $p_n^0(t,P)$

$$(4) \qquad p_n^0(t,P) = \begin{cases} t^n \prod_{i=1}^{n} (P - x_{n,i}(t)I) & \text{if} \quad t \in (-\varepsilon,0)\cup(0,1+\varepsilon)\;, \\ \prod_{i=1}^{n-1} (P - x_{n,i}(0)I) & \text{if} \quad t=0, \end{cases}$$

where $P = \text{circ}\,(0,\,1,\,0,\,\ldots,\,0)$ . Then we will first consider elementary factors of the form $p(x) = x - \alpha$ .

THEOREM 7. *Let* $p(x) = x - \alpha$ *and* $\alpha \in \mathbb{C}$ . *If* $P = \text{circ}\,(0,1,0,\ldots,0)$ *is of order* $N$ , *then*

(i) $\quad ||p(P)||_\infty = 1 + |\alpha|$ ,

(ii) $\quad p(P)$ *is invertible if and only if* $\alpha^N \neq 1$ . *In this case*

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$$p(P)^{-1} = \frac{1}{1-\alpha^N} \sum_{i=0}^{N-1} \alpha^{N-1-i} P^i$$

*and*

$$||p(P)^{-1}||_\infty = \frac{1}{|1-\alpha^N|} \sum_{i=0}^{N-1} |\alpha|^{N-1-i} = \begin{cases} \dfrac{1-|\alpha|^N}{|1-\alpha^N|(1-|\alpha|)} & \text{if } |\alpha| \neq 1 , \\ \dfrac{N}{|1-\alpha^N|} & \text{if } |\alpha| = 1 . \end{cases}$$

**Proof.** (i) Obvious. (ii) See Davis [1, p.89]. □

COROLLARY 8. *Let* $p(x) = x - \alpha$ *and* $\alpha \in \mathbb{R}$ . *If* $P = \text{circ}\,(0,1,0,\ldots,0)$ *then*

$$||p(P)||_\infty = \begin{cases} |p(-1)| & \text{if } \alpha \geq 0 , \\ p(1) & \text{if } \alpha \leq 0 . \end{cases}$$ □

COROLLARY 9. *Let* $p(x) = x - \alpha$, $\alpha \in \mathbb{R}$ *and* $P = \text{circ}\,(0,1,0,\ldots,0)$ *is of order* $N$.

*(i)* *If* $|\alpha| \neq 1$ *then* $p(P)$ *is invertible and*

$$||p(P)^{-1}||_\infty \leq \begin{cases} \dfrac{1}{|p(1)|} & \text{if } \alpha \geq 0 , \\ \dfrac{1}{|p(-1)|} & \text{if } \alpha \leq 0 . \end{cases}$$

*(ii)* *If* $\alpha = 1$ *then the matrix* $p(P) = P - I$ *of order* $N$ *is of rank* $N - 1$ .

*(iii)* *If* $\alpha = -1$ *then* $p(P) = P + I$ .

*(a)* *If* $N$ *is odd then* $p(P)$ *is invertible,*

$p(P)^{-1} = \tfrac{1}{2}\,\text{circ}\,(1,-1,\ldots,-1,1)$ *and* $||p(P)^{-1}||_\infty = N/2$ .

*(b)* *If* $N$ *is even then the matrix* $p(P)$ *of order* $N$ *is of rank* $N - 1$ . □

When the matrix $p(P)$ is not invertible, we can obtain the generalized inverse $p(P)^+$ of $p(P)$ using the formula

$p(P)^{+} = U^T(UU^T)^{-1}(L^TL)^{-1}L^T$ where $p(P) = LU$ is a full rank factorization of $p(P)$ . Using this method for $p(P) = P + I$ and $N$ even, we have

$$L = \begin{bmatrix} I \\ \hline \nu^T \end{bmatrix}_{N\times N-1} , \quad U = \begin{bmatrix} 1 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 1 \end{bmatrix}_{N-1\times N} \quad \textit{and} \quad \nu = \begin{bmatrix} 1 \\ -1 \\ \cdot \\ \cdot \\ \cdot \\ -1 \\ 1 \end{bmatrix}_{N-1\times 1} ,$$

and $L^TL = I + \nu\nu^T$, $(L^TL)^{-1} = I - \frac{1}{N}\ \nu\nu^T$ and

$$UU^T = \begin{bmatrix} 2 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 2 \end{bmatrix}$$

A similar decomposition can be done for $p(P) = P - I$ .

Now we apply these results to the matrix $p_n^k(t,P)$ and obtain the following results.

THEOREM 10. *Let* $t \in [0,1]$, *then* $||p_n^0(t,P)||_\infty = p_n^0(t,1) = n!$ *and for* $k = 0,\dots,n$ *we have* $||p_n^k(t,P)||_\infty \le 2^k(n-k)!$

Proof. From (3) it follows that the coefficients of $p_n^0(t,x)$ are nonnegative for all $t \in [0,1]$ , then the results follow from Theorem 3 (i) and from Theorem 2 (iii). $\square$

THEOREM 11. *Let* $P = \text{circ}\ (0,1,0,\dots,0)$ *be of order* $N \ge n + 1$ , *where* $n \ge 1$ .

*(i)* *If* $t \in [0,1]$ *and* $\begin{cases} n \ \textit{ is odd and } \ t \ne \frac{1}{2} \\ \textit{or} \\ n \ \textit{ is even and } \ t \ne 0 \ \textit{ and } \ t \ne 1 , \end{cases}$

*then* $p_n^0(t,P)$ *is invertible and* $||p_n^0(t,P)^{-1}||_\infty \le 1/|p_n^0(t,-1)|$ .

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(ii) If $\begin{cases} n \text{ is odd and } t = \frac{1}{2}, \\ \text{or} \\ n \text{ is even and } t = 0 \text{ or } t = 1, \end{cases}$ then $p_n^0(t,P) = (P+I)q_n^0(t,P)$

where $q_n^0(t,P)$ is invertible, $||q_n^0(t,P)^{-1}||_\infty \le 1/|q_n^0(t,-1)|$ and $q_n^0(t,-1) = -\frac{1}{2}\, p_{n+1}^0(t,-1)$.

(a) If $N$ is odd then $P + I$ is invertible, $(P + I)^{-1} = \frac{1}{2}\,\mathrm{circ}\,(1,-1,\ldots,-1,1)$, $p_n^0(t,P)^{-1} = (P+I)^{-1}\, q_n^0(t,P)^{-1}$ and $||p_n^0(t,P)^{-1}||_\infty \le N/|p_{n+1}^0(t,-1)|$.

(b) If $N$ is even then $P + I$ is not invertible, but

$$p_n^0(t,P)^+ = (P+I)^+\, q_n^0(t,P)^{-1}$$

**Proof.** The two situations come from the decomposition (4), Corollary 5 with $k = 0$ and the fact that all the roots of $p_n^0(t,x)$ are non-positive when $t \in [0,1]$. Then we obtain part (i) from Corollary 9 (i). To obtain $q_n^0(t,-1) = -\frac{1}{2}\, p_{n+1}^0(t,-1)$ we use (2) and Corollary 5. To complete the proof of the part (ii) we use parts (i) and (iii) of Corollary 9 and the fact that the generalized inverse of a circulant matrix is a circulant matrix [1, p.87]. □

We observe that the bound of part (i) of the Theorem 11 is a minimum when the denominator is a maximum. From Corollary 6, this happens when

(i) $n$ is odd and $t = 0$ (or $1$), (see also [4]), then

$$p_n^0(0,-1) = (-2)^n E_n(0) = 2^{n+1}\,\frac{2^{n+1} - 1}{n+1}\, B_{n+1},$$

(ii) $n$ is even and $t = \frac{1}{2}$, then

$$p_n^0(\tfrac{1}{2},-1) = (-2)^n E_n(\tfrac{1}{2}) = E_n,$$

where we used part (ii) of Theorem 3 and where $B_n$ and $E_n$ are Bernoulli

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and Euler numbers. The first situation corresponds to the odd degree spline interpolation at the knots and the second situation corresponds to the even degree spline interpolation at midknots.

For part (ii) of Theorem 11 we have

(i) $n$ is odd and $t = \frac{1}{2}$ , then

$$q_n^0(\tfrac{1}{2},-1) = -\frac{p_{n+1}^0(\tfrac{1}{2},-1)}{2} = -\frac{E_{n+1}}{2}$$

(ii) $n$ is even and $t = 0$ (or $1$ ), (see also [2]), then

$$q_n^0(0,-1) = -\frac{p_{n+1}^0(0,-1)}{2} = -2^{n+1}\,\frac{2^{n+2}-1}{n+2}\,B_{n+2}$$

and these two situations correspond to the odd degree spline interpolation at midknots and even degree spline interpolation at the knots.

Finally, the spline and its derivatives can be given by

$$s_n^{(k)} = p_n^{(k)}(u,P)\;p_n^0(v,P)^{-1}\;b$$

when $p_n^0(v,P)$ is invertible and by

$$s_n^{(k)} = p_n^{(k)}(u,P)\;p_n^0(v,P)^{+}\;b$$

when $p_n^0(v,P)$ is not invertible. This last situation corresponds to a least squares problem. These representations of the spline are useful in obtaining convergence results when the data $b$ comes from a regular function $f$ .

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