ANZIAM J. 44(2002), 247–259

NEWTON'S DIAGRAM METHOD FOR NONLINEAR EQUATIONS WITH SEVERAL SMALL PARAMETERS

PETER AIZENGENDLER[1]

(Received 1 April, 1999; revised 20 November, 1999)

Abstract

In this article, we generalise Newton's diagram method for finding small solutions $\xi(\lambda)$ of equations $f(\xi, \lambda) = 0$ ($f(0,0) = 0$) with f analytic (see [1, 2, 4, 6]) to the case of a multi-dimensional function f, unknown variable ξ and small parameter λ. This method was briefly described in [1]. The method has many different applications and allows one to solve some inflexible problems. In particular, the method can be used in very difficult bifurcation problems, for example, for systems with small imperfections.

1. Introduction

In this paper, we propose a new method for finding all the small solutions[2] (SS) of equations $f(\xi, \lambda) = 0$ ($f(0,0) = 0$) with f analytic, where the small parameter λ is multi-dimensional: $\lambda = (\lambda_1, \dots, \lambda_n)$. This method has many different applications and allows one to solve a number of inflexible problems. In particular, it can be very useful in problems related to dynamic systems (bifurcations of equilibrium states, boundedness and stability of limit cycles), deformed systems, *etc.*, which can be reduced to the above defined problem. Some examples of typical problems of static systems are presented in [5] and in [6, pp. 454–469].

In this article, the method is described in three sections. In Section 2 the method is explicated for two small parameters ($n = 2$). In Section 3 we consider the critical situation when the method does not give the SS but, nevertheless, the problem can be solved by means of some additional considerations. In Section 4 the method is generalised for n ($n > 2$) small parameters.

[1]Professor Peter Aizengendler, late of Pscov University, Russia, died in November 2000. This paper, his last mathematical testament, is published with the kind consent of his son, Dr Mark Aizengendler, 11 Varram Way, West Lakes Shore, SA 5020; e-mail: mark_15jan@yahoo.com.au.

[2]The continuous solution $\xi(\lambda)$ is defined to be an SS if $\xi(\lambda) \to 0$ as $\lambda \to 0$.

2. Case of two parameters ($n = 2$)

First, we describe this method for an equation with two small scalar parameters: λ_1 and λ_2, that is, when λ is two-dimensional:

$$\Gamma(\xi, \lambda_1, \lambda_2) \equiv \sum_{m+n+k=1}^{\infty} \varphi_{mnk}(\xi)\lambda_1^n\lambda_2^k = 0, \tag{1}$$

where $\xi = (\xi_1, \dots, \xi_r)$, $\varphi_{mnk}(\xi) = \{\varphi_{mnk}^{(1)}(\xi), \dots, \varphi_{mnk}^{(r)}(\xi)\}$, $\varphi_{mnk}^{(i)}(\xi)$ is a scalar homogeneous polynomial relative to $\xi_1, \dots, \xi_r$, of degree m with real coefficients or a zero polynomial. Let us assume that $\varphi_{m00}(\xi) = 0$ at $m < m_0$ and $\varphi_{m_000}(\xi) \neq 0$. It is also assumed that the series Γ converges in some neighbourhood of zero. We are studying the problem about SS $\xi(\lambda_1, \lambda_2)$ of (1) for (λ_1, λ_2) near the straight line $\lambda_1 = 0$ but away from the line $\lambda_2 = 0$. The indicated neighbourhood T must satisfy the condition that $(0, 0) \in \bar{T}$ and $|\lambda_1/\lambda_2^s|$ is bounded in T; $s \in \mathbb{N}$. The minimal value of s is found from the method.

Let us denote $OMNK$ to be a Cartesian coordinate system in $\mathbb{R}^3$; π_1 and π_2 to be operators of the orthogonal projection of $\mathbb{R}^3$ on the coordinate planes OMN and OMK respectively; $Q = \{(m, n, k) : \varphi_{mnk}(\xi) \neq 0\}$ and $Q_0 = \{(m, n, k) \in Q : m \leq m_0\}$ to be integer lattices; $R_1 = \pi_1(Q_0)$ and $R_2 = \pi_2(Q_0)$. First of all, we construct Newton's diagram (denoted by D_1) for R_1 on the plane OMN.[3] We denote its links and slopes as $L^1, \dots, L^x$ and $p^{(1)}, \dots, p^{(x)}$ respectively. Moreover, it is assumed that the links are numbered in order of the increase of their slopes (the count is made from right to left) so that $0 \leq p^{(1)} < \cdots < p^{(x)}$. Further, let us construct Newton's diagram on the plane OMK for the lattice $\pi_2(\pi_1^{-1}(L^i) \cap Q_0)$, $i \in \{1, \dots, x\}$. Let us also denote its links and slopes as $\{L^{i,\theta}\}$ and $\{p_{i,\theta}\}$ ($\theta = 1, \dots, \alpha_i$) respectively. If it has been done for all links L^i of D_1, we get a broken line (a diagram) D_2 on the plane OMK.

Two cases can arise. The first one (the standard case), when D_2 is the standard diagram for R_2, that is, Newton's diagram for R_2 with a unique attenuation—the slopes corresponding to the adjacent links of D_1 can coincide; $p_{i,\alpha_i} = p_{i+1,1}$, $i \in \{1, \dots, x - 1\}$ and the second case (the general case) arises when D_2 is not a standard diagram for R_2.

Let us now show that by the conversion

$$\lambda_1 = \mu_1\lambda_2^s, \quad \lambda_2 = \lambda_2, \tag{2}$$

[3]Newton's diagram D_1 for R_1 is a convex broken line without vertical links. All tops of D_1 belong to R_1 and the links of D_1 satisfy the minimal condition, that is, there are no points from R_1 situated below the support lines of links. If the equation of the support line of a link is $y = k(x - x_0)$, then the slope of this link is equal to $-k$. All links of D_1 (except maybe the right link) have positive slopes. The slope of the right link can be equal to 0. The Newton's diagram method of construction is presented widely in the literature, see for example [6, pp. 10–15].

where s is a natural number, the general case reduces to the standard case. Indeed, making the substitution (2) in (1), we obtain

$$\bar{\Gamma}(\xi, \mu_1, \lambda_2) \equiv \sum_{m+n+k\geq 1} \varphi_{mnk}(\xi)\mu_1^n\lambda_2^{ns+k} = 0, \tag{1'}$$

where $\varphi_{mnk}(\xi)\mu_1^n\lambda_2^{ns+k}$ is the image of the term $\varphi_{mnk}(\xi)\lambda_1^n\lambda_2^k$. Therefore (2) induces a linear conversion of the space of multi-indices (m, n, k) with the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & s & 1 \end{bmatrix}.$$

Under that conversion the diagram D_1 remains unchanged but the diagram $D_2 = \{L^{i,\theta}\}$ transforms into the diagram $\bar{D}_2 = \{\bar{L}^{i,\theta}\}$ and the slope $\bar{p}_{i,\theta}$ of the link $\bar{L}^{i,\theta}$ satisfies the relation

$$\bar{p}_{i,\theta} = p_{i,\theta} + sp^{(i)}. \tag{3}$$

From (3) it follows that if we take s as the least non-negative entire number s_0 satisfying the inequalities

$$s_0 \geq \frac{p_{i,\alpha_i} - p_{i+1,1}}{p^{(i+1)} - p^{(i)}}, \quad i = 1, \dots, \varkappa - 1, \tag{4}$$

then all the links $\bar{D}_2$ will be arranged in non-decreasing order of their slopes (the count is made from right to left):

$$\bar{p}_{1,1} < \cdots < \bar{p}_{1,\alpha_1} \leq \bar{p}_{2,1} < \cdots < \bar{p}_{\varkappa-1,\alpha_{\varkappa-1}} \leq \bar{p}_{\varkappa,1} < \cdots < \bar{p}_{\varkappa,\alpha_\varkappa}.$$

NOTE 1. If instead of s_0 we take $s_0 + 1$, then all links of $\bar{D}_2$ will be aranged in order of the increase of their slopes.

Let us suppose that $P_i(m_i, k_i)$ $(i = 1, \dots, \alpha)$ is the set of all points from R_2 situated under D_2. For each point P_i let us put in accordance the point (m_i, n_i', k_i') such that $(m_i, n_i') \in D_1$ and $(m_i, k_i') \in D_2$. By n_i we denote the least ordinate of the points of the set $\pi_2^{-1}(P_i) \cap Q_0$, $\Delta_1^i = n_i - n_i'$ and $\Delta_2^i = k_i' - k_i$. Let us use the conversion A. Then the points (m_i, n_i, k_i) and (m_i, n_i', k_i') map to the points $(m_i, n_i, \bar{k}_i)$ and $(m_i, n_i', \bar{k}_i')$ accordingly, where $\bar{k}_i = k_i + sn_i$ and $\bar{k}_i' = k_i' + sn_i'$. Thus we obtain the relation

$$\bar{\Delta}_2^i = \Delta_2^i - s\Delta_1^i \quad (\bar{\Delta}_2^i = \bar{k}_i' - \bar{k}_i). \tag{5}$$

From (5) it follows that if s is replaced by the least natural number $\bar{s}$ satisfying the inequalities

$$\bar{s} \geq \Delta_2^i/\Delta_1^i, \quad i = 1, \dots, \alpha, \tag{6}$$

then all points from $\bar{R}_2 = \pi_2(A\,Q_0)$ will not be situated under the diagram $\bar{D}_2$.

NOTE 2. If instead of $\bar{s}$ we take $\bar{s}+1$, then all points from R_2 lying below D_2 map to points situated above $\bar{D}_2$. If $s = \max(s_0, \bar{s})$, then by the conversion (2) the equation (1) is standardised, that is, it transforms into that for the standard case.

Let the equation (1) be standard. Let it be represented in the form

$$\Gamma(\xi, \lambda_1, \lambda_2) \equiv \sum{}' \varphi_{mnk}(\xi)\lambda_1^n\lambda_2^k + \sum{}'' \varphi_{mnk}(\xi)\lambda_1^n\lambda_2^k = 0, \qquad (1'')$$

where $\sum'$ denotes the sum of all non-zero terms of (1) such that $(m, n) \in L^i$ and $(m, k) \in L^{i,\theta}$ (L^i is any link of D_1 and $L^{i,\theta}$ is one of the links of D_2 corresponding to L^i) and $\sum''$ is the sum of the rest of the terms of (1). We denote also by r_1/s_1 the slope of L^i, by r_2/s_2 the slope of $L^{i,\theta}$ and suppose that $(r_1, s_1) = (r_2, s_2) = 1$ ((a, b) denotes the greates common divisor (GCD) for a and b). If $r_i = 0$, then we assume $s_i = 1$, $i \in \{1, 2\}$.

Let us make the substitution

$$\xi = \eta\mu_1^{r_1}\mu_2^{r_2}, \quad \lambda_1 = \mu_1^{s_1}, \quad \lambda_2 = \mu_2^{s_2}. \qquad (7)$$

Then (1″) has been reduced to the form

$$\sum{}' \varphi_{mnk}(\eta)\mu_1^{mr_1+ns_1}\mu_2^{mr_2+ks_2} + \sum{}'' \varphi_{mnk}(\eta)\mu_1^{mr_1+ns_1}\mu_2^{mr_2+ks_2} = 0. \qquad (*)$$

From the convexity of D_1 and the minimal condition of its links, it follows that $mr_1 + ns_1 = l_1$ for all $(m, n) \in L^i \cap R_1$, where l_1 is a non-negative integer, and for each $(m, n) \in \pi_1(Q) \setminus L^i$ the inequality $mr_1 + ns_1 > l_1$ is true. Analogously, for all $(m, k) \in L^{i,\theta} \cap R_2$ the relation $mr_2 + ks_2 = l_2$ holds (l_2 is a non-negative integer) and for all $(m, k) \in \pi_2(Q) \setminus L^{i,\theta}$ the inequality $mr_2 + ks_2 > l_2$ is true. Moreover, $l_1 + l_2 > 0$. From this point, $(*)$ is transformed into

$$F(\eta, \mu_1, \mu_2) \equiv P(\eta) + \sum{}'' \varphi_{mnk}(\eta)\mu_1^{mr_1+ns_1-l_1}\mu_2^{mr_2+ks_2-l_2} = 0, \qquad (8)$$

where $P(\eta) = \sum' \varphi_{mnk}(\eta)$.[4]

Let us suppose that η_0 is a simple non-zero root of $P(\eta)$. Then $F(\eta_0, 0, 0) = 0$, $\det F'_\eta(\eta_0, 0, 0) \neq 0$ and according to the implicit function theorem, (8) has a single solution in a neighbourhood of the point $(\eta_0, 0, 0)$:

$$\eta = \eta_0 + \sum_{i+k\geq 1} \eta_{ik}\mu_1^i\mu_2^k. \qquad (9)$$

All coefficients $\{\eta_{ik}\}$ are defined using the indefinite coefficients (IC) method. Moreover, if $\eta_0 \in \mathbb{R}^r$ and all coefficients of (1) are real, then all $\eta_{ik} \in \mathbb{R}^r$. Substituting (9)

[4]We name the polynomial $P(\eta)$ (respectively, the equation $P(\eta) = 0$) and the defining polynomial (DP) (respectively, the defining equation (DE)) for the link $L^{i,\theta}$.

https://doi.org/10.1017/S1446181100013924 Published online by Cambridge University Press

into the first relation of (7) and using the second and third parts of (7), we arrive at the SS (or rēal SS) of (1):

$$\xi = \eta_0 \lambda_1^{r_1/s_1} \lambda_2^{r_2/s_2} + \sum_{i+k\geq 1} \eta_{ik} \lambda_1^{(r_1+i)/s_1} \lambda_2^{(r_2+k)/s_2}. \tag{10}$$

(When we write $a^{1/s}$ ($s \in \mathbb{N}$), we are taking into account only the arithmetic value of $\sqrt[s]{a}$.) Thus every simple non-zero root of the DP provides a single SS of (1) and this solution is represented in the form of the convergent series (10).

Let η_0 be a multiple root of $P(\eta)$. Then assuming $\eta = \eta_0 + u$ in (8), we obtain a new equation $\bar{F}(u, \mu_1, \mu_2) = 0$ and the problem is reduced to obtaining the SS of the last equation. Standardising this equation (if required) and then repeating the same deduction as for (1′), we obtain a unique SS for every non-zero simple root of its DP. For every multiple root the procedure is repeated. If after a finite number of steps the process of obtaining multiple roots is stopped, then every SS of the problem, corresponding to η_0, is represented as a convergent series for integer or fractional non-negative powers of the parameters λ_2 and t, where $\lambda_1 = t\lambda_2^{\sigma}$ ($\sigma \geq 0$). If the process of obtaining the multiple roots is nonfinite, that is, when the SS is a multiple solution, then for a scalar equation all SS also have the structure described above. This is established in just the same way as for a scalar equation with a single small parameter (see [1]).

From Weierstrass' preparation theorem ([6, p. 39]) it follows that for the scalar equation (1), the total number of non-zero SS is finite and equal to $m_0 - l_0$ (each solution is counted as many times as its multiplicity).[5] Using all links of D_1 and $\bar{D}_2$ and all DP, the described method gives all non-zero SS because the total number of all non-zero roots of all DP is equal to $m_0 - l_0$ (each root is counted according to its multiplicity).

EXAMPLE 1. As an illustration of this method, we consider the problem for the scalar equation

$$\Gamma(\xi, \lambda_1, \lambda_2) \equiv \lambda_1^5 + 2\lambda_2\lambda_1^4\xi + \lambda_1^3\lambda_2^2\xi^2 - \lambda_2^2\xi^8 + 4\lambda_2\xi^9 + \xi^{11} + \sum_{m+n\geq 11} L_{mnk}\xi^m\lambda_1^n\lambda_2^k = 0. \tag{11}$$

SOLUTION. The diagram D_1 is represented in Figure 1 (a). It consists of 3 links: A_1A_3 with slope 1, A_3A_4 with slope 1/2 and A_4A_6 with slope 0. Two points A_2 and A_5 from R_1 are also shown in Figure 1 (a); they belong to D_1. The other part of R_1 is not situated below the straight line (U) passing through points (11; 0) and (0; 11). The diagram D_2 is represented in Figure 1 (b). It consists of 4 links: B_1B_3 with slope -1,

[5]The number l_0 is equal to the multiplicity of the zero root of the DP for the left link of $\bar{D}_2$.

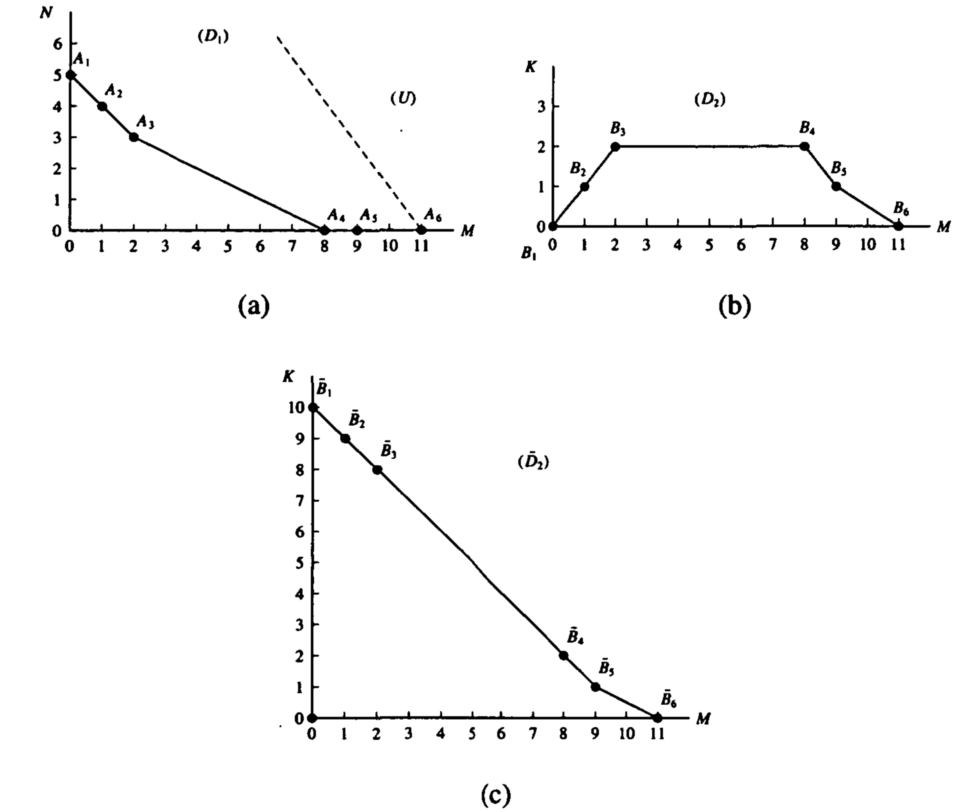

FIGURE 1. The diagram for (11) (see Example 1).

B_3B_4 with slope 0, B_4B_5 with slope 1 and B_5B_6 with slope 1/2. The diagram D_2 is not standard. For standardising we use (4). For the point B_4 and B_5, the right side of (4) is equal to 2; we have $s_0 = 2$. Using (6), Figures 1 (a) and (b), we define $\bar{s} = 1$. Thus $s = \max(2; 1)$ and by the substitution $\lambda_1 = \mu_1\lambda_2^2$, (11) is standardised. Further, we construct the diagram $\bar{D}_2$ (see Figure 1 (c)). It consists of 4 links: $\bar{B}_1\bar{B}_3$ (the slope is equal to 1, the DP is $\eta^2 + 2\eta + 1$, its roots are $-1, -1$ (double root)); $\bar{B}_3\bar{B}_4$ (the slope is 1, the DP is $-\eta^8 + \eta^2$, its non-zero roots $a_0^{(k)}$ ($k = 1, \dots, 6$) are the system of all roots $\sqrt[6]{1}$, each root is simple); $\bar{B}_4\bar{B}_5$ (the slope is 1, the DP is $4\eta^9 - \eta^8$; the unique non-zero root of the DP is $b_0 = 1/4$ (simple)); $\bar{B}_5\bar{B}_6$ (the slope is 1/2, the DP is $\eta^{11} + 4\eta^9$, the non-zero roots of this polynomial are simple, they are $c_0^{(1)} = 2i$, $c_0^{(2)} = -2i$).

We obtain for (A_3A_4, $\bar{B}_3\bar{B}_4$), six SS: $\xi^{(k)} = a_0^{(k)}\mu_1^{1/2}\lambda_2 + \sum_{m+n\geq 1} a_{mn}^{(k)}\mu_1^{(m+1)/2}\lambda_2^{n+1}$, $k = 1, \dots, 6$; for (A_4A_6, $\bar{B}_4\bar{B}_5$), one SS: $\xi = b_0\lambda_2 + \sum_{m+n\geq 1} b_{mn}\mu_1^m\lambda_2^{n+1}$; for ($A_4A_6$, $\bar{B}_5\bar{B}_6$), two SS: $\xi^{(\nu)} = c_0^{(\nu)}\lambda_2^{1/2} + \sum_{m+n\geq 1} c_{mn}^{(\nu)}\mu_1^m\lambda_2^{(n+1)/2}$, $\nu = 1, 2$, where

$\lambda_1 = \mu_1\lambda_2^2$.

For obtaining the SS corresponding to $(A_1A_3, \bar{B}_1\bar{B}_3)$, we use the substitution $\xi = (v-1)\mu_1\lambda_2$. We get the equation

$$v^2 - \mu_1^3 + \cdots = 0, \tag{12}$$

where the left-hand side does not include the terms of the form $a\lambda_2^k$ and $b\lambda_2^k v$ (k is a natural number). Equation (12) is standard. The diagram D_1 for (12) consists of a unique link joined to the points (0; 3) and (2; 0) (the slope is 3/2); the diagram D_2 also consists of one link denoted as C_1C_2 ($C_1(0;0)$ and $C_2(2;0)$). Both roots of the DP for C_1C_2 are simple and equal to 1 and -1. Equation (12) has two SS; they are of the form

$$v^{(k)} = f_0^{(k)}\mu_1^{3/2} + \sum_{m+n\geq 1} f_{mn}^{(k)}\mu_1^{(m+3)/2}\lambda_2^n, \quad \left(f_0^{(1)} = 1,\ f_0^{(2)} = -1;\ k = 1, 2\right).$$

Finally, the pair of links $(A_1A_3, \bar{B}_1\bar{B}_3)$ provides two SS: $\xi^{(k)} = -\mu_1\lambda_2\left(1 - v^{(k)}\right)$, where $k = 1, 2$, and $\lambda_1 = \mu_1\lambda_2^2$.

3. Critical situation

If (1) is vectorial ($r > 1$), then the described method may not always obtain all the SS. For example, the two-dimensional equation

$$\varphi_{200}(\xi) + \varphi_{300}(\xi) + \varphi_{010}\lambda_1 + \varphi_{001}\lambda_2 = 0, \tag{13}$$

where $\xi = (\xi_1, \xi_2)$, $\varphi_{200}(\xi) = \{3\xi_1^2 - \xi_2^2; 0\}$, $\varphi_{300}(\xi) = \{0; (\xi_1 - \xi_2)^3\}$, $\varphi_{010} = \varphi_{001} = \{1; 8\}$, has six SS. However, if we use Newton's diagram method, we get the following. The diagram D_1 for (13) consists of one link with ends (0; 0) and (2; 0) ($p^{(1)} = 0$); the diagram D_2 consists of a unique link AB ($A(0;1)$, $B(2;0)$) with slope 1/2. The substitution $\lambda_1 = \mu_1\lambda_2$ reduces (13) to the standard case:

$$\varphi_{200}(\xi) + \varphi_{300}(\xi) + \tilde{\varphi}_{011}\mu_1\lambda_2 + \varphi_{001}\lambda_2 = 0, \tag{13'}$$

where $\tilde{\varphi}_{011} = \varphi_{010}$. The view of $\bar{D}_2$ is the same as D_2, but $\bar{D}_2$ is a standard diagram. The DP for AB of $\bar{D}_2$ is $\{3\eta_1^2 - \eta_2^2 + 1; 8\}$. This polynomial is not solvable in $\mathbb{C}^2$ and therefore Newton's diagram method gives nothing.

In order that the total number of SS be finite for the vectorial equation (1) and all the SS be obtainable by using Newton's diagram method, the following condition is sufficient: all the fields, corresponding to the right ends of the links of $\bar{D}_2$ (for all stages of using this method), are non-degenerate in $\mathbb{C}^r$. The above condition is a corollary from the following theorem [2].

THEOREM. *Let*

$$Q_i(x_1, \dots, x_n) + \Omega_i(x_1, \dots, x_n) = 0, \quad i = 1, \dots, n, \tag{14}$$

be an algebraic system with complex coefficients, where Q_i is a homogeneous polynomial of degree n_i ($n_i \geq 1$) and Ω_i is a zero polynomial or a polynomial of degree $m_i < n_i$. Then the following is true: if the field $\{Q_1, \dots, Q_n\}$ is non-degenerate in $\mathbb{C}^n$, then (14) *is regular in $\mathbb{C}^n$, that is,* (14) *is solvable in $\mathbb{C}^n$ and the total number of its solutions is finite.*

The non-degenerating condition for polynomial fields in $\mathbb{C}^n$ is also described in [2]. For (13) the sufficient condition for using Newton's diagram method is infringed, because the field φ_{200} is degenerate (the point $(2, 0, 0) \in Q_0$ provides the right end of the link $AB \in \bar{D}_2$). Moreover, the DE for AB is not solvable in $\mathbb{C}^2$. The point of Q providing the right end link of $\bar{D}_2$ is called a bad point of Q if the DE for this link is not regular in $\mathbb{C}^r$. In the case when Q_0 has bad points, we can offer two methods. The first one is the combination of Newton's diagram method and a new method called the method of removal of the bad corner points. The main idea is the following: if $(m, n, k) \in Q$ is a bad point, we remove it from Q and construct for the rest of Q the diagrams D_1 and $\bar{D}_2$.[6] In this situation we use the diagram only to define the possible exponents for the small parameters. For obtaining the coefficients of the expansion of the SS, we use a substitution in the form

$$\xi = \eta\mu_1^{p_i}\lambda_2^{p_{\nu,i}} + x(\mu_1, \lambda_2)\mu_1^{p_i}\lambda_2^{p_{\nu,i}}, \tag{15}$$

where $\eta = (\eta_1, \dots, \eta_r)$, $x = (x_1, \dots, x_r)$ and $x(\mu_1, \lambda_2) \to 0$ as $(\mu_1, \lambda_2) \to (0, 0)$.

To illustrate this method we choose (13). The lattice for (13) is

$$Q = \{(2, 0, 0), (3, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

We showed before that $(2, 0, 0)$ is a bad point of Q. According to this method, we remove $(2, 0, 0)$ from Q and construct the diagrams D_1 and D_2 for the lattice $\{(3, 0, 0), (0, 1, 0), (0, 0, 1)\}$: the diagram D_1 consists of a unique link with slope 0; the diagram D_2 also consists of a unique link joining $(0; 1)$ and $(3; 0)$ with slope $1/3$. Using the substitution $\lambda_1 = \mu_1\lambda_2$, we get the standard diagram $\bar{D}_2$. The view of $\bar{D}_2$ is the same as D_2. Thus the possible exponents are 0 and $1/3$.

Now, for obtaining the SS of (13), we use (15), where $p_i = 0$ and $p_{\nu,i} = 1/3$. We get the system

$$\begin{aligned} -x_2^2 + 3\eta_1^2 - \eta_2^2 + 6\eta_1 x_1 - 2\eta_2 x_2 + 3x_1^2 + \mu_1\lambda_2^{1/3} + \lambda_2^{1/3} &= 0, \\ [(\eta_1 - \eta_2) + (x_1 - x_2)]^3 + 8 + 8\mu_1 &= 0. \end{aligned} \tag{16}$$

[6] When we say "Newton's diagram" for $\pi_1(Q)$ or $\pi_2(Q)$, we have in view only their decreasing branches.

Passing to the limit as $(\mu_1, \lambda_2) \to (0, 0)$ in (16), we obtain the DE for η:

$$3\eta_1^2 - \eta_2^2 = 0, \quad (\eta_1 - \eta_2)^3 + 8 = 0. \tag{17}$$

The system (17) is regular and has exactly six simple roots $\eta^i = \left(\eta_1^{(i)}, \eta_2^{(i)}\right)$, $i = 1, \dots, 6$ (only two of them are real). For obtaining $x(\mu_1, \lambda_2)$, we substitute $\lambda_2 = \mu_2^3$ into (16) and for each root η^i obtain the system

$$\begin{aligned} 6\eta_1^{(i)}x_1 - 2\eta_2^{(i)}x_2 + 3x_1^2 - x_2^2 + \mu_1\mu_2 + \mu_2 = 0, \\ 3\bar{\eta}_i^2(x_1 - x_2) + 3\bar{\eta}_i(x_1 - x_2)^2 + (x_1 - x_2)^3 + 8\mu_1 = 0. \end{aligned} \tag{18_i}$$

where $\bar{\eta}_i = \eta_1^{(i)} - \eta_2^{(i)}$. Again using the method for each (18_i), we obtain the first term of the expressions for x. However, the roots η^i are simple and we can use the IC method.

Consequently, the problem for (13) has exactly six SS and they have the view

$$\xi^i = \eta^i\mu_2 + \mu_2 \sum_{k+\nu=1}^{\infty} a^i\mu_1^k\mu_2^\nu, \quad (\lambda_1 = \mu_1\mu_2^3,\ \lambda_2 = \mu_2^3;\ i = 1, \dots, 6).$$

Two SS among them belong to $\mathbb{R}^2$.

The second method, called the NDE method, is a combination of Newton's diagram method for scalar equations and the method of elimination. In the case of a single scalar small parameter it was described in [4, 6]. For several small parameters the way to obtain the SS is analogous. According to the NDE method, we get the scalar equations for each component of the unknown variables and use Newton's diagram for each equation. The main difference is in the following: in the case of a single parameter we use the classical Newton's diagram method, in the case of several parameters we use the method described in this article.

Note here that the necessary and sufficient condition for the general system with a single small parameter to be regular is presented in [4, 6]. In the case of several parameters the condition is the same.

4. Case of n ($n > 2$) parameters

In this section, Newton's diagram method is extended to the general case, when the equation includes more than two small parameters. We investigate the equation

$$\Gamma(\xi, \lambda_1, \dots, \lambda_n) \equiv \sum_{m+k_1+\dots+k_n=1}^{\infty} \varphi_{mk_1\cdots k_n}(\xi)\lambda_1^{k_1}\cdots\lambda_n^{k_n} = 0, \tag{19}$$

where $\lambda_1, \dots, \lambda_n$ are small parameters ($n > 2$), $\varphi_{m\dots}(\xi)$ is a vectorial homogeneous polynomial of power m or a zero polynomial. Let us assume that $\varphi_{m0\cdots0}(\xi) = 0$ at $m < m_0$ and $\varphi_{m_00\cdots0}(\xi) \neq 0$. We are studying the problem about the SS $\xi(\lambda_1, \dots, \lambda_n)$ of (19) for $(\lambda_1, \dots, \lambda_n)$ near $(0, \dots, 0)$ and near the hyperplane $\lambda_1 = 0$ without $\lambda_2 = \cdots = \lambda_n = 0$. Let us denote $Q = \{(m, k_1, \dots, k_n) : \varphi_{mk_1\cdots k_n}(\xi) \neq 0\}$, $Q_0 = \{(m, k_1, \dots, k_n) \in Q : m \leq m_0\}$, $R_i = \pi_i(Q_0)$, $i = 1, \dots, 6$, where π_i is the orthogonal projection of $\mathbb{R}^{n+1}$ on the coordinate plane OMK_i. On the plane OMK_1, we construct Newton's diagram $D_1 = \{L^{i_1} \mid i_1 = 1, \dots, \alpha_1\}$ for R_1. After that we construct, on the plane OMK_2, Newton's diagram for each lattice $\pi_2(\pi_1^{-1}(L^{i_1}) \cap Q_0)$, $i_1 = 1, \dots, \alpha_1$. We get a diagram $D_2 = \{L^{i_1, i_2} \mid i_2 = 1, \dots, \alpha_2^{(i_1)};\ i_1 = 1, \dots, \alpha_1\}$. If D_2 is Newton's diagram for the lattice R_2 and it has no points from $R_2 \setminus \bigcup_{i_1} \pi_2(\pi_1^{-1}(L^{i_1}) \cap Q_0)$, then,[7] using D_2 by analogy with D_1, we construct the diagram D_3 on the plane OMK_3. If D_2 does not satisfy these conditions we first reduce D_2 to $\bar{D}_2$ (to Newton's diagram without additional points), substituting $\lambda_1 = \mu_1 \lambda_2^{s_1}$ (s_1 is chosen according to (4) and (6), and Notes 1 and 2) and only after that we use $\bar{D}_2$ and construct D_3. This procedure is continued until the diagram D_n comes onto the plane OMK_n. If D_n is not a standard diagram, we use the substitution $\lambda_{n-1} = \mu_{n-1} \lambda_n^{s_{n-1}}$ and reduce D_n to the standard form. For the last diagram it is assumed that the slopes of its links make up only a non-decreasing sequence and these links can include points of R_n with prototypes not belonging to $\bar{D}_{n-1}$.

For example, we consider the equation

$$\varphi_{0201}(\xi)\lambda_1^2\lambda_3 + \varphi_{2111}(\xi)\lambda_1\lambda_2\lambda_3 + \varphi_{4022}(\xi)\lambda_2^2\lambda_3^2 + \varphi_{6101}(\xi)\lambda_1\lambda_3 + \varphi_{8001}(\xi)\lambda_3 + \varphi_{10;000}(\xi) + \sum_{m+k_1 \geq 11} \varphi_{mk_1k_2k_3}(\xi)\lambda_1^{k_1}\lambda_2^{k_2}\lambda_3^{k_3} = 0 \qquad (\star)$$

with three parameters $\lambda_1^{k_1}, \lambda_2^{k_2}, \lambda_3^{k_3}$. We get the following.

The diagram D_1 (Figure 2 (a)) consists of two links: A_1A_3 with slope $1/2$ and A_3A_6 with slope 0. The points A_2, A_4 and A_5 from R_1 also belong to D_1. The other points of R_1 are situated on the right of the line connecting the points $(0;10)$ and $(10;0)$.

The diagram D_2 is presented in Figure 2 (b). Using the substitution $\lambda_1 = \mu_1\lambda_2^3$, the diagram D_2 is reduced to Newton's diagram $\bar{D}_2$ without additional points. The diagram $\bar{D}_2$ (Figure 2 (c)) consists of three links: $\bar{B}_1\bar{B}_3$ with slope 1, $\bar{B}_3\bar{B}_5$ with slope $1/2$ and $\bar{B}_5\bar{B}_6$ with slope 0. The other points of $\bar{R}_2$ are situated on the right of the line connecting the points $(0;30)$ and $(10;0)$.

The diagram D_3 is represented in Figure 2 (d). The substitution $\lambda_2 = \mu_2\lambda_3^2$ reduces D_3 to the standard diagram $\bar{D}_3$ (Figure 3).

Let us suppose that D_1 for (19) is Newton's diagram; $D_2, \dots, D_{n-1}$ are Newton's diagrams without additional points and D_n is a standard diagram, that is, we have the

[7]We name this diagram Newton's diagram without additional points.

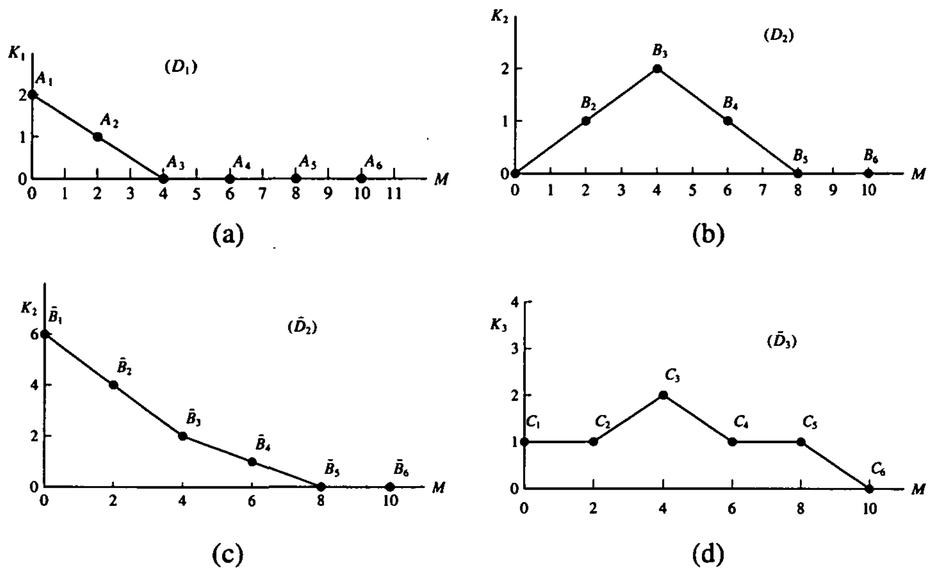

FIGURE 2. The diagrams D_1, D_2, $\bar{D}_2$ and D_3 for $(\star)$.

standard case.[8] We write (19) in the form

$$\sum{}' \varphi_{mk_1\cdots k_n}(\xi)\lambda_1^{k_1}\cdots\lambda_n^{k_n} + \sum{}'' \varphi_{mk_1\cdots k_n}(\xi)\lambda_1^{k_1}\cdots\lambda_n^{k_n} = 0, \qquad (19')$$

where $\sum'$ denotes the sum of all non-zero terms which satisfy the condition:

$$(m, k_1) \in L^{i_1}, \quad (m, k_2) \in L^{i_1, i_2}, \quad \ldots, \quad (m, k_n) \in L^{i_1,\ldots,i_n}.$$

Here L^{i_1} is any link of D_1, L^{i_1,i_2} is any link of D_2 corresponding to L^{i_1} and *etc.*; at last, $L^{i_1,\ldots,i_n}$ is any link of D_n corresponding to the link $L^{i_1,\ldots,i_{n-1}}$. The sum $\sum''$ denotes the sum of all other terms of (19). We assume also that r_k/s_k ($(r_k, s_k) = 1$) is the slope of $L^{i_1,\ldots,i_k}$, $k = 1, \ldots, n$.

Let us make the substitutions $\xi = \eta\mu_1^{r_1}\cdots\mu_n^{r_n}$; $\lambda_1 = \mu_1^{s_1}, \ldots, \lambda_n = \mu_n^{s_n}$ in (19′). Then the equation is reduced to the form

$$P(\eta) + \sum{}'' \varphi_{mk_1\cdots k_n}(\eta)\mu_1^{\nu_1}\cdots\mu_n^{\nu_n} = 0,$$

where $\nu_1, \ldots, \nu_n$ are non-negative numbers, $\nu_1 + \cdots + \nu_n > 0$ and $P(\eta) = \sum' \varphi_{mk_1\cdots k_n}(\eta)$ is the DP for the link $L^{i_1,\ldots,i_n}$. The further operations, reasons and conclusions are analogous to the case of two small parameters.

Finally we can make some additional remarks.

[8]If not, we, at first, reduce (19) to the standard case.

https://doi.org/10.1017/S1446181100013924 Published online by Cambridge University Press

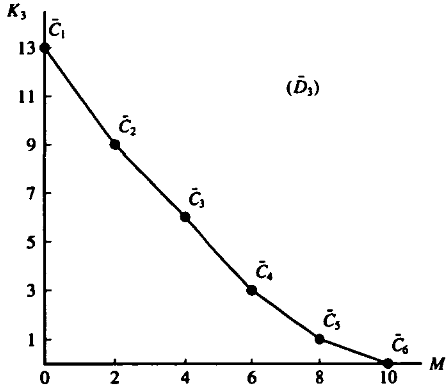

FIGURE 3. The diagram $\bar{D}_3$ for ($\star$).

REMARK 1. The stated method allows us to find SS defined in a neighbourhood of a hypersurface $\lambda_n = g(\lambda_1, \dots, \lambda_{n-1})$, where g is an analytical function relative to $\lambda_1^{1/p_1}, \dots, \lambda_{n-1}^{1/p_{n-1}}$, ($p_1, \dots, p_{n-1}$ are natural numbers) and $g \to 0$ as $(\lambda_1, \dots, \lambda_{n-1}) \to (0, \dots, 0)$.

Indeed, letting $\lambda_1 = \mu_2^{p_1}, \dots, \lambda_{n-1} = \mu_n^{p_{n-1}}, \lambda_n = g + \mu_1$ in (19), we reduce this problem to the problem about the SS for $(\mu_1, \dots, \mu_n)$ defined in a neighbourhood of the hyperplane $\mu_1 = 0$.

REMARK 2. If (19) is represented by the standard case and all roots of all the DP are simple, then all the SS of (19) are defined in some full neighbourhood of $\lambda_1 = \dots = \lambda_n = 0$.

If the diagrams D_i ($i = 1, \dots, k$; $k < n$) are Newton's diagrams without additional points and all root of the DP are simple, then the SS are defined in a neighbourhood of the hyperplane $\lambda_1 = \dots = \lambda_k = 0$. In this situation the number k is called the coefficient of standardisation. This coefficient is important in some applications.

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