A NOTE ON TENSOR PRODUCTS OF REFLEXIVE ALGEBRAS

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In this short note, we obtain a concrete description of rank-one operators in $\text{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$. Based on this characterisation, we give a simple proof of the tensor product formula:

$$\text{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n) = \text{Alg} L_1 \otimes_w \cdots \otimes_w \text{Alg} L_n$$

if $\text{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$ is weakly generated by rank-one operators in itself and $\mathcal{L}_i$ ($i = 1, \ldots, n$) are subspace lattices.

1. Introduction

One of the central results in the theory of tensor products of von Neumann algebras is Tomita’s commutation formula:

$$\mathcal{M}' \otimes_w \mathcal{N}' = (\mathcal{M} \otimes \mathcal{N})',$$

where $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras. It was observed in [1] that if we let $\mathcal{L}_1$ and $\mathcal{L}_2$ denote the projection lattices of $\mathcal{M}$ and $\mathcal{N}$ respectively, then (1) can be rewritten as

(2) $$\text{Alg} \mathcal{L}_1 \otimes_w \text{Alg} \mathcal{L}_2 = \text{Alg}(\mathcal{L}_1 \otimes \mathcal{L}_2).$$

This version of Tomita’s theorem makes sense for any pair of reflexive algebras $\text{Alg} \mathcal{L}_1$ and $\text{Alg} \mathcal{L}_2$. It remains a deep open question whether the tensor product formula (2) is valid for general reflexive algebras, or even general commutative subspace lattice algebras. However, (2) has been proved in a number of special cases ([1, 2, 3, 4]). In particular, it is known that if $\mathcal{L}_1$ is a commutative subspace lattice that is either completely distributive ([4]) or finite width ([2]), then (2) is valid for $\mathcal{L}_1$ and any subspace lattice $\mathcal{L}_2$. The main purpose of this paper is to study the $n$-fold tensor product formula of reflexive algebras. The technique employed in this note is simple and different from the other papers about tensor products. We use rank-one operators to investigate tensor products and the technique shows its power in this note.

Let us introduce some notation and terminology. Throughout, $\mathcal{H}$ represents a complex separable Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$. A sublattice
The projection lattice of $B(H)$ is said to be a subspace lattice if it contains $0$ and $I$ and is strongly closed, where we identify projections with their ranges. If the elements of $\mathcal{L}$ pairwise commute, $\mathcal{L}$ is a commutative subspace lattice. A nest is a totally ordered subspace lattice. If $\mathcal{L}$ is a subspace lattice, $\text{Alg}\mathcal{L}$ denotes the set of operators in $B(H)$ that leave the elements of $\mathcal{L}$ invariant. If $\mathcal{L}$ is a commutative subspace lattice, $\text{Alg}\mathcal{L}$ is said to be a commutative subspace lattice algebra. If $\mathcal{L}$ is a nest, $\text{Alg}\mathcal{L}$ is said to be a nest algebra.

If $A$ is a subset of $B(H)$ then $\text{Lat}A$, the set of projections left invariant by each element of $A$, is a subspace lattice. A subalgebra $A$ of $B(H)$ is reflexive if $A = \text{Alg}\text{Lat}A$. The reflexive algebras are precisely the algebras of the form $\text{Alg}\mathcal{L}$, where $\mathcal{L}$ is a subspace lattice. If $\mathcal{L}_i \subseteq B(H_i)(i = 1, \ldots, n)$ are subspace lattices, $\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n$ is the subspace lattice in $B(H_1 \otimes \cdots \otimes H_n)$ generated by $\{L_1 \otimes \cdots \otimes L_n : L_i \in \mathcal{L}_i, i = 1, \ldots, n\}$. If $S_i \subseteq B(H_i)(i = 1, \ldots, n)$ are subspaces, then $S_1 \otimes \cdots \otimes S_n$ denotes the linear span of $\{S_1 \otimes \cdots \otimes S_n : S_i \in S_i\}$; $S_1 \otimes_w \cdots \otimes_w S_n$ denotes the weak closure of $S_1 \otimes \cdots \otimes S_n$ in $B(H_1 \otimes \cdots \otimes H_n)$.

2. Tensor products of reflexive algebras

For $x, y \in H$, the operator $xy^*$ is defined by the equation

$$(xy^*)(z) = (z, y)x, \quad \text{for all } z \in H.$$

If $\mathcal{L}$ is a subspace lattice and $L \in \mathcal{L}$, we write $L_-$ for the projection $\bigvee\{E \in \mathcal{L} : L \notin E\}$. The following result of Longstaff [6] is essential.

**Lemma 1.** Let $\mathcal{L}$ be a subspace lattice. Then $xy^* \in \text{Alg}\mathcal{L}$ if and only if there is an element $L \in \mathcal{L}$ such that $x \in L$ and $y \in L_-^\perp$.

Let $\mathcal{L}_i \subseteq B(H_i)(i = 1, \ldots, n)$ be subspace lattices. For $1 \leq j \leq n$, let $I_1 \otimes \cdots \otimes L_j \otimes \cdots \otimes I_n = \{I_1 \otimes \cdots \otimes L_j \otimes \cdots \otimes I_n : L_j \in \mathcal{L}_j\}$; certainly, it is a subspace lattice.

**Lemma 2.** Let $\mathcal{L}_j \subseteq B(H_i)(1 \leq j \leq n)$ be subspace lattices. Suppose that $N_j \in \mathcal{L}_j$, then

$$(I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n)_- = I_1 \otimes \cdots \otimes N_j_- \otimes \cdots \otimes I_n$$

and

$$(I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n)_+^- = I_1 \otimes \cdots \otimes N_j^- \otimes \cdots \otimes I_n$$

in $I_1 \otimes \cdots \otimes \mathcal{L}_j \otimes \cdots \otimes I_n$.

**Proof:** We first show that $I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n \subseteq I_1 \otimes \cdots \otimes L_j \otimes \cdots \otimes I_n$ if and only if $N_j \leq L_j$. For the forward implication choose unit vectors $x_i \in H_i(i \neq j)$. For any $x_j \in H_j$,

$$0 \leq \left\langle (I_1 \otimes \cdots \otimes (L_j - N_j) \otimes \cdots \otimes I_n)(x_1 \otimes \cdots \otimes x_n), x_1 \otimes \cdots \otimes x_n \right\rangle$$

$$= \left\langle x_1 \otimes \cdots \otimes (L_j - N_j)x_j \otimes \cdots \otimes x_n, x_1 \otimes \cdots \otimes x_n \right\rangle$$

$$= \left\langle (L_j - N_j)x_j, x_j \right\rangle.$$
So $N_j \leq L_j$. The converse implication is also natural. Thus $I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n \not\subseteq I_1 \otimes \cdots \otimes L_j \otimes \cdots \otimes I_n$ if and only if $N_j \not\subseteq L_j$. Hence

$$(I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n) = \bigvee \{I_1 \otimes \cdots \otimes L_j \otimes \cdots \otimes I_n : I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n \not\subseteq I_1 \otimes \cdots \otimes L_j \otimes \cdots \otimes I_n\}$$

$\not\subseteq I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n$.

(The proof of the third equality is routine). Since

$$(I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n)(I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n) = 0,$$

we have

$$I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n \leq (I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n)^{\perp}.$$

If $(I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n)^{\perp} \not= I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n$, we can choose a non-zero vector $z \in (I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n)^{\perp} \otimes (I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n)$. Thus

$$z = (I_1 \otimes \cdots \otimes I_j \otimes \cdots \otimes I_n)z = (I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n)z + (I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n)z = 0.$$

This is a contradiction. So

$$(I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n)^{\perp} = (I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n)^{\perp} = I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n.$$

**Lemma 3.** Let $\mathcal{L}_i \subseteq B(H_i)(i = 1, \ldots, n)$ be subspace lattices. Then a rank-one operator $xy^* \in \text{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$ if and only if there exist $N_i \in \mathcal{L}_i$ such that $x \in N_1 \otimes \cdots \otimes N_n$ and $y \in N_n^\perp \otimes \cdots \otimes N_n^\perp$.

**Proof:** Set $\mathcal{F}_i = I_1 \otimes \cdots \otimes \mathcal{L}_i \otimes \cdots \otimes I_n$. Thus

$$\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n = \mathcal{F}_1 \lor \cdots \lor \mathcal{F}_n,$$

and

$$\text{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n) = (\text{Alg} \mathcal{F}_1) \cap \cdots \cap (\text{Alg} \mathcal{F}_n).$$

Now suppose that $xy^* \in \text{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$. Since $\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n \supseteq \mathcal{F}_i$, $\text{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n) \subseteq \text{Alg} \mathcal{F}_i$. Thus $xy^* \in \text{Alg} \mathcal{F}_i$ and, by the definition of $\mathcal{F}_i$ and Lemma 1 and Lemma 2, there is an element $N_i \in \mathcal{L}_i$ such that $x \in I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n$ and $y \in (I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n)^{\perp} = I_1 \otimes \cdots \otimes N_i^\perp \otimes \cdots \otimes I_n$. This is valid for each $i = 1, \ldots, n$, whence

$$x \in N_1 \otimes \cdots \otimes N_n \text{ and } y \in N_n^\perp \otimes \cdots \otimes N_n^\perp.$$
For the converse, if \( x \in N_1 \otimes \cdots \otimes N_n \) and \( y \in N_1^\perp \otimes \cdots \otimes N_n^\perp \), then, in particular, \( x \in I_1 \otimes \cdots \otimes I_n \) and \( y \in I_1 \otimes \cdots \otimes I_n \). Lemma 1 and Lemma 2 imply that \( xy^* \in \Alg F_i \) for each \( i \). Hence
\[
xy^* \in \bigcap_{i=1}^n \Alg F_i = \Alg (L_1 \otimes \cdots \otimes L_n).
\]

**Proposition 4.** Let \( \mathcal{L}_i \subseteq B(\mathcal{H}_i)(i = 1, \ldots, n) \) be subspace lattices. If \( L \in \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n \) and \( L \not\subseteq L_- \), then
\[
L = \bigvee \{N_1 \otimes \cdots \otimes N_n : N_1 \otimes \cdots \otimes N_n \leq L\}.
\]

**Proof:** Suppose that \( 0 \neq x \in L \). Since \( L \not\subseteq L_- \), \( L_- \neq I_1 \otimes \cdots \otimes I_n \). For any \( 0 \neq y \in L_1^\perp \), Lemma 1 shows that the rank-one operator \( xy^* \in \Alg (L_1 \otimes \cdots \otimes L_n) \).

By Lemma 3, there exist \( N_i \in \mathcal{L}_i (i = 1, \ldots, n) \) such that \( x \in N_1 \otimes \cdots \otimes N_n \) and \( y \in N_1^\perp \otimes \cdots \otimes N_n^\perp \). If \( N_1 \otimes \cdots \otimes N_n \not\subseteq L \), it follows from the definition of \( (N_1 \otimes \cdots \otimes N_n)_- \) that \( L \leq (N_1 \otimes \cdots \otimes N_n)_- \). By virtue of Lemma 2, we then have
\[
L \leq (N_1 \otimes \cdots \otimes N_n)_-
= (I_1 \otimes \cdots \otimes I_n)_-
= I_1 \otimes \cdots \otimes I_n
\]
and
\[
L_1^\perp \geq I_1 \otimes \cdots \otimes N_1^\perp \otimes \cdots \otimes I_n, \quad \text{for each } i.
\]

So \( L_1^\perp \geq N_1^\perp \otimes \cdots \otimes N_n^\perp \) and we have shown that \( y \in L_1^\perp \). Thus for any \( y \in L_1^\perp \), we show that \( y \in L_1^\perp \). This implies that \( L_- \leq L_1^\perp \) and \( L \leq L_- \). This contradicts our hypothesis. Hence \( N_1 \otimes \cdots \otimes N_n \leq L \) and for any \( x \in L \),
\[
x \in \bigvee \{N_1 \otimes \cdots \otimes N_n : N_1 \otimes \cdots \otimes N_n \leq L\}.
\]
Thus
\[
L \leq \bigvee \{N_1 \otimes \cdots \otimes N_n : N_1 \otimes \cdots \otimes N_n \leq L\}.
\]
The converse inequality is obvious and this completes the proof.

**Lemma 5.** Let \( x_i, y_i \in \mathcal{H}_i (i = 1, \ldots, n) \). Then
\[
(x_1 \otimes \cdots \otimes x_n)(y_1 \otimes \cdots \otimes y_n)^* = (x_1 y_1^*) \otimes \cdots \otimes (x_n y_n^*).
\]

**Proof:** For any \( z_i \in \mathcal{H}_i \), it follows from the definition that
\[
\left[(x_1 \otimes \cdots \otimes x_n)(y_1 \otimes \cdots \otimes y_n)^*\right](z_1 \otimes \cdots \otimes z_n)
= \langle z_1 \otimes \cdots \otimes z_n, y_1 \otimes \cdots \otimes y_n \rangle (x_1 \otimes \cdots \otimes x_n)
= \langle z_1, y_1 \rangle \cdots \langle z_n, y_n \rangle (x_1 \otimes \cdots \otimes x_n)
= \langle (x_1 y_1^*) z_1 \rangle \otimes \cdots \otimes \langle (x_n y_n^*) z_n \rangle
= \left[(x_1 y_1^*) \otimes \cdots \otimes (x_n y_n^*)\right](z_1 \otimes \cdots \otimes z_n).
\]
Since the linear span of simple tensors is everywhere dense in $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$, so

$$(x_1 \otimes \cdots \otimes x_n)(y_1 \otimes \cdots \otimes y_n)^* = (x_1 y_1^*) \otimes \cdots \otimes (x_n y_n^*).$$

**Theorem 6.** Suppose that $L_i \subseteq B(\mathcal{H}_i)(i = 1, \ldots, n)$ are subspace lattices and $\text{Alg}(L_1 \otimes \cdots \otimes L_n)$ is weakly generated by rank-one operators in itself. Then

$$\text{Alg}(L_1 \otimes \cdots \otimes L_n) = \text{Alg} L_1 \otimes_w \cdots \otimes_w \text{Alg} L_n.$$  

**Proof:** Each of the operators which generate $\text{Alg} L_1 \otimes_w \cdots \otimes_w \text{Alg} L_n$ leaves invariant each of the projections which generate $L_1 \otimes \cdots \otimes L_n$; therefore

$$\text{Alg}(L_1 \otimes \cdots \otimes L_n) \supseteq \text{Alg} L_1 \otimes_w \cdots \otimes_w \text{Alg} L_n.$$  

It remains to show that $\text{Alg}(L_1 \otimes \cdots \otimes L_n) \subseteq \text{Alg} L_1 \otimes_w \cdots \otimes_w \text{Alg} L_n$. Since $\text{Alg}(L_1 \otimes \cdots \otimes L_n)$ is weakly generated by rank-one operators in itself, it suffices to show that each rank-one operator in $\text{Alg}(L_1 \otimes \cdots \otimes L_n)$ belongs to $\text{Alg} L_1 \otimes_w \cdots \otimes_w \text{Alg} L_n$. Now for any $N_i \in L_i$ and $x_i, y_i \in \mathcal{H}_i$, we have that

$$(N_1 \otimes \cdots \otimes N_n)((x_1 \otimes \cdots \otimes x_n)(y_1 \otimes \cdots \otimes y_n)^*)(N_1^\perp \otimes \cdots \otimes N_n^\perp) = (N_1 \otimes \cdots \otimes N_n)[(x_1 y_1^*) \otimes \cdots \otimes (x_n y_n^*)](N_1^\perp \otimes \cdots \otimes N_n^\perp) = N_1(x_1 y_1^*)N_1^\perp \otimes \cdots \otimes N_n(x_n y_n^*)N_n^\perp \in \text{Alg} L_1 \otimes_w \cdots \otimes_w \text{Alg} L_n.$$  

For any rank-one operator $zw^* \in \text{Alg}(L_1 \otimes \cdots \otimes L_n)$, it follows from Lemma 3 that there exist $N_i \in L_i(i = 1, \ldots, n)$ such that $z \in N_1 \otimes \cdots \otimes N_n$ and $w \in N_1^\perp \otimes \cdots \otimes N_n^\perp$. Since $z, w \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$, there exist sequences $\{z_m\}$ and $\{w_m\}$ such that

$$z_m \overset{\text{w}}{\rightarrow} z \quad \text{and} \quad w_m \overset{\text{w}}{\rightarrow} w,$$

where $z_m, w_m$ are finite linear combinations of simple tensors. It is routine to show that

$$(N_1 \otimes \cdots \otimes N_n)(z_m w_m^*)(N_1^\perp \otimes \cdots \otimes N_n^\perp) \overset{\text{w}}{\rightarrow} (N_1 \otimes \cdots \otimes N_n)(zw^*)(N_1^\perp \otimes \cdots \otimes N_n^\perp) = zw^*.$$  

The preceding paragraph shows that

$$(N_1 \otimes \cdots \otimes N_n)(z_m w_m^*)(N_1^\perp \otimes \cdots \otimes N_n^\perp) \in \text{Alg} L_1 \otimes_w \cdots \otimes_w \text{Alg} L_n,$$

so $zw^* \in \text{Alg} L_1 \otimes_w \cdots \otimes_w \text{Alg} L_n$. This completes the proof.  

**Corollary 7.** Let $L_i \subseteq B(\mathcal{H}_i)(i = 1, \ldots, n)$ be subspace lattices. If $\text{Alg}(L_1 \otimes \cdots \otimes L_{n-1})$ is weakly generated by rank-one operators in itself, then

$$\text{Alg}(L_1 \otimes \cdots \otimes L_n) = \text{Alg} L_1 \otimes_w \cdots \otimes_w \text{Alg} L_n.$$
PROOF: It follows from [4, Proposition 1.1 and Theorem 2.1] that

\[ \text{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n) = \text{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_{n-1}) \otimes_w \text{Alg} \mathcal{L}_n. \]

By virtue of Theorem 6, we obtain that

\[ \text{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n) = \text{Alg} \mathcal{L}_1 \otimes_w \cdots \otimes_w \text{Alg} \mathcal{L}_n. \]

The following corollary is one of the main results in [3].

**COROLLARY 8.** ([3, Theorem 17].) Let \( \mathcal{L}_i \subseteq \mathcal{B}(\mathcal{H}_i)(i = 1, \ldots, n) \) be completely distributive commutative subspace lattices. Then

\[ \text{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n) = \text{Alg} \mathcal{L}_1 \otimes_w \cdots \otimes_w \text{Alg} \mathcal{L}_n. \]

**PROOF:** It follows from [3, Theorem 10] that \( \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n \) is a completely distributive commutative subspace lattice. Thus, by virtue of [5, Theorem 3], \( \text{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n) \) is weakly generated by the rank-one operators in itself. So the corollary follows from Theorem 6.

**COROLLARY 9.** ([1, Theorem 2.6].) Let \( \mathcal{N}_i(i = 1, \ldots, n) \) be nests. Then

\[ \text{Alg}(\mathcal{N}_1 \otimes \cdots \otimes \mathcal{N}_n) = \text{Alg} \mathcal{N}_1 \otimes_w \cdots \otimes_w \text{Alg} \mathcal{N}_n. \]

If \( \mathcal{L} \) is a subspace lattice, let \( \mathcal{R}(\mathcal{L}) \) denote the linear span of rank-one operators in \( \text{Alg} \mathcal{L} \) and \( \overline{\mathcal{R}(\mathcal{L})} \) the norm closure of \( \mathcal{R}(\mathcal{L}) \). If \( S_i \subseteq \mathcal{B}(\mathcal{H}_i)(i = 1, \ldots, n) \) are subspaces, \( S_1 \otimes \cdots \otimes S_n \) denotes the norm closure of \( S_1 \otimes \cdots \otimes S_n \).

**PROPOSITION 10.** Let \( \mathcal{L}_i \subseteq \mathcal{B}(\mathcal{H}_i)(i = 1, \ldots, n) \) be subspace lattices. Then

\[ \overline{\mathcal{R}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)} = \overline{\mathcal{R}(\mathcal{L}_1) \otimes \cdots \otimes \mathcal{R}(\mathcal{L}_n)}. \]

**PROOF:** The result is essentially implied in the proof of Theorem 6.

Note that in Proposition 10 we do not need the hypothesis that \( \text{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n) \) is weakly generated by rank-one operators in itself.

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