ON CIRCULANT MATRICES FOR CERTAIN PERIODIC SPLINE
AND HISTOSPINE PROJECTIONS

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We present a unified treatment of the band circulant matrices which
occur in the periodic spline and histospline projection theory with
equispaced knots. Explicit bounds for the norm of these matrices
are given.

1. Introduction

For the \( n \geq 1 \) degree periodic spline and histospline projections
on a uniform partition of a periodic function we have to consider linear
systems of the form

\[
\frac{p_n^O(v, P)}{s_n^{(k)}} = \frac{p_n^k(u, P)}{b} \quad (k = 0, \ldots, n).
\]

The column vectors \( s_n^{(k)} \) and \( b \) of \( \mathbb{R}^N \) correspond respectively to the \( k^{th} \)
derivative of the spline and to the data, and the circulant matrices
\( p_n^k(t, P) (k = 0, \ldots, n, t \in [0, 1]) \) of order \( N \geq n + 1 \) are generated by
the polynomials \( p_n^k(t, x) \) and a permutation matrix \( P \). These systems
come from the linear dependence relationships that exist between a spline

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and its $k^{th}$ derivative (see [7]).

The regularity properties of the matrices $p_k^n(v,P)$ are useful in establishing existence results and, together with bounds for the uniform matrix norm of the inverses, in obtaining convergence results (see [5]).

The object of this paper is to review the properties of the polynomials $p_k^n(t,x)$ and, using elementary facts about circulant matrices, to present a unified treatment of the matrices $p_k^n(t,P)$.

Throughout this paper $||A||_\infty$ is the uniform matrix norm of the matrix $A$ and $A = \text{circ}(a_1, a_2, \ldots, a_N)$ means that $A$ is a circulant matrix of order $N$ with $a_1, a_2, \ldots, a_N$ on its first row [1, p.66].

2. Definition and Examples

Let $t \in \mathbb{R}$. The polynomials $p_k^n(t,x)$ are defined as follows

$$(1) \quad p_k^n(t,x) = \sum_{j=0}^{n-k} a_k^n(t,j) x^j$$

where $a_k^n(t,j) = (-1)^k \sqrt{n+1} [((j+1-t)^{n-k} x_{[0,\infty)}(j)]$ and $\triangledown$ is the backward difference operator.

From (1) we have $p_k^n(t,x) = \frac{1}{(n-k)!} \frac{\partial^k}{\partial t^k} p_0^n(t,x)$ and it is known (see [3] or [7]) that

$$p_0^n(t,x) = \frac{x}{(1-x)^{n+1}} \sum_{k=0}^{n} \frac{x^k}{k!} (1-t)^n, \quad |x| < 1.$$ 

It follows that $p_0^n(t,\cdot) \equiv 1$, and for $n \geq 1$ we have recurrence relations for the polynomials $p_0^n(t,\cdot)$

$$(2) \quad p_n^0(t,x) = [(1-t) + (n-1+t)x] p_{n-1}^0(t,x) + x(1-x) \frac{\partial}{\partial x} p_{n-1}^0(t,x)$$

and for the coefficients $a_n^0(t,j)$

$$(3) \quad a_n^0(t,j) = (n-j+t) a_{n-1}^0(t,j-1) + (j+1-t) a_{n-1}^0(t,j).$$
The circulant matrix \( p_n^0(t, P) \) is generated by the permutation matrix \( P = \text{circ}(0, 1, 0, \ldots, 0) \) of order \( N > n + 1 \) and the polynomial \( p_n^0(t, x) \).

For example, \( p_0^0(t, P) = I \) and for \( n = 1, \ldots, 8 \) and \( t = 0 \) or \( t = \frac{1}{2} \), we have

(i) \( t = 0 \):

\[
\begin{align*}
p_0^0(0, P) &= \text{circ}(1, 0, \ldots, 0), \\
p_3^0(0, P) &= \text{circ}(1, 4, 1, 0, \ldots, 0), \\
p_4^0(0, P) &= \text{circ}(1, 11, 11, 1, 0, \ldots, 0), \\
p_5^0(0, P) &= \text{circ}(1, 26, 56, 26, 1, 0, \ldots, 0), \\
p_6^0(0, P) &= \text{circ}(1, 57, 302, 302, 57, 1, 0, \ldots, 0), \\
p_7^0(0, P) &= \text{circ}(1, 120, 1191, 2416, 1191, 120, 1, 0, \ldots, 0), \\
p_8^0(0, P) &= \text{circ}(1, 247, 4293, 15619, 15619, 4293, 247, 1, 0, \ldots, 0).
\end{align*}
\]

(ii) \( t = \frac{1}{2} \):

\[
\begin{align*}
p_2^0\left(\frac{1}{2}, P\right) &= \left(\frac{1}{2}\right)^2 \text{circ}(1, 6, 1, 0, \ldots, 0), \\
p_3^0\left(\frac{1}{2}, P\right) &= \left(\frac{1}{2}\right)^3 \text{circ}(1, 23, 23, 1, 0, \ldots, 0), \\
p_4^0\left(\frac{1}{2}, P\right) &= \left(\frac{1}{2}\right)^4 \text{circ}(1, 76, 230, 76, 1, 0, \ldots, 0), \\
p_5^0\left(\frac{1}{2}, P\right) &= \left(\frac{1}{2}\right)^5 \text{circ}(1, 237, 1682, 1682, 237, 1, 0, \ldots, 0), \\
p_6^0\left(\frac{1}{2}, P\right) &= \left(\frac{1}{2}\right)^6 \text{circ}(1, 722, 10543, 23548, 10543, 722, 1, 0, \ldots, 0), \\
p_7^0\left(\frac{1}{2}, P\right) &= \left(\frac{1}{2}\right)^7 \text{circ}(1, 3179, 60657, 259723, 259723, 60657, 3179, 1, 0, \ldots, 0), \\
p_8^0\left(\frac{1}{2}, P\right) &= \left(\frac{1}{2}\right)^8 \text{circ}(1, 6552, 331612, 2485288, 4675014, 2485288, 331612, 6552, 1, 0, \ldots, 0).
\end{align*}
\]
Remark 1. If $c_n^k(t,0) = 0$ or $c_n^k(t,n) = 0$ we can consider $P$ of order $N \geq n$ instead of order $N \geq n + 1$.

3. Properties of the polynomials $p_n^k(t, \cdot)$

The polynomials $p_n^k(t, \cdot)$ have been analyzed by several authors (see [3], [6], [7], [8] and [9]) and are closely related to the exponential Euler polynomials. In this section we recall their properties without proof.

**Theorem 2.** $p_n^0(t, x) = 1$ and for $n \geq 1$

(i) $p_n^0(t, \cdot)$ is a polynomial of degree \begin{align*}
&\begin{cases}
  n & \text{if } t \in (0,1], \\
  n-1 & \text{if } t = 0;
\end{cases}
\end{align*}

(ii) $p_n^0(1, x) = x p_n^0(0, x)$;

(iii) $p_n^k(t, x) = (x-1)^k p_{n-k}^0(t, x)$ for $k = 0, \ldots, n$;

(iv) $p_n^0(t, x) = x^n p_n^0(1-t, 1/x)$.

**Theorem 3.** For all $n \geq 0$ we have

(i) $p_n^0(t, 1) = n!$,

(ii) $p_n^0(t, -1) = (-2)^n E_n(t)$

where $E_n(\cdot)$ is the Euler polynomial degree $n$.

The next theorem has been obtained by several authors for $t \in [0,1]$ (see [6], [7], [9]) and the extension to $t \in (-\epsilon, 1+\epsilon)$ is given in [3].

**Theorem 4.** For all $n \geq 1$ there exist a strictly positive real number $\epsilon$ and $n$ functions, denoted $x_n^i(\cdot)$ for $i = 1, \ldots, n$, such that $x_n^i(t)$ is the $i$th root of $p_n^0(t, x)$ for all $t \in (-\epsilon, 1+\epsilon)$ (except for $i = n$ when $t = 0$). These functions are such that
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(i) $x_{n,1}(\cdot) \in C^\infty((-\epsilon, 1+\epsilon); \mathbb{R})$, $x_{n,1}(\cdot)$ is strictly increasing over $(-\epsilon, 1]$ and strictly increasing (respectively decreasing), over $[1, 1+\epsilon)$ when $n$ is odd (respectively even);

(ii) for $i = 2, \ldots, n-1$, $x_{n,i}(\cdot) \in C^\infty((-\epsilon, 1+\epsilon); \mathbb{R})$ and $x_{n,i}(\cdot)$ is strictly increasing over $(-\epsilon, 1+\epsilon)$;

(iii) $x_{n,n}(\cdot) \in C^\infty((-\epsilon, 0) \cup (0, 1+\epsilon); \mathbb{R})$, $x_{n,n}(\cdot)$ is strictly increasing (respectively decreasing) over $(-\epsilon, 0)$ if $n$ is odd (respectively even), and

$$\lim_{t \to 0^-} x_{n,n}(t) = -\infty \quad \text{and} \quad \lim_{t \to 0^+} x_{n,n}(t) = \begin{cases} -\infty & \text{if } n \text{ is odd,} \\ +\infty & \text{if } n \text{ is even.} \end{cases}$$

Moreover $x_{n,n+1-i}(t) = 1/x_{n,i}(1-t)$ when the two roots exist, $x_{n,1}(1) = 0$ and for $i = 2, \ldots, n$ we have

$$x_{n,i-1}(0) = x_{n,i}(1)$$

and

$$x_{n,i}(t) < x_{n-1,i-1}(t) < x_{n,i-1}(t)$$

for all $t \in (0,1)$.

It is important to observe that for $t \in [0,1]$ the roots of $p_n^0(t,x)$ are real, distinct and nonpositive. This result can be obtained from the fact that the coefficients of the polynomials form a Polya frequency sequence (see [9]). But this is not the case when $t \notin [0,1]$.

From these theorems we can prove the following consequences.

COROLLARY 5. Let $n \geq 1$ and $k \in \{0, \ldots, n\}$. There exist a strictly positive real number $\epsilon$ such that

$$t \in (-\epsilon, 1+\epsilon) \quad \text{and} \quad p_n^k(t,-1) = 0$$

if and only if
In this case we have \( p_n^k(t, x) = (x + 1) q_n^k(t, x) \) where
\[
q_n^k(t, x) = \sum_{j=0}^{n-1} d_n^j(t, j) x^j \quad \text{and} \quad d_n^j(t, j) = \sum_{l=0}^j (-1)^{j-l} c_n^l(t, l).
\]
Moreover for \( n \) odd \( x \left( \frac{n+1}{2} \right) = -1 \) and for \( n \) even \( x \left( \frac{n}{2} \right) = \frac{n}{2} + 1 \)

\[
x \left( 0 \right) = \begin{cases} -1 & \text{if } n \text{ odd} \\ \frac{n}{2} & \text{if } n \text{ even} \end{cases}
\]

**Corollary 6.** For each \( n \geq 2 \) there exist a strictly positive real number \( \varepsilon \) such that
\[
\max_{t \in (-\varepsilon, 1+\varepsilon)} \left| p_n^0(t, -1) \right| = \left| p_n^0(0, -1) \right| = \left| p_n^0(1, -1) \right| \quad \text{if } n \text{ is odd},
\]
\[
\left| p_n^0 \left( \frac{n}{2}, -1 \right) \right| \quad \text{if } n \text{ is even}.
\]

**4. Regularity properties and explicit bounds**

From the factorization of the polynomial \( p_n^0(t, x) \) we obtain the following decomposition for the matrix \( p_n^0(t, P) \)

\[
p_n^0(t, P) = \begin{cases} \sum_{i=1}^{\frac{n}{2}} (P - x_n, i(t)I) & \text{if } t \in (-\varepsilon, 0) \cup (0, 1+\varepsilon), \\ \sum_{i=1}^{\frac{n-1}{2}} (P - x_n, i(0)I) & \text{if } t = 0,
\end{cases}
\]

where \( P = \text{circ} (0, 1, 0, \ldots, 0) \). Then we will first consider elementary factors of the form \( p(x) = x - a \).

**Theorem 7.** Let \( p(x) = x - a \) and \( a \in \mathbb{C} \). If \( P = \text{circ} (0, 1, 0, \ldots, 0) \) is of order \( N \), then

\[
(i) \quad \|p(P)\|_\infty = 1 + |a|,
\]

\[
(ii) \quad p(P) \text{ is invertible if and only if } a^N \neq 1. \text{ In this case}
\]
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\[ p(P)^{-1} = \frac{1}{1-\alpha} \sum_{i=0}^{N-1} \alpha^{N-1-i} p^i \]

and

\[ ||p(P)^{-1}||_{\infty} = \frac{1}{|1-\alpha|} \sum_{i=0}^{N-1} |\alpha^{N-1-i}| = \begin{cases} \frac{1-|\alpha|^N}{|1-\alpha|(1-|\alpha|)} & \text{if } |\alpha| \neq 1, \\ \frac{N}{|1-\alpha|} & \text{if } |\alpha| = 1. \end{cases} \]

Proof. (i) Obvious. (ii) See Davis [1, p.89].

COROLLARY 8. Let \( p(x) = x - \alpha \) and \( \alpha \in \mathbb{R} \). If \( P = \text{circ}(0,1,0,...,0) \) then

\[ ||p(P)||_{\infty} = \begin{cases} |p(-1)| & \text{if } \alpha \geq 0, \\ p(1) & \text{if } \alpha < 0. \end{cases} \]

COROLLARY 9. Let \( p(x) = x - \alpha, \alpha \in \mathbb{R} \) and \( P = \text{circ}(0,1,0,...,0) \) is of order \( N \).

(i) If \( |\alpha| \neq 1 \) then \( p(P) \) is invertible and

\[ ||p(P)^{-1}||_{\infty} \leq \begin{cases} \frac{1}{|p(1)|} & \text{if } \alpha > 0, \\ \frac{1}{|p(-1)|} & \text{if } \alpha < 0. \end{cases} \]

(ii) If \( \alpha = 1 \) then the matrix \( p(P) = P - I \) of order \( N \) is of rank \( N - 1 \).

(iii) If \( \alpha = -1 \) then \( p(P) = P + I \).

(a) If \( N \) is odd then \( p(P) \) is invertible, \( p(P)^{-1} = \frac{1}{2} \text{circ}(1,-1,...,-1,1) \) and \( ||p(P)^{-1}||_{\infty} = N/2 \).

(b) If \( N \) is even then the matrix \( p(P) \) of order \( N \) is of rank \( N - 1 \).

When the matrix \( p(P) \) is not invertible, we can obtain the generalized inverse \( p(P)^+ \) of \( p(P) \) using the formula
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\[ p(P)^+ = U^T (U^T U)^{-1} (L^T L)^{-1} L^T \]
where \( p(P) = LU \) is a full rank factorization
of \( p(P) \). Using this method for \( p(P) = P + I \) and \( N \) even, we have

\[
L = \begin{bmatrix}
I \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
V^T
\end{bmatrix}_{N \times N-1}, \quad U = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
1 & 1 & 1
\end{bmatrix}_{N-1 \times N}
\]
and \( v = \begin{bmatrix}
1 \\
-1 \\
\vdots \\
-1 \\
1
\end{bmatrix}_{N-1 \times 1} \)

and \( L^T L = I + vv^T \), \((L^T L)^{-1} = I - \frac{1}{N} vv^T\)

A similar decomposition can be done for \( p(P) = P - I \).

Now we apply these results to the matrix \( p(t,P) \) and obtain the
following results.

**THEOREM 10.** Let \( t \in \{0,1\} \), then \( \|p_n^0(t,P)\|_\infty = p_n^0(t,1) = n! \) and
for \( k = 0, \ldots, n \) we have \( \|p_n^k(t,P)\|_\infty \leq 2^k(n-k)! \).

**Proof.** From (3) it follows that the coefficients of \( p_n^0(t,x) \) are
nonnegative for all \( t \in [0,1] \), then the results follow from Theorem 3 (i)
and from Theorem 2 (iii).

**THEOREM 11.** Let \( P = \text{circ}(0,1,0,\ldots,0) \) be of order \( N \geq n + 1 \),
where \( n \geq 1 \).

(i) If \( t \in [0,1] \) and

\[
\begin{cases}
n \text{ is odd and } t \neq \frac{1}{2} \\
n \text{ is even and } t \neq 0 \text{ and } t \neq 1,
\end{cases}
\]

then \( p_n^0(t,P) \) is invertible and \( \|p_n^0(t,P)^{-1}\|_\infty \leq 1/|p_n^0(t,-1)| \).
(ii) If \( n \) is odd and \( t = \frac{k}{n} \), then
\[
p_n^0(t, P) = (P + I)q_n^0(t, P)
\]
where \( q_n^0(t, P) \) is invertible, \(|q_n^0(t, P)^{-1}|_\infty \leq 1/|q_n^0(t, -1)|\) and
\[
q_n^0(t, -1) = -\frac{1}{n}p_{n+1}^0(t, -1).
\]

(a) If \( n \) is odd then \( P + I \) is invertible, \((P + I)^{-1} = \frac{1}{n} \text{circ}(1, -1, \ldots, -1, 1)\), \( p_n^0(t, P)^{-1} = (P + I)^{-1}q_n^0(t, P)^{-1} \) and
\[
|p_n^0(t, P)^{-1}|_\infty \leq N/|p_{n+1}^0(t, -1)|.
\]

(b) If \( n \) is even then \( P + I \) is not invertible, but
\[
p_n^0(t, P) = (P + I)^+q_n^0(t, P)^{-1}
\]

Proof. The two situations come from the decomposition (4), Corollary 5 with \( k = 0 \) and the fact that all the roots of \( p_n^0(t, x) \) are non-positive when \( t \in [0, 1] \). Then we obtain part (i) from Corollary 9 (i).

To obtain \( q_n^0(t, -1) = -\frac{1}{n}p_{n+1}^0(t, -1) \) we use (2) and Corollary 5. To complete the proof of the part (ii) we use parts (i) and (iii) of Corollary 9 and the fact that the generalized inverse of a circulant matrix is a circulant matrix [7, p.87].

We observe that the bound of part (i) of the Theorem 11 is a minimum when the denominator is a maximum. From Corollary 6, this happens when

(i) \( n \) is odd and \( t = 0 \) (or \( 1 \)), (see also [4]), then
\[
p_n^0(0, -1) = (-2)^nE_n(0) = \frac{2^{n+1}}{n + 1} B_{n+1}
\]
(ii) \( n \) is even and \( t = \frac{k}{n} \), then
\[
p_n^0(k, -1) = (-2)^nE_n(k) = E_n
\]
where we used part (ii) of Theorem 3 and where \( B_n \) and \( E_n \) are Bernoulli
and Euler numbers. The first situation corresponds to the odd degree spline interpolation at the knots and the second situation corresponds to the even degree spline interpolation at midknots.

For part (ii) of Theorem 11 we have

(i) \( n \) is odd and \( t = \frac{1}{2} \), then

\[
q_n^0(\frac{1}{2},-1) = - \frac{p_{n+1}^0(\frac{1}{2},-1)}{2} = - \frac{E_{n+1}}{2}
\]

(ii) \( n \) is even and \( t = 0 \) (or 1), (see also [2]), then

\[
q_n^0(0,-1) = - \frac{p_{n+1}^0(0,-1)}{2} = - 2^{n+1} \frac{n+2}{n+2} \frac{B_{n+2}}{B_{n+2}}
\]

and these two situations correspond to the odd degree spline interpolation at midknots and even degree spline interpolation at the knots.

Finally, the spline and its derivatives can be given by

\[
s_n^{(k)} = p_n^{(k)}(u,P) \ p_n^0(v,P)^{-1} \ b
\]

when \( p_n^0(v,P) \) is invertible and by

\[
s_n^{(k)} = p_n^{(k)}(u,P) \ p_n^0(v,P)^{+} \ b
\]

when \( p_n^0(v,P) \) is not invertible. This last situation corresponds to a least squares problem. These representations of the spline are useful in obtaining convergence results when the data \( b \) comes from a regular function \( f \).

References

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