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ON THE MODULI OF CONTINUITY OF H^p FUNCTIONS WITH $0<p<1$

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We prove two inequalities which relate the L^p modulus of continuity of n-th order, $\omega_n(f,\cdot)_p$, of an H^p function f with the p-th mean values of the n-th derivative $f^{(n)}$. Using these inequalities we extend classical results of Hardy and Littlewood [**5**], Gwiliam [**4**], Zygmund [**13**] and Taibleson [**12**] as well as a recent result of Oswald [**6**].

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1. Introduction

In this paper we consider connections between the L^p modulus of continuity of an H^p function and the p-th mean value of the corresponding derivative. We shall mainly be concerned with the case where $0<p<1$. As a consequence of our main results we have the following two inequalities, valid for $f\in H^p$, $0<p<1$ (for the case $p\geqq 1$ see [**8**]):

$$M_p(D^n f,r)\leqq C(1-r)^{-n}\omega_n(f,1-r)_p,\quad 0<r<1, \tag{1.1}$$

$$\omega_n(f,t)_p^p\leqq C\int_{1-t}^{1}(1-r)^{np-1}M_p^p(D^n f,r)\,dr,\quad 0<t<1, \tag{1.2}$$

where

$$(D^n f)(re^{i\theta})=\frac{\partial^n f}{\partial\theta^n}(re^{i\theta})\quad (n=1,2,\ldots).$$

(Here C denotes a positive real constant depending only on p and n.)

In the case $n=1$ the inequality (1.1) was provied by Storoženko [**11**].

In Section 2 we state the main results and apply them to deduce a generalization of a result of Gwiliam [**4**]. The proofs are in Sections 3 and 4. Section 5 contains a simple proof of a result, due to Taibleson [**12**] and Oswald [**6**], from the theory of Lipschitz spaces.

Notation. Throughout the paper n denotes a fixed positive integer and p a positive real number. For a complex-valued function h, defined on the real line, let $\Delta_t^n h$ denote the n-th difference with step t:

$$(\Delta_t^1 h)(\theta)=\Delta_t^1 h(\theta)=h(\theta+t)-h(\theta),$$

$$\Delta_t^n h=\Delta_t^1\Delta_t^{n-1}h \quad (n\geqq 2).$$

If g is a function defined on the unit circle T, then $\Delta_t^n g$ is defined by

$$\Delta_t^n g(e^{i\theta})=\Delta_t^n h(\theta), \quad h(\theta)=g(e^{i\theta}).$$

For a fixed t, Δ_t^n is a linear operator which preserves the classical spaces such as L^p, $0<p<\infty$. The L^p modulus of continuity of order n is defined by

$$\omega_n(g,t)_p=\sup\{\|\Delta_s^n g\|_p: |s|\leqq t\},\ t>0,\ g\in L^p(T),$$

where $\|\cdot\|_p$ stands for the "norm" in $L^p(T)$:

$$\|g\|_p=\left\{\frac{1}{2\pi}\int_0^{2\pi}|g(e^{i\theta})|^p\,d\theta\right\}^{1/p}.$$

The p-th mean values of a continuous function f, defined in the unit disc Δ, are defined by

$$M_p(f,r)=\|f_r\|_p, \quad 0<r<1,$$

where

$$f_r(w)=f(rw), \quad w\in T.$$

The Hardy space H^p consists of all f analytic in Δ for which

$$\|f\|_p:=\sup\{M_p(f,r): 0<r<1\}<\infty.$$

It is well-known that each $f\in H^p$ has the radial limits almost everywhere on T and that the L^p norm of the boundary function equals the H^p norm of f. This fact enables us to treat H^p as a subspace of $L^p(T)$. See [1].

2. Main results

Theorem 2.1. *Let $f\in H^p$, $0<p<\infty$, $0<q<\infty$ and let ψ be a non-negative function such that $\psi\in L^1(0,1)$ and*

$$\psi(2x)\leqq K\psi(x), \quad 0<x<1/2, \tag{2.1}$$

where K is a positive constant. Then

$$\int_0^1 M_p^q(D^n f, r)\psi(1-r)\,dr \leqq C\int_0^1 [t^{-n}\|\Delta_t^n f\|_p]^q \psi(t)\,dt, \tag{2.2}$$

where C depends only on p, q, n and K.

Corollary 1. *If* $\alpha > -1$, *then*

$$\int_r^1 M_p^q(D^n f, \rho)(1-\rho)^\alpha\,d\rho \leqq C \int_0^{1-r} [t^{-n}\|\Delta_t^n f\|_p]^q t^\alpha\,dt \tag{2.3}$$

$(0<r<1)$, *where C is independent of r and f.*

Proof. For a fixed $r, 0<r<1$, we consider the function

$$\psi(x) = \begin{cases} x^\alpha, & 0<x<1-r, \\ 0, & 1-r<x<1. \end{cases}$$

Then ψ satisfies (2.1) with $K=2^\alpha$, and K is independent of r. Now (2.3) follows from (2.2).

Corollary 2. *If* $\alpha > -1$, *then*

$$M_p(D^n f, r) \leqq C\left\{(1-r)^{-\alpha-1} \int_0^{1-r} [t^{-n}\|\Delta_t^n f\|_p]^q t^\alpha\,dt\right\}^{1/q}, \tag{2.4}$$

where C is independent of r and f.

Proof. By the increasing property of $M_p(D^n f, \rho)$,

$$M_p^q(D^n f, r)\int_r^1 (1-\rho)^\alpha\,d\rho \leqq \int_r^1 M_p^q(D^n f, \rho)(1-\rho)^\alpha\,d\rho,$$

which, together with (2.3), gives (2.4).

As a special case we have

$$M_p(D^n f, r) \leqq C(1-r)^{-n-1} \int_0^{1-r} \|\Delta_t^n f\|_p\,dt,$$

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which implies (1.1).

Our second result is the following.

Theorem 2.2. *If f is analytic in $\Delta, 0<p<1$ and*

$$\int_0^1 (1-r)^{np-1} M_p^p(D^n f, r)\, dr < \infty, \tag{2.5}$$

then f belongs to H^p and satisfies (1.2).

Remark. That the condition (2.5) (which is independent of n) implies that $f \in H^p$ was proved by Flett [**2**]. For more information see [**3**].

As an application of our main results we have an extension of Theorem 3 of [**8**].

Theorem 2.3. *Let ψ be a positive non-decreasing function on the interval $[1, \infty)$ such that for some $\beta < n$ the function $\psi(x)/x^\beta$, $x \geqq 1$, is decreasing. If $0<p<1$ and f is analytic in Δ, then the following are equivalent:*

(a) *f is in H^p and $\omega_n(f,t)_p = 0(t^n \psi(1/t))$, $t \to 0$;*

(b) *$M_p(D^n f, r) = 0(\psi(1/(1-r)))$, $r \to 1^-$.*

Proof. The implication (a)$\Rightarrow$(b) is a direct consequence of the inequality (1.1) and is independent of the hypothesis on ψ.

Assuming (b) we have

$$\int_{1-t}^1 (1-r)^{np-1} M_p^p(D^n f, r)\, dr \leqq \int_{1/t}^\infty x^{-np-1} \psi(x)^p\, dx, \quad 0<t \leqq 1.$$

Using the inequality $\psi(x)x^{-\beta} \leqq \psi(1/t)(1/t)^{-\beta}$, $x > 1/t$, we see that the last integral is dominated by

$$\psi(1/t)^p t^{\beta p} \int_{1/t}^\infty x^{(\beta-n)p-1}\, dx = t^{np} \psi(1/t)^p/(n-\beta)p.$$

Now the assertion (a) follows from Theorem 2.2.

By taking $\psi(x) = x^{n-\alpha}$, where $0<\alpha \leqq n$, we get the following result of Gwiliam [**4**] and Oswald [**6**]. (In fact, Gwiliam considered the case $n=1$, and Oswald considered the case $\alpha < n$.)

Corollary. *Let $0<p<1$ and $0<\alpha \leqq n$. An analytic function f is in H^p with $\omega_n(f,t)_p = 0(t^\alpha)$ if and only if $M_p(D^n f, r) = 0((1-r)^{\alpha - n})$.*

Remarks. 1. It is easily seen that Theorem 2.3 remains true if we replace "0" by "o".

2. It is well known that the corollary holds for $p \geqq 1$. If $n=1$, this is the famous

theorem of Hardy and Littlewood [5]. And if $n=2$ and $\alpha=1$, this result is due to Zygmund [13] and is the first result concerning the moduli of continuity of higher orders.

3. Proof of Theorem 2.1

Throughout this and the following section we shall use the notation

$$f_r(\theta)=f(re^{i\theta}).$$

Lemma 3.1. *Let f be analytic in Δ, $0<p<\infty$, $0<r<\rho<1$, $\lambda>0$ and*

$$F(\theta)=\sup\{|f_r(\theta+y)|: 0\leqq y\leqq\lambda(\rho-r)\},\quad 0\leqq\theta\leqq 2\pi.$$

Then

$$\|F\|_p\leqq CM_p(f,\rho),$$

where C depends only on λ and p.

Proof. By the subharmonicity of $|f|^p$,

$$|f_r(\theta+y)|^p\leqq\frac{1}{2\pi}\int_0^{2\pi}|f_\rho(t)|^p\,\frac{\rho^2-r^2}{|\rho e^{it}-re^{i\theta+iy}|^2}\,dt. \tag{3.1}$$

If $0\leqq y\leqq\lambda(\rho-r)$, then

$$|\rho e^{it}-re^{i\theta}|\leqq|\rho e^{it}-re^{i(\theta+y)}|+|re^{i(\theta+y)}-re^{i\theta}|.$$

Since

$$\begin{aligned}|re^{i(\theta+y)}-re^{i\theta}|&=r|e^{iy}-1|\\&\leqq ry\leqq\lambda(\rho-r)\\&\leqq\lambda|\rho e^{it}-re^{i(\theta+y)}|,\end{aligned}$$

we get

$$|\rho e^{it}-re^{i\theta}|\leqq(1+\lambda)|\rho e^{it}-re^{i(\theta+y)}|.$$

From this and (3.1) it follows that

$$F(\theta)^p \leqq (1+\lambda)^2 \frac{1}{2\pi} \int_0^{2\pi} |f_\rho(t)|^p \frac{\rho^2 - r^2}{|\rho e^{it} - re^{i\theta}|^2} \, dt.$$

Now integration yields the desired inequality with $C = (1+\lambda)^{2/p}$.

Proof of Theorem 2.1. Without loss of generality we can assume that f is analytic in $|z| < R$, for some $R > 1$. For a fixed r, $0 < r < 1$, let $h(\theta) = f_r(\theta)$. By induction,

$$(\Delta_t^n h)(\theta) = \int_{tE} h^{(n)}(\theta + x_1 + \cdots + x_n) \, dx_1 \ldots dx_n, \tag{3.2}$$

where tE is the n-dimensional cube $[0, t]^n$. Hence

$$\begin{aligned}(D^n f)(re^{i\theta}) t^n &= h^{(n)}(\theta) t^n \\ &= (\Delta_t^n h)(\theta) - \int_{tE} (h^{(n)}(\theta + x_1 + \cdots + x_n) - h^{(n)}(\theta)) \, dx_1 \ldots dx_n.\end{aligned}$$

This implies that

$$\begin{aligned}|(D^n f)(re^{i\theta})| t^n &\leqq |\Delta_t^n f_r(\theta)| + \int_{tE} \sup_{0<y<nt} |D^{n+1} f(re^{i(\theta+y)})| (x_1 + \cdots + x_n) \, dx_1 \ldots dx_n \\ &= |\Delta_t^n f_r(\theta)| + (n/2) \sup_{0<y<nt} |D^{n+1} f(re^{i(\theta+y)})| t^{n+1}.\end{aligned}$$

Hence, by Lemma 3.1,

$$M_p^p(D^n f, r) t^{np} \leqq \|\Delta_t^n f_r\|_p^p + C t^{(n+1)p} M_p^p(D^{n+1} f, (3r+1)/4)$$

provided that $0 < t < 1 - r$. Combining this with the familiar inequalities $\|\Delta_t^n f_r\|_p \leqq \|\Delta_t^n f\|_p$ and

$$M_p(D^{n+1} f, (3r+1)/4) \leqq C(1-r)^{-1} M_p(D^n f, (1+r)/2)$$

(see [1]), we obtain

$$M_p^p(D^n f, r) \leqq t^{-np} \|\Delta_t^n f\|_p^p + C t^p (1-r)^{-p} M_p^p(D^n f, (1+r)/2) \tag{3.3}$$

where $0 < t < 1 - r$.

Let $0 < q < \infty$ and

$$A(r) = M_p^q(D^n f, r) \psi(1-r), \quad 0 < r < 1.$$

It follows from (3.3) and (2.1) that

$$A(r) \leqq C_1 t^{-nq} \|\Delta_t^n f\|_p^q \psi(1-r) + C_2 K t^q (1-r)^{-q} A((1+r)/2)$$

for $0<t<1-r$. Let m be the smallest integer such that $2^{mq} C_2 K \leqq 1/4$ and take $t = a(1-r)$, $a = 2^{-m}$. Then we have

$$A(r) \leqq C_3 (1-r)^{-nq} \phi(a(1-r)) \psi(1-r) + (1/4) A((1+r)/2),$$

where $\phi(t) = \|\Delta_t^n f\|_p^q$. Hence, by integration,

$$\int_0^1 A(r)\,dr \leqq C_3 \int_0^1 t^{-nq} \phi(at) \psi(t)\,dt + (1/2) \int_{1/2}^1 A(r)\,dr,$$

and hence

$$\begin{aligned}(1/2)\int_0^1 A(r)\,dr &\leqq \int_0^1 A(r)\,dr - (1/2)\int_{1/2}^1 A(r)\,dr \\ &\leqq C_3 \int_0^1 t^{-nq} \phi(at) \psi(t)\,dt \\ &= C_3 a^{nq-1} \int_0^a t^{-nq} \phi(t) \psi(2^m t)\,dt \\ &\leqq C_3 a^{nq-1} K^m \int_0^1 t^{-nq} \phi(t) \psi(t)\,dt.\end{aligned}$$

This concludes the proof of Theorem 2.1.

Remark. The last step in the proof is correct because the function $A(r)$ is integrable over $(0,1)$: the function $M_p^q(D^n f, r)$ is bounded near 1 (because $f(z)$ is analytic in $|z|<R$, $R>1$) and the function $\psi(1-r)$ is integrable.

4. Proof of Theorem 2.2

If f satisfies (2.5), then $f \in H^p$, by Flett's result. In order to prove (1.2) we can assume that f is analytic in $|z|<R$, for some $R>1$. Then (1.2) is equivalent to

$$\|\Delta_t^n f_1\|_p^p \leqq C \int_{1-t}^1 (1-s)^{np-1} M_p^p(D^n f, s)\,ds. \tag{4.1}$$

In proving this we can also assume that $0<t<1/4$. Then let $r=1-2t$, i.e. $t=(1-r)/2$. It follows from (3.2) and Lemma 3.1 that

$$\|\Delta_t^n f_r\|_p^p \leqq Ct^{np} M_p^p(D^n f, (1+r)/2),$$

which is dominated by the right hand side of (4.1) because of the increasing property of the p-th mean values. Since

$$\|\Delta_t^n f_1\|_p^p \leqq \|\Delta_t^n f_r\|_p^p + \|\Delta_t^n (f_1 - f_r)\|_p^p$$

it remains to estimate $\|\Delta_t^n(f_1-f_r)\|_p^p$.

Using the identity

$$1-r^j = r^j \sum_{k=1}^{n-1} \frac{j^k}{k!}\left(\log\frac{1}{r}\right)^k + \frac{j^n}{(n-1)!}\int_r^1 \left(\log\frac{1}{s}\right)^{n-1} s^{j-1}\,ds$$

one shows that

$$f_1(\theta) - f_r(\theta) = \frac{H(\theta)}{(n-1)!} + \sum_{k=1}^{n-1} \frac{1}{k!}\left(\log\frac{1}{r}\right)^k h_k(\theta),$$

where

$$H(\theta) = i^{-n}\int_r^1 \frac{1}{s}\left(\log\frac{1}{s}\right)^{n-1} D^n f(se^{i\theta})\,ds,$$

$$h_k(\theta) = i^{-k} D^k f(re^{i\theta}).$$

We have

$$\begin{aligned}\|\Delta_t^n h_k\|_p^p &\leqq 2^k \|\Delta_t^{n-k} h_k\|_p^p \\ &\leqq Ct^{(n-k)p} M_p^p(D^{n-k} D^k f, (1+r)/2) = Ct^{(n-k)p} M_p^p(D^n f, 1-t).\end{aligned}$$

Hence

$$\begin{aligned}\|\Delta_t^n(f_1-f_r)\|_p^p &\leqq C_1 \|\Delta_t^n H\|_p^p + C_2 \sum_{k=1}^{n-1} (1-r)^{kp} \|\Delta_t^n h_k\|_p^p \\ &\leqq C_1 2^n \|H\|_p^p + Ct^{np} M_p^p(D^n f, 1-t).\end{aligned}$$

Finally, we have to prove that $\|H\|_p^p$ is dominated by the right hand side of (4.1). We have

$$|H(\theta)| \leqq C \int_r^1 (1-s)^{n-1} g(s,\theta)\, ds,$$

where

$$g(s,\theta) = \sup\{|D^n f(\rho e^{i\theta})| : 0<\rho<s\}.$$

Since $g(s,\theta)$ increases with s,

$$|H(\theta)|^p \leqq C \int_r^1 (1-s)^{np-1} g(s,\theta)^p\, ds.$$

(See Lemma 4.1 below.) Integrating this inequality from 0 to 2π and using the Hardy–Littlewood complex maximal theorem, we get

$$\begin{aligned} \|H\|_p^p &\leqq C \int_r^1 (1-s)^{np-1} M_p^p(D^n f, s)\, ds \\ &\leqq C \int_r^1 (1-s)^{np-1} M_p^p(D^n f, (1+s)/2)\, ds \\ &= C 2^{np} \int_{1-t}^1 (1-s)^{np-1} M_p^p(D^n f, s)\, ds, \end{aligned}$$

and this completes the proof.

Lemma 4.1. *If φ is an increasing non-negative function on $[0,1)$ and $0<p<1$, then*

$$\left(\int_r^1 (1-s)^{n-1} \varphi(s)\, ds\right)^p \leqq C \int_r^1 (1-s)^{np-1} \varphi(s)^p\, ds, \quad 0 \leqq r < 1,$$

where $C=(np)^{1-p}$.

Proof. Let

$$\int_r^1 (1-s)^{np-1} \varphi(s)^p\, ds = 1$$

for a fixed r. Since φ is increasing we have

$$\varphi(\rho)^p \int_\rho^1 (1-s)^{np-1}\, ds \leqq 1 \; (r<\rho<1),$$

whence

$$\varphi(\rho) \leqq (np)^{1/p}(1-\rho)^{-n}.$$

Thus

$$\int_r^1 (1-s)^{n-1}\,\varphi(s)\,ds = \int_r^1 (1-s)^{n-1}\,\varphi(s)^p\,\varphi(s)^{1-p}\,ds$$

$$\leqq (np)^{(1-p)/p}\int_r^1 (1-s)^{n-1}\,\varphi(s)^p(1-s)^{-n(1-p)}\,ds$$

$$= (np)^{(1-p)/p}.$$

This proves the lemma.

5. Further applications to Lipschitz spaces

Let ϕ be a positive increasing function on $(0,1]$. We define $\mathrm{Lip}_n(\phi,p,q)$ $(0<p,q<\infty)$ to be the class of functions $f\in L^p(T)$ for which the function $F(t)=\|\Delta_t^n f\|_p/\phi(t)$, $0<t\leqq 1$, belongs to the Lebesgue space $L^q(dt/t)$. If $\phi(t)=t^\alpha$ $(0<\alpha<n)$, then these spaces, denoted by $\mathrm{Lip}_n(\alpha,p,q)$, coincide with the classical Lipschitz spaces as defined by Taibleson [**12**]. Taibleson generalized the theorems of Hardy and Littlewood and of Zygmund to the case of $\mathrm{Lip}_n(\alpha,p,q)$ with $p\geqq 1$ by showing that the function $F(t)$ in the above definition can be replaced by $t^{n-\alpha}M_p(D^nP[f],\,1-t)$, where $P[f]$ is the Poisson integral of f. In [**6**], Oswald extended Taibleson's result to $H^p\cap\mathrm{Lip}_n(\alpha,p,q)$ with $p<1$. In this section we apply Theorems 2.1 and 2.2 to prove a generalized version of Oswald's result.

Let $H\Lambda_n(\phi,p,q)$ denote the space of functions f analytic in the unit disc for which the function

$$r\mapsto(1-r)^n M_p(D^nf,r)/\phi(1-r),\quad 0<r<1,$$

belongs $L^q(dr/(1-r))$. These spaces are generalizations of the spaces $H\Lambda(\alpha,p,q)$ introduced by Flett [**3**]. It follows from Theorem 2.1 that if

$$\int_0^1 (t^n/\phi(t))^q\,dt/t<\infty, \tag{5.1}$$

then $H^p\cap\mathrm{Lip}_n(\phi,p,q)\subset H\Lambda_n(\phi,p,q)$.

Theorem 5.1. *The inclusion $H\Lambda_n(\phi,p,q)\subset H^p\cap\mathrm{Lip}_n(\phi,p,q)$ holds if the function ϕ satisfies the following condition*:

(L) *There exists a constant* $\alpha>0$ *such that* $\phi(t)/t^\alpha$ *is increasing for* $0<t<1$.

Corollary. *If* ϕ *satisfies (5.1) and* (L), *then* $H\Lambda_n(\phi,p,q)=H^p\cap \mathrm{Lip}_n(\phi,p,q)$.

Proof. Let $f\in H\Lambda_n(\phi,p,q)$ $(0<p<1)$, where ϕ satisfies (L). Since then $\phi(t)\leqq\phi(1)t^\alpha$, we have $H\Lambda_n(\phi,p,q)\subset H\Lambda(\alpha,p,q)\subset H^p$ (see [3] for various relations between H^p and $H\Lambda(\alpha,p,q)$). Thus $f\in H^p$, and we have to prove that $f\in \mathrm{Lip}_n(\phi,p,q)$.

Let $q\leqq p$. It follows from 1.2 and an obvious modification of Lemma 4.1 that

$$\omega_n f,t)_p^q\leqq C\int_{1-t}^{1}(1-r)^{nq-1}\varphi(r)^q\,dr,$$

where $\varphi(r)=M_p(D^n f,r)$. Multiplying both sides of this inequality by $\phi(t)^{-q}t^{-1}$, then integrating the resulting inequality and using Fubini's theorem, we obtain

$$\int_0^1[\omega_n(f,t)_p/\phi(t)]^q\,dt/t\leqq C\int_0^1(1-r)^{nq-1}\varphi(r)^q\,dr\int_{1-r}^{1}\phi(t)^{-q}t^{-1}\,dt.$$

Now the result follows from the inequality

$$\int_x^1\phi(t)^{-q}t^{-1}\,dt\leqq C\phi(x)^{-q},\quad 0<x<1,$$

which is a consequence of the condition (L).

Assuming that $q>p$ we have, by Jensen's inequality,

$$\left\{\alpha pt^{-\alpha p}\int_{1-t}^{1}(1-r)^{(n-\alpha)p}(r)^p(1-r)^{\alpha p-1}\,dr\right\}^{q/p}\leqq\alpha pt^{-\alpha p}\int_{1-t}^{1}(1-r)^{(n-\alpha)q}\varphi(r)^q(1-r)^{\alpha p-1}\,dr.$$

From this and (1.2) it follows that

$$\omega_n(f,t)_p^q\leqq Ct^\varepsilon\int_{1-t}^{1}(1-r)^{nq-\varepsilon-1}\varphi(r)^q\,dr,$$

where $\varphi(r)=M_p(D^n f,r)$ and $\varepsilon=\alpha(q-p)$. Hence

$$\int_0^1[\omega_n(f,t)_p/\phi(t)]^q\,dt/t\leqq C\int_0^1(1-r)^{nq-\varepsilon-1}\varphi(r)^q\,dr\int_{1-r}^{1}t^{\varepsilon-1}\phi(t)^{-q}\,dt.$$

Using the inequality $t^\alpha/\phi(t)\leqq(1-r)^\alpha/\phi(1-r)$, $1-r\leqq t$, one shows that the inner integral is dominated by $(1-r)^\varepsilon/\phi(1-r)^q$, which completes the proof.

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