ON THE $n$-PARAMETER ABSTRACT CAUCHY PROBLEM

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Let $H_i (i = 1, 2, \ldots, n)$, be closed operators in a Banach space $X$. The generalised initial value problem

$$
\begin{aligned}
& \frac{\partial}{\partial t_i} u(t_1, t_2, \ldots, t_i, \ldots, t_n) = H_i u(t_1, \ldots, t_n), \quad t_i \in (0, T_i) \quad i = 1, 2, \ldots, n \\
& u(0) = x, \quad x \in \bigcap_{i=1}^{n} D(H_i),
\end{aligned}
$$

of the abstract Cauchy problem is studied. We show that the uniqueness of solution $u : [0, T_1] \times [0, T_2] \times \cdots \times [0, T_n] \to X$ of this $n$-abstract Cauchy problem is closely related to $C_0$-$n$-parameter semigroups of bounded linear operators on $X$. Also as another application of $C_0$-$n$-parameter semigroups, we prove that many $n$-parameter initial value problems cannot have a unique solution for some initial values.

1. INTRODUCTION

Suppose $X$ is a Banach space and $A$ is a linear operator from $D(A) \subseteq X$ into $X$. Given $x \in X$, the abstract Cauchy problem for $A$ with the initial value $x$, consists of finding a solution $u(t)$ to the initial value problem

$$
\begin{aligned}
& \frac{du(t)}{dt} = Au(t) \quad t \in (0, T] \\
& u(0) = x
\end{aligned}
$$

where by a solution we mean an $X$-valued function $u : [0, T] \to X$ which is continuous for $t \geq 0$, continuously differentiable for $t > 0$, $u(t) \in D(A)$ for $t \in (0, T]$ and (1) is satisfied.

A one-parameter semigroup of operators is a homomorphism $T : (\mathbb{R}_+, +) \to B(X)$ for which $T(0) = I$, where $\mathbb{R}_+ = [0, \infty)$ and $B(X)$ is the Banach space of all bounded linear operators on $X$. The one-parameter semigroup $\{T(t)\}_{t \geq 0}$ is called strongly continuous (or $C_0$-continuous) if $\lim_{t \to 0} T(t)x = x$, for each $x \in X$ and is called uniformly continuous if $\lim_{t \to 0} T(t) = I$ in $B(X)$. The linear mapping $A$ defined by

$$
A(x) = \lim_{t \to 0} \frac{T(t)x - x}{t},
$$

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where \( D(A) = \{ x : \lim_{t \to 0} (T(t)x - x)/t \text{ exist} \} \), is called the infinitesimal generator of \((T, \mathbb{R}_+, X)\).

The following Theorem which is due to Hille [5], shows the close relation of abstract Cauchy problem with semigroup theory (see also [4]).

**Theorem 1.1.** Let \( A \) be a closed linear operator in Banach space \( X \), then the following are equivalent:

(a) For each \( x \in D(A) \) there exists a unique solution for \((1)\).

(b) The part \( A_1 = A|_{X_1} \) of \( A \) in \( X_1 := (D(A), \| \cdot \|_A) \) is the infinitesimal generator of a \( C_0 \)-one-parameter semigroup of operators on the Banach space \( X_1 \), where \( \| \cdot \|_A \) is the graph norm on \( D(A) \).

**Proof:** [2, II.6.6].

The previous theorem has many applications in inhomogeneous initial value problems and evaluation systems. One can see some more applications of abstract Cauchy problem in [3, 7].

Let \( \mathbb{R}_n^+ = \{(t_1, t_2, \ldots, t_n) : t_i \geq 0, i = 1, 2, \ldots, n\} \). By an \( n \)-parameter semigroup of operators we mean a homomorphism \( W : (\mathbb{R}_n^+, +) \to B(X) \) for which \( W(0) = I \) and denote it by \((W, \mathbb{R}_n^+, X)\). Suppose \( H_i \) is the infinitesimal generator of the one-parameter semigroup \( \{W(te_i)\}_{t \geq 0} \) where \( \{e_i\}_{i=1}^n \) is the standard basis of \( \mathbb{R}^n \), we shall think of \((H_1, H_2, \ldots, H_n)\) as the infinitesimal generator of \( W \). As in the one-parameter case, \((W, \mathbb{R}_n^+, X)\) is called strongly continuous (or \( C_0 \)-continuous ) if for each \( x \in X, \lim_{t \to 0} W(t)x = x, \) and is called uniformly continuous if \( \lim_{t \to 0} W(t) = I, \) where \( t \to 0 \) in \( \mathbb{R}_n^+ \). It is not hard to see that \((W, \mathbb{R}_n^+, X)\) is a \( C_0 \)-continuous (respectively uniformly continuous) if and only if for each \( i = 1, 2, \ldots, n \), \( \{W(te_i)\}_{t \geq 0} \) is strongly (respectively uniformly) continuous. The following useful proposition which states some basic properties of \( n \)-parameter semigroups can be found in [1] as is described in [6].

**Proposition 1.2.** Suppose \((W, \mathbb{R}_n^+, X)\) is a \( C_0 \)-\( n \)-parameter semigroup then

(a) If \( x \in D(H_i) \), so does \( W(t)x, \) for each \( t \in \mathbb{R}_n^+ \) and

\[ H_i W(t)x = W(t)H_i x \quad (i = 1, 2, \ldots, n). \]

(b) \( \bigcap_{i=1}^n D(H_i) \) is dense in \( X \), and \( X_1 = \left( \bigcap_{i=1}^n D(H_i), \| \cdot \|_1 \right) \) is a Banach space, where for \( x \in \bigcap_{i=1}^n D(H_i), \|x\|_1 = \|x\| + \sum_{i=1}^n \|H_i(x)\|. \)

(c) For each \( 1 \leq i, j \leq n \), \( D(H_i) \cap D(H_iH_j) \subseteq D(H_jH_i), \) and for \( x \in D(H_i) \cap D(H_iH_j) \),

\[ H_iH_j(x) = H_jH_i(x). \]

In the rest of this note we shall state an extension of one-parameter abstract Cauchy problem and establish its relation with \( C_0 \)-\( n \)-parameter semigroups of operators.
another application of $C_0$-$n$-parameter semigroups we shall show that some $n$-parameter initial valued problems cannot have a unique solution. The abstract Cauchy problem also admits another natural generalisation which is discussed in [5, 6, 8].

2. THE MAIN RESULTS

Suppose as before $X$ is a Banach space, $H_i$ are closed linear operators from $D(H_i) \subseteq X$ into $X$ and $T_i > 0$, ($i = 1, 2, \ldots, n$). Then, a continuous $X$-valued function $u : [0, T_1] \times \cdots \times [0, T_n] \rightarrow X$ with continuous partial derivatives which satisfy the following $n$-parameter abstract Cauchy problem ($n$-abstract Cauchy problem)

$$
\begin{align*}
\frac{\partial}{\partial t_i} u(t_1, t_2, \ldots, t_i, \ldots, t_n) &= H_i u(t_1, \ldots, t_n), \quad i = 1, 2, \ldots, n \quad t_i \in (0, T_i] \\
u(0) &= x, \quad x \in \bigcap_{i=1}^{n} D(H_i),
\end{align*}
$$

is called a solution of the initial value problem (2).

For convenience in the rest of this note we denote by $I_T$ the positive $n$-cell $[0, T_1] \times [0, T_2] \times \cdots \times [0, T_n]$ where $T = (T_1, T_2, \ldots, T_n) \in \mathbb{R}^n_+$ and $T_i > 0$. As mentioned in the previous section, we shall illustrate that (2) is closely related to $C_0$-$n$-parameter semigroups of operators. In the following theorem we prove that if $I_T$ is arbitrary and $(H_1, H_2, \ldots, H_n)$ is the infinitesimal generator of a $C_0$-$n$-parameter semigroup $(W, \mathbb{R}^n_+, X)$, then (2) has the unique solution $u(t_1, t_2, \ldots, t_n) = W(t_1, t_2, \ldots, t_n)x$, for each $x \in \bigcap_{i=1}^{n} D(H_i)$, where $(t_1, t_2, \ldots, t_n) \in I_T$.

**THEOREM 2.1.** Suppose $I_T$ is a positive $n$-cell corresponding to $T \in \mathbb{R}^n_+$, and $(H_1, H_2, \ldots, H_n)$ is the infinitesimal generator of the $C_0$-$n$-parameter semigroup $(W, \mathbb{R}^n_+, X)$ of operators, then for each $x \in \bigcap_{i=1}^{n} D(H_i)$ the $n$-abstract Cauchy problem (2) has a unique solution.

**PROOF:** Let $I_T$ be arbitrary, $\{e_i\}_{i=1}^{n}$ be the standard basis of $\mathbb{R}^n$ and $H_i$ be the infinitesimal generator of the $C_0$-$n$-parameter semigroup $\{W(t_i)\}_{i \geq 0}$. For $x \in \bigcap_{i=1}^{n} D(H_i)$, define $u : I_T \rightarrow X$ by $u(t) = W(t)x$. One can easily see that $u(t)$ is a solution of $n$-abstract Cauchy problem (2) for the initial value $x \in \bigcap_{i=1}^{n} D(H_i)$. For proving the uniqueness of solution it is enough to show that (2) has no proper (that is, nonzero) solution for the initial value $x = 0$. Theorem 1.1 shows that for each $i = 1, 2, \ldots, n$, the initial value problem

$$
\begin{align*}
\frac{du^i(s)}{ds} &= H_i u^i(s), \quad s \in (0, T_i] \\
u^i(0) &= x, \quad x \in D(H_i)
\end{align*}
$$

is unique.
has a unique solution for each \( x \in D(H_i) \). By definition of solution we know that for \( t \in I_T \), \( u(t) \) which is a solution of (2) for \( x = 0 \), is in \( \bigcap_{i=1}^n D(H_i) \), so for the initial value \( x = u(t_1, t_2, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n) \in D(H_i) \), \( u^i(s) = u(t_1, \ldots, t_{i-1}, s, t_{i+1}, \ldots, t_n) \) and \( v^i(s) = W(se_i)u(t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n) \) is a solution of (3) for \( x \). Uniqueness of solution of (3) implies that

\[
W(se_i)u(t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n) = v^i(s)
\]

(4)

\[
u^i(s) = u(t_1, \ldots, t_{i-1}, s, t_{i+1}, \ldots, t_n),
\]

for each \( i = 1, 2, \ldots, n \), \( 0 \leq s \leq T_i \) and \( 0 \leq t_j \leq T_j \), \( i \neq j = 1, 2, \ldots, n \). Using (4) for \( t = \sum_{i=1}^n t_ie_i \in I_T \), shows that

\[
u(t) = u(t_1, t_2, \ldots, t_n) = W(t_1e_1)u(0, t_2, \ldots, t_n) \quad (i = 1, s = t_1)
\]

\[
= W(t_1e_1)(W(t_2e_2)u(0, 0, t_3, \ldots, t_n) \quad (i = 2, s = t_2)
\]

\[
(W \text{ in } n \text{-parameter}) = W\left(\sum_{i=1}^n t_ie_i\right)u(0, 0, \ldots, 0) = W(t)(0) = 0
\]

Hence \( u(t) = 0 \) and (2) cannot have a proper solution for \( x = 0 \), or equivalently (2) has a unique solution for each \( x \in \bigcap_{i=1}^n D(H_i) \).

Now let \( H_i \)’s \( (i = 1, 2, \ldots, n) \) from \( D(H_i) \subseteq X \) into \( X \) be closed operators. Similarly to Proposition 1.2 (b) one can see \( X_1 = \left( \bigcap_{i=1}^n D(H_i), \| \cdot \|_i \right) \), where \( \| x \|_i = \| x \| + \sum_{i=1}^n \| H_i(x) \| \), \( (x \in \bigcap_{i=1}^n D(H_i)) \) is a Banach space. In the next theorem we are going to show that for positive \( n \)-cells \( I_T \) and \( I_{T'} \), where \( I_T \subseteq I_{T'} \), if (2) has a unique solution for each \( x \in X_1 \) then there exist a \( C_0 \)-\( n \)-parameter semigroup \((W, \mathbb{R}_+^n, X_1)\) with the infinitesimal generator \((K_1, K_2, \ldots, K_n)\) for which \( W(t)x = u(t; x) \), the unique solution of (2) for \( x \in X_1 \) and \( t \in I_T \), also for \( x \in D(K_i) \), \( K_i(x) = H_i(x) \).

**Theorem 2.2.** Suppose \( H_i \)’s \( (i = 1, 2, \ldots, n) \) are closed linear operators and for positive \( n \)-cells \( I_T \) and \( I_{T'} \), where \( I_T \subseteq I_{T'} \), the \( n \)-abstract Cauchy problem (2) has a unique solution for each \( x \in X_1 \), then there exist a \( C_0 \)-\( n \)-parameter semigroup \((W, \mathbb{R}_+^n, X_1)\) of linear bounded operators with the infinitesimal generator \((K_1, K_2, \ldots, K_n)\) such that for \( t \in I_T \) and \( x \in X_1 \), \( W(t)x = u(t; x) \) where \( u(t; x) \) is the unique solution of (2) for the initial value \( x \), and for \( x \in D(K_i) \), \( K_i(x) = H_i(x) \).

**Proof:** Let \( u(t; x) \) be the unique solution of (2) for \( x \in X_1 \). For \( t \in I_T \), we define the operator \( W_1(t) : X_1 \to X_1 \) by \( W_1(t)x = u(t; x) \). Trivially \( W_1(t) \) is well-defined and a linear operator, since the solution is unique. We are going to show that \( W_1(t) \) is bounded. Define the mapping \( \Phi : X_1 \to C^1(I_T, X_1) \) by \( \Phi(x)(t) = W_1(t)(x) \), where \( C^1(I_T, X_1) \) is the Banach space of all continuous \( X_1 \)-valued functions on \( I_T \) with continuous partial
derivative, equipped with the supremum norm. $\Phi$ is linear, we prove it is closed. Suppose $x_m \to x$ in $X_1$ and $\Phi(x_m) \to f$ in $C^1(I_T, X_1)$, integrating of (2) implies that for each $i = 1, 2, \ldots, n$, $m \in \mathbb{N}$ and $t = (t_1, \ldots, t_n) \in I_T$,

(5) $W_1(t_1, \ldots, t_n) x_m = W_1(t_1, t_2, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n) x_m$

$+ \int_0^{t_i} H_i W_1(t_1, t_2, \ldots, t_{i-1}, s, t_{i+1}, \ldots, t_n) x_m \, ds.$

Let $m \to \infty$, so $\sup_{t \in I_T} \|\Phi(x_m)(t) - f(t)\|_1 \to 0$, this, (5), together with the closedness of $H_i$, imply that for each $i = 1, 2, \ldots, n$,

(6) $f(t_1, \ldots, t_n) = f(t_1, t_2, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n)$

$+ \int_0^{t_i} H_i f(t_1, t_2, \ldots, t_{i-1}, s, t_{i+1}, \ldots, t_n) \, ds.$

Thus (6) and the fact that $f \in C^1(I_T, X_1)$ show that

$$\begin{cases}
\frac{\partial}{\partial t_i} f(t_1, t_2, \ldots, t_i, \ldots, t_n) = H_i f(t_1, \ldots, t_n), & i = 1, 2, \ldots, n \quad t_i \in (0, T_i] \\
f(0) = \lim_{m \to \infty} \Phi(x_m)(0) = \lim_{m \to \infty} W_1(0)(x_m) = x.
\end{cases}$$

Hence $f$ is a solution of (2) for the initial value $x$, the uniqueness of solution gives

$f(t) = W_1(t)x = \Phi(x)t, \quad t \in I_T,$

it means that $\Phi$ is closed operator from the Banach space $X_1$ into the Banach space $C^1(I_T, X_1)$ and the closed graph theorem tell us

$$\sup_{\|x\|_1 \leq 1} \|W_1(\cdot)x\|_\infty = \sup_{\|x\|_1 \leq 1} \left( \sup_{t \in I_T} \|W_1(t)x\|_1 \right) = M < \infty.$$
where \( r = (r_1, r_2, \ldots, r_n) \in I_{T'} \). By the previous parts of proof the operators in the right hand side of the last equality commute and are bounded linear operators on \( X_1 \). One can easily see that \((W, \mathbb{R}_n^+, X_1)\) is an \( n \)-parameter semigroup of operators and the fact that \( \lim_{t \to 0} W(t)x = x \) (by continuity of \( u(t; x) \)) show that \( W \) is strongly continuous. Also for \( s \in I_T \), \( W(s)x = W(s)x \), since

\[
\frac{\partial}{\partial s_i} W(s)x = \frac{\partial}{\partial r_i} W_1(r) \left[ \Pi_{i=1}^n (W_1(T'_i e_i))^{m_i} \right] (x) = H_i W_1(r) \left[ \Pi_{i=1}^n (W_1(T'_i e_i))^{m_i} \right] (x) = H_i W(s)x
\]

and the equality holds from the uniqueness of solution in \( I_T \). If \((K_1, K_2, \ldots, K_n)\) is the generator of \( W \) and \( x \in D(K_i) \subseteq X_1 = \bigcap_{i=1}^n D(H_i) \), then

\[
\|W(te_t)x - x\|_1 \to 0 \quad \text{as} \quad t \to 0
\]

which implies \( \lim_{t \to 0} (W(te_t)x - x)/t = K_i(x) \), but \( x \in D(H_i) \) and so

\[
\lim_{t \to 0} \frac{W(te_t)x - x}{t} = \frac{\partial}{\partial t_i} W(0,0,\ldots,0)x = H_i W(0)x = H_i(x)
\]

Thus \( K_i(x) = H_i(x) \) and this complete the proof of theorem.

In the previous Theorem we could replace the assumption of existence of a unique solution for (2) in \( I_T \) and \( I_{T'} \), by the assumption that (2) has a unique solution in \( I_T \) and whole of \( \mathbb{R}_n^+ \), which seems stronger than our hypothesis. As another application of \( C_0-n \)-parameter semigroups, we shall show that for a closed linear operator \( A : D(A) \subseteq X \to X \), the \( n \)-parameter initial value problem

\[
\begin{cases}
  \sum_{i=1}^n \frac{\partial}{\partial t_i} u(t_1, t_2, \ldots, t_n) = Au(t_1, t_2, \ldots, t_n), & t = (t_1, t_2, \ldots, t_n) \in I_T \\
u(0) = x, & x \in D(A)
\end{cases}
\]

does not have a unique solution in both \( I_T \) and \( I_{T'} \) for each \( x \in D(A) \), for which \( I_{T'} \subseteq I_T \).

The initial value problem (7) can have a solution, for example if \((H_1, H_2, \ldots, H_n)\) is generator of a \( C_0-n \)-parameter semigroup \((W, \mathbb{R}_n^+, X)\) and \( A = H_1 + H_2 + \cdots + H_n \), then obviously \( u(t) = W(t)x \) is a solution of (7) in any positive \( n \)-cell \( I_T \), for the initial value \( x \in \bigcap_{i=1}^n D(H_i) \subseteq D(A) \).

Before proving our claim we need the following lemmas.

**Lemma 2.3.** Suppose \( \{T(t)\}_{t \geq 0} \) is a \( C_0 \)-one parameter semigroup of operators with the infinitesimal generator \( A \), and \( B \in B(X) \), then \( A + B \) is the infinitesimal generator of a \( C_0 \)-semigroup \( S(t) \) on \( X \) satisfying

\[
S(t)x = T(t)x + \int_0^t T(t-s)BS(s)x \, ds, \quad x \in X.
\]
![Image](https://www.cambridge.org/core/terms). https://doi.org/10.1017/S0004972700036169

PROOF: See [7, III.1.1 and III.1.2].

Also the next lemma which provide a necessary and sufficient condition for the composition of \( C_0 \)-one parameter semigroups to be a \( C_0 \)-\( n \)-parameter semigroup, has a principal role in the next theorem.

Recall that for linear operator \( H \) in Banach space \( X \), \( \rho(H) \) denotes the resorvent set of \( H \) and for \( \lambda \in \rho(H) \), \( R(\lambda; H) \) is used for \((\lambda I - H)^{-1}\).

**Lemma 2.4.** Suppose \( \{U_i(t)\}_{s \geq 0} \) is a \( C_0 \)-one-parameter semigroup of operators on Banach space \( X \) with the infinitesimal generator \( H_i \), \( (i = 1, 2, \ldots, n) \), then \( W(t_1, t_2, \ldots, t_n) = U_1^1(t_1)U_2^2(t_2) \ldots U_n^n(t_n) \) is a \( C_0 \)-\( n \)-parameter semigroup of operators if and only if there is a \( \omega > 0 \) such that for each \( i = 1, 2, \ldots, n \), \( [\omega, \infty) \subseteq \rho(H_i) \) and for each integers \( 0 \leq i, j \leq n \) and \( \lambda, \lambda' \geq \omega \), we have

\[
R(\lambda'; H_j)R(\lambda'; H_i) = R(\lambda; H_i)R(\lambda'; H_j).
\]

**Proof:** First suppose \( W \) is a \( C_0 \)-\( n \)-parameter semigroup of operators. Since \( H_i \) is the infinitesimal generator of \( \{u_i(t)\}_{t \geq 0} \), by the Hille-Yosida Theorem ([7, I.5.3]), there is an \( \omega_i > 0 \) such that for each \( \lambda \geq \omega_i \), \( R(\lambda; H_i) \) exist and are bounded operators. Let \( \omega = \max\{\omega_i : i = 1, 2, \ldots, n\} \). If \( \lambda \geq \omega \), from [7, I.5.4]

\[
R(\lambda; H_i)(x) = \int_0^\infty e^{-\lambda t}U_i^i(s)(x) \, ds.
\]

Also we know that for each integers \( 0 \leq i, j \leq n \),

\[
U_i^i(s)U_j^j(t) = W(s_e_i)W(t_e_j) = W(t_e_j)W(s_e_i) = U_j^j(t)U_i^i(s),
\]

so

\[
R(\lambda; H_i)(U_j^j(t)x) = \int_0^\infty e^{-\lambda t}U_j^j(t)U_i^i(s)x \, ds
\]

\[
= \int_0^\infty e^{-\lambda t}U_j^j(t)U_i^i(s)x \, ds = U_j^j(t)\int_0^\infty e^{-\lambda t}U_i^i(s)x \, ds
\]

\[
= u_j^j(t)R(\lambda; H_i)x.
\]

Now let \( \lambda' \geq \omega \), we know \( R(\lambda; H_i) \) is bounded so

\[
R(\lambda; H_i)R(\lambda'; H_j)x = R(\lambda; H_i)\int_0^\infty e^{-\lambda' t}U_j^j(t)x \, dt
\]

\[
= \int_0^\infty e^{-\lambda' t}U_j^j(t)R(\lambda; H_i)x \, dt
\]

\[
= R(\lambda'; H_j)R(\lambda; H_i)x
\]

and this prove the necessary part of lemma.

For the converse suppose there is an \( \omega > 0 \) such that for each \( \lambda, \lambda' > 0 \), \( R(\lambda; H_i) \) and \( R(\lambda'; H_j) \) exist and commute. So we have \( H_i^j H_j^i = H_j^i H_i^j \) where \( H_i^j = \lambda^2 R(\lambda; H_i) - \lambda I \)
and \( H_i^\lambda = \lambda^2 R(\lambda; H_i) - \lambda I \) are the Yosida approximation of \( H_i \) and \( H_j \) respectively. Applying [7, 1.3.5] we have \( U_j(t) x = \lim_{\lambda' \to \infty} e^{t H_i^\lambda} x \) and \( U_j(t) x = \lim_{\lambda' \to \infty} e^{t H_i^\lambda} x \), thus

\[
U_i(s) U_j(t) x = \lim_{\lambda, \lambda' \to \infty} e^{s H_i^\lambda} e^{t H_j^\lambda} x = \lim_{\lambda, \lambda' \to \infty} e^{s H_i^\lambda} e^{t H_j^\lambda} x = \lim_{\lambda, \lambda' \to \infty} U_j(t) e^{s H_i^\lambda} x = U_j(t) U_i(s) x
\]

Hence \( W(t_1, t_2, \ldots, t_n) = U_1(t_1) U_2(t_2) \ldots U_n(t_n) \) is a \( C_0 \)-\( n \)-parameter semigroup of operators.}

Now we are ready for this theorem.

**THEOREM 2.5.** Suppose \( A \) is a closed operator from \( D(A) \subseteq X \) into \( X \) and \( f \) and \( f' \) is given. Then the initial value problem (7) cannot have a unique solution for each \( x \in D(A) \) in both \( I_T \) and \( I_{T'} \).

**PROOF:** Suppose to the contrary (7) has a unique solution for each \( x \in D(A) \) in both \( I_T \) and \( I_{T'} \). As in Theorem 2.2 we are going to show that if \( u(t; x) \) is the unique solution of (7) for \( x \in D(A) \) and \( t \in I_T \), then \( W_1(t) x = u(t; x) \) can be extended to a \( C_0 \)-\( n \)-parameter semigroup of operators, and using previous lemma we shall get a contradiction.

Obviously uniqueness of solution shows that \( W_1(t) x = u(t; x) \) is a well-defined linear operator on Banach space \( X_1 = (D(A), \| \cdot \|_A) \) where \( \| \cdot \|_A \) is the graph norm on \( X_1 \). Before proving the boundedness of \( W_1(t) \) we notice that \( Y = (C^1(I_T, X_1), \| \cdot \|') \), where

\[
\| f \|' = \| f \|_\infty + \sum_{i=1}^n \left\| \frac{\partial}{\partial t_i} f \right\|_\infty
\]

is a Banach space. Next we show that the mapping \( \Phi : X_1 \to Y \) defined by \( \Phi(x)(t) = W_1(t) \) is closed, for; suppose \( x_m \to x \) in \( X_1 \) and \( \Phi(x_m) \to f \) in \( Y \). Integrating of (7) for initial value \( x_m \), we have

\[
W_1(t_1, t_2, \ldots, t_n) x_m = W_1(0, t_2, \ldots, t_n) x_m - \sum_{i=2}^n \int_0^{t_1} \frac{\partial}{\partial t_i} W_1(s, t_2, \ldots, t_n) x_m ds + \int_0^{t_1} A W_1(s, t_2, \ldots, t_n) x_m ds.
\]

As \( m \to \infty \) by our choosing of the norm and the closeness of \( A \) we get

\[
\left\| \frac{\partial}{\partial t_i} W_1(\cdot) x_m - \frac{\partial}{\partial t_i} f(\cdot) \right\|_\infty \to 0, \quad \text{as} \quad m \to \infty, \quad i = 1, 2, \ldots, n
\]
and

\[
\|W_1(\cdot)x_m - f(\cdot)\|_{\infty} = \sup_{t \in I_T} \left( \|W_1(t)x_m - f(t)\|_A \right)
\]

\[
= \sup_{t \in I_T} \left( \|W_1(t)x_m - f(t)\| + \|AW_1(t)x_m - Af(t)\| \right) \to 0
\]
as \(m \to \infty\). Hence

\[
f(t_1, t_2, \ldots, t_n) = f(0, t_2, \ldots, t_n) - \sum_{i=2}^{n} \int_{0}^{t_1} \frac{\partial}{\partial t_i} f(s, t_2, \ldots, t_n) \, ds
\]

\[
+ \int_{0}^{t_1} Af(s, t_2, \ldots, t_n) \, ds.
\]
It gives

\[
\begin{align*}
\frac{n}{\sum_{i=1}^{n} \frac{\partial}{\partial t_i} f(t_1, t_2, \ldots, t_n) = Au(t_1, t_2, \ldots, t_n) \\
\end{align*}
\]

\[
f(0) = \lim_{m \to \infty} W_1(0)x_m = x.
\]
So \(f\) is a solution of (7) and by the uniqueness of solution we conclude \(f(t) = W_1(t)x\), equivalently \(f\) is closed and by closed graph theorem \(\Phi\) is bounded, thus \(\sup_{t \in I_T} \|W_1(t)\| < \infty\).

As in Theorem 2.2 \(W_1(t)\) can be extended to a \(C_0\)-\(n\)-parameter semigroup \((W, \mathbb{R}_+, X_1)\). Let \((H_1, H_2, \ldots, H_n)\) be the infinitesimal generator of \(W\), for \(x \in \bigcap_{i=1}^{n} D(H_i) \subseteq D(A)\) we have

\[
\frac{\partial}{\partial t_i} W(t)x = H_iW(t)x.
\]
Thus

\[
\sum_{i=1}^{n} \frac{\partial}{\partial t_i} W(t)x = \left( \sum_{i=1}^{n} H_i \right) W(t)x = AW(t)x.
\]
From the continuity of \(\frac{\partial}{\partial t_i} W(t)x\) and strong continuity of \(W(t)x\), the fact that \(\sum_{i=1}^{n} H_iW(t)x = W(t) \sum_{i=1}^{n} H_i x\) (Proposition 1.2), and the closedness of \(A\) as \(t \to 0\), the last equality yields

\[
\sum_{i=1}^{n} H_i(x) = A(x), \text{ for each } x \in \bigcap_{i=1}^{n} D(H_i).
\]
Applying Lemma 2.4 shows that there is \(\omega > 0\) such that for each \(\lambda, \lambda' \geq \omega\), we have

\[
R(\lambda'; H_j)R(\lambda; H_i) = R(\lambda; H_i)R(\lambda'; H_j).
\]
Now let $H'_1 = H_1 + I$ and $H'_2 = H_2 - I$, if $\omega' = \omega + 1$ and $\lambda, \lambda' \geq \omega'$, we have $\lambda + 1, \lambda' - 1 \geq \omega$ and

$$R(\lambda'; H'_1)R(\lambda'; H'_2) = R(\lambda' - 1; H'_3)R(\lambda + 1; H_2)$$
$$= R(\lambda + 1; H_2)R(\lambda' - 1; H'_1)$$
$$= R(\lambda; H'_2)R(\lambda'; H'_1).$$

Similarly $R(\lambda; H'_i)R(\lambda'; H'_j) = R(\lambda'; H'_j)R(\lambda; H'_i)$, for $\lambda, \lambda' \geq \omega'$, $i = 1, 2$, and $j = 3, 4, \ldots, n$. By Lemma 2.3 $H'_1$ and $H'_2$ are the infinitesimal generators of two $C_0$-one-parameter semigroups of operators. With the above equalities and Lemma 2.4, this shows that $(H'_1, H'_2, H_3, \ldots, H_n)$ is the infinitesimal generator of a $C_0$-$n$-parameter semigroup, say $(W', \mathbb{R}^n_+, X_1)$. So by Lemma 2.3, for each $x \in X_1$,

$$W'(te_1)x = W(te_1)x + \int_0^t W((t - \mu)e_1)W'(\mu e_1)x d\mu,$$

and

$$W'(te_2)x = W(te_2)x - \int_0^t W((t - \nu)e_2)W'(\nu e_2)x d\nu.$$

Also $W'(te_i) = W(te_i)$, for $i > 2$. We conclude that for $x \in \bigcap_{i=1}^n D(H_i)$,

$$\frac{\partial}{\partial t_i} W'(t_1, t_2, \ldots, t_n)x = \begin{cases} H'_i W'(t_1, t_2, \ldots, t_n) & i = 1, 2 \\ H_i W'(t_1, t_2, \ldots, t_n) & i > 2. \end{cases}$$

Hence by (8)

$$\begin{cases} \frac{\partial}{\partial t_i} W'(t) = (H'_1 + H'_2 + H_3 + \cdots + H_n)W'(t) = \sum_{i=1}^n H_i W'(t) = AW'(t)x \\ W'(0) = x. \end{cases}$$

But the solution of (7) is unique, and so for $i = 1, \ldots, n$ and $0 \leq t \leq T_i$,

$$W'(te_i) = W(te_i).$$

This implies that

$$W(te_1)x = W'(te_1)x = W(te_1)x + \int_0^t W((t - \mu)e_1)W'(\mu e_1)x d\mu$$
$$= W(te_1)x + \int_0^t W(te_1)x d\mu$$
$$= W(te_1)x + tW(te_1)x.$$

So $tW(te_1)x = 0$ or $W(te_1)x = 0$. This is a contradiction, because $0 = \lim_{t \to 0} W(te_1)x = x \neq 0$. Thus (7) cannot have a unique solution for each $x \in D(A)$. \[\Box\]
Remark 2.6. Our technique for proving Theorem 2.2 and a part of Theorem 2.5 is based on Hille’s technique for one-parameter case [5]. \( C_0 \)-\( n \)-parameter semigroups are solutions of many initial value problems contain partial derivative and as in previous Theorem, \( C_0 \)-\( n \)-parameter semigroups can be used for showing that these initial value problems cannot have a unique solution. As another example for second order initial value problems, consider the two-parameter initial value problem

\[
\begin{align*}
\frac{\partial}{\partial s} \frac{\partial}{\partial t} u(s,t) &= A u(s,t) \\
(s,t) &\in [0,S] \times [0,T] \\
u(0,0) &= x, \quad x \in D(A)
\end{align*}
\]

where \( A \) is a closed operator. If this problem has a unique solution for each \( x \in D(A) \) in both \( I(s,T) \) and \( I(s',T') \) for which \( I(s,T) \subseteq I(s',T') \), then \( W_i(s,t) = u(s,t;x) \) can be extended to a \( C_0 \)-two-parameter semigroup on Banach space \( X_1 = (D(A), \| \cdot \|_A) \), with the infinitesimal generator \( (H, K) \). We know \( D(HK) \cap D(KH) = D(A) \), (it can be proved completely similarly to the proof of Proposition 1.2 (b)), so \( D(HK) \cap D(KH) \neq 0 \). Now for \( x \in D(HK) \cap D(KH) \), by Proposition 1.2 \( HK(x) = KH(x) \) and one can see that this is equal to \( A(x) \). Also it can be checked that \( (H/2, 2K) \) is the generator of \( W'(s,t) = W(s/2,2t) \neq W(s,t) \). So for \( x \in D(HK) \cap D(KH) \),

\[
\begin{align*}
\frac{\partial}{\partial s} \frac{\partial}{\partial t} W'(s,t) &= \left( \frac{1}{2}H \right)(2K)W'(s,t)x = HKW(s/2,t)x = A W'(s,t) \\
W'(0,0)x &= x
\end{align*}
\]

and this is a contradiction with the uniqueness of solution.

References


